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Stochastic bifurcation analysis in Brusselator system with white noise

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Abstract

In this paper, we mainly study the stochastic stability and stochastic bifurcation of Brusselator system with multiplicative white noise. Firstly, by a polar coordinate transformation and a stochastic averaging method, the original system is transformed into an Itô averaging diffusion system. Secondly, we apply the largest Lyapunov exponent and the singular boundary theory to analyze the stochastic local and global stability. Thirdly, by means of the properties of invariant measures, the stochastic dynamical bifurcations of stochastic averaging Itô diffusion equation associated with the original system is considered. And we investigate the phenomenological bifurcation by analyzing the associated Fokker–Planck equation. We will show that, from the view point of random dynamical systems, the noise "destroys" the deterministic stability. Finally, an example is given to illustrate the effectiveness of our analyzing procedure.

Keywords: Stochastic bifurcation; Stochastic averaging; Largest Lyapunov exponent; Singular boundary theory; Brusselator system

1 Introduction

The determined Brusselator is a coupled differential equation written as

$$\begin{cases} u' = A - (B+1)u + u^2 v, \\ v' = Bu - u^2 v, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ are the concentrations of reactants at time t, respectively. $A > 0$ and $B > 0$ are external system parameters describing the (constant) supply of "reservoir" chemicals. As a theoretical model for a type of autocatalytic reaction, the Brusselator model has attracted the attention of many scholars since it was proposed by Prigogine and Lefever [1]. Abundant dynamical behaviors have been reported for several decades; see [2–6].

Unfortunately, it is generally recognized that the effects of "external fluctuations" (environmental noises) are inevitable in dynamical systems due to various factors, such as possible changes of system parameters, variations in excitations, errors in modeling schemes. Therefore, to be more realistic, there is need to take into account stochastic systems which consider external influences in complex physical systems. There are a lot of papers about the stochastic Brusselator model. Tu and You [7] considered the existence of random attractor. In [8], Arnold et al. performed mainly a numerical study of the bifurcation behavior of the Brusselator under parametric white noise. For other results, see [9–11].

There exist two ways to develop a stochastic model corresponding to a deterministic model to study the effect of fluctuating environment on its dynamical behavior. The first one is to replace the time independent parameters involved with the deterministic model system by some random parameters. For example, according to the orthogonal polynomial approximation in Hilbert space, Ma [12] considered the stochastic Hopf bifurcation of the following stochastic Brusselator model:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = (\bar{B}-1)x + A^2y + \frac{\bar{B}x^2}{A} + 2Axy + x^2y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -\bar{B}x - A^2y - \frac{\bar{B}x^2}{A} - 2Axy - x^2y, \end{cases}$$

where $\bar{B}$ is a random parameter. The obtained critical value of the stochastic Hopf bifurcation is $B_c = A^2 + 1 - \frac{\sqrt{3}\delta}{2}$.

And the second one is to add randomly fluctuating driving force directly to the deterministic equations without affecting any particular parameter to incorporate the effect of randomly fluctuating environment ([13, 14]). This type of stochastic perturbation technique to study the effect of fluctuating environment was first introduced by Beretta et al. [15] on a population model system, and Shaikhet [16] and then other researchers used it [17].

In this paper, following the second method, we assume that stochastic perturbations of the state variables around their steady-state values $E^*(u^*, v^*)$ are of Gaussian white noise type which are proportional to the distances of u, v from their steady-state values E^*, respectively. So, in order to study the effect of environmental fluctuation, the deterministic model system (1.1) can be extended to a stochastic differential equation system as follows:

$$\begin{cases} u' = (A - (B+1)u + u^2v)\,\mathrm{d}t - \delta(u-u^*)\,\mathrm{d}W(t), \\ v' = (Bu - u^2v)\,\mathrm{d}t + \delta(v-v^*)\,\mathrm{d}W(t), \end{cases} \tag{1.2}$$

where $(u^*, v^*) = (A, \frac{B}{A})$, A, B and δ are the deterministic parameters; $W(t)$ is standard real-valued Wiener process on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Therefore, in this paper, by applying the stochastic average method, singular boundary theory and invariant measure theory, we aim to analyze stochastic stability and bifurcation of system (1.2).

The organization of the rest of this paper is as follows: In Sect. 2, we state some preliminary results needed in later sections. Moreover, by applying polar coordinate transformation and stochastic averaging method, we obtain a stochastic averaging Itô diffusion equation. In Sect. 3, we establish our main results ensuring the stochastic local stability and global stability of system (1.2) by using the largest Lyapunov exponent and the singular boundary theory. In Sect. 4, the stochastic dynamical and phenomenological bifurcations of stochastic averaging Itô diffusion equation associated with system (1.2) will be discussed by means of the properties of invariant measures and by considering the Fokker–Planck equation, the phenomenological bifurcation of system (1.2) is shown. In Sect. 5, numerical simulation results are given to illustrate the effectiveness of our analytical results. Finally, we present our discussions and conclusions.

2 Preliminaries

For the reader's convenience, firstly, we present the following results about singular boundary theory. To learn more about the classification of singular boundary, see [18] and [19], the reference cited therein.

According to [18] and [19], denote by x_s the boundaries of one-dimensional Itô stochastic differential equation

$$\mathrm{d}\rho = m(\rho)\,\mathrm{d}t + \sigma(\rho)\,\mathrm{d}W_\rho(t).$$

The subscript s can be either l or r, denoting the left or right boundary.

Singular Boundary of the First Kind: Consider a singular boundary x_s of the first kind, namely, the diffusion term $\sigma(x_s) = 0$. The boundary is said to be a shunt if $m(x_s) \neq 0$, and a trap if $m(x_s) = 0$.

We introduce the following definitions.

(i) α_s is the diffusion exponent of x_s, if

$$\sigma^2(x) = O|x - x_s|^{\alpha_s}(x \to x_s), \quad \alpha_s \geq 0.$$

(ii) β_s is the drift exponent of x_s, if

$$m(x) = O|x - x_s|^{\beta_s}(x \to x_s), \quad \beta_s \geq 0.$$

(iii) c_s is the drift exponent of x_s, given by

$$c_l = \lim_{x\to x_l^+} \frac{2m(x)(x - x_l)^{\alpha_l - \beta_l}}{\sigma^2(x)}, \qquad c_r = -\lim_{x\to x_r^-} \frac{2m(x)(x_r - x)^{\alpha_r - \beta_r}}{\sigma^2(x)}.$$

Singular Boundary of the Second Kind: We call the boundary x_s the singular boundary of the second kind if the drift term $m(x_s)$ is unbounded and $|x_s| < \infty$.

(i) α_s is the diffusion exponent of x_s, if

$$B^2(x) = O|x - x_s|^{-\alpha_s}(x \to x_s), \quad \alpha_s \geq 0.$$

(ii) β_s is the drift exponent of x_s, if

$$m(x) = O|x - x_s|^{-\beta_s}(x \to x_s), \quad \beta_s \geq 0.$$

(iii) c_s is the drift exponent of x_s, given by

$$c_l = \lim_{x\to x_l^+} \frac{2m(x)(x - x_l)^{\beta_l - \alpha_l}}{\sigma^2(x)}, \qquad c_r = -\lim_{x\to x_r^-} \frac{2m(x)(x_r - x)^{\beta_r - \alpha_r}}{\sigma^2(x)}.$$

We also give some definitions as regards stochastic stability and bifurcation.

Definition 2.1 (D-Bifurcation [18, 20]) Dynamical bifurcation is concerned with a family random dynamical systems which is differential and has invariant measure μ_α. If there exists a constant α_D satisfying the condition in any neighborhood of α_D, there exist another constant α and the corresponding invariant measure $\nu_\alpha \neq \mu_\alpha$ satisfying $\nu_\alpha \to \mu_\alpha$ as $\alpha \to \alpha_D$. Then the constant α_D is a point of dynamical bifurcation.

Definition 2.2 (P-Bifurcation [18, 20]) Phenomenological bifurcation is concerned with the change in the shape of stationary probability density of a family random dynamical systems with the change of the parameter. If there exists a constant α_0 satisfying the condition in any neighborhood of α_D, there exist two other constants α_1, α_2 and their corresponding invariant measures $p_{\alpha_1}, p_{\alpha_2}$ where p_{α_1} and p_{α_2} are not equivalent. Then the constant α_0 is a point of phenomenological bifurcation.

Definition 2.3 (Stochastically Stable [18–20]) The trivial solution $x(t, t_0, x_0)$ of stochastic differential equation is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $\alpha > 0$, there exists a $\delta = \delta(\varepsilon, \delta, t_0) > 0$ such that

$$P\{|x(t, t_0, x_0)| < \alpha, \forall t \geq t_0\} \geq 1 - \varepsilon,$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

Next, we will apply polar coordinate transformation and stochastic averaging method to obtain an averaging Itô diffusion equation from system (1.2).

Note that system (1.2) has the same steady-state values $E^*(A, \frac{B}{A})$ as system (1.1). By applying the translation

$$\begin{cases} x = u - A, \\ y = v - \frac{B}{A}, \end{cases}$$

system (1.2) becomes

$$\begin{cases} \mathrm{d}x = ((B-1)x + A^2 y + 2Axy + x^2 y + \frac{B}{A}x^2)\,\mathrm{d}t - \delta x\,\mathrm{d}W(t), \\ \mathrm{d}y = (-Bx - A^2 y - 2Axy - x^2 y - \frac{B}{A}x^2)\,\mathrm{d}t + \delta y\,\mathrm{d}W(t). \end{cases} \tag{2.1}$$

Hence, to analyze stochastic stability and bifurcation of system (1.2) at E^* is equivalent to consider the stochastic stability and bifurcation of system (1.2) at $(0, 0)$.

By the polar coordinate transformation $x = \rho\cos\theta$, $y = \rho\sin\theta$ and Itô formula, we have

$$\begin{cases} \mathrm{d}\rho = f_1(\rho,\theta)\,\mathrm{d}t + g_{11}(\rho,\theta)\,\mathrm{d}W(t), \\ \mathrm{d}\theta = f_2(\rho,\theta)\,\mathrm{d}t + g_{21}(\rho,\theta)\,\mathrm{d}W(t), \end{cases} \tag{2.2}$$

where

$$\begin{cases} f_1(\rho,\theta) = (A^2(\cos\theta\sin\theta - \sin^2\theta) + B(\cos^2\theta - \sin\theta\cos\theta) - \cos^2\theta)\rho \\ \qquad + (2A(\sin\theta\cos^2\theta - \sin^2\theta\cos\theta) + \frac{B\cos^2\theta}{A}(\cos\theta - \sin\theta))\rho^2 \\ \qquad + (\sin\theta\cos^3\theta - (\sin\theta\cos\theta)^2)\rho^3, \\ f_2(\rho,\theta) = (-(A^2\sin\theta + B\cos\theta)(\cos\theta + \sin\theta) + \cos\theta\sin\theta) \\ \qquad - ((2A\sin\theta\cos\theta + \frac{B}{A}\cos^2\theta)(\cos\theta + \sin\theta))\rho \\ \qquad - (\sin\theta\cos^2\theta(\cos\theta + \sin\theta))\rho^2, \\ g_{11}(\rho,\theta) = -\delta\rho\cos 2\theta, \\ g_{21}(\rho,\theta) = \delta\sin 2\theta. \end{cases} \tag{2.3}$$

According to the Khasminskii limiting theorem [18–20], the stochastic response process $\rho(t), \theta(t)$ of system (2.2) weakly converges to a two-dimensional Markov diffusion process. Therefore, by using the stochastic averaging method introduced in Chap. 5, Sect. 5.1 of Ref. [18], we obtain the averaged Itô stochastic differential equation for system (2.2):

$$\begin{cases} \mathrm{d}\rho = m_1(\rho)\,\mathrm{d}t + \sigma_1(\rho)\,\mathrm{d}W_\rho(t), \\ \mathrm{d}\theta = m_2(\rho)\,\mathrm{d}t + \sigma_2(\rho)\,\mathrm{d}W_\theta(t), \end{cases} \tag{2.4}$$

where $W_\rho(t)$ and $W_\theta(t)$ are independent and standard Wiener processes, the drift coefficients $m_i(\rho)$ $(i = 1, 2)$ and the square of diffusion coefficients $\sigma_i(\rho)$ $(i = 1, 2)$ are

$$\begin{aligned} m_1(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{f_1(\rho,\theta) + \frac{1}{2}\left[\frac{\partial g_{11}(\rho,\theta)}{\partial\rho}g_{11}(\rho,\theta) + \frac{\partial g_{11}(\rho,\theta)}{\partial\theta}g_{21}(\rho,\theta)\right]\right\}\mathrm{d}\theta \\ &= \frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho - \frac{1}{8}\rho^3, \\ \sigma_1^2(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\bigl(g_{11}(\rho,\theta)\bigr)^2\,\mathrm{d}\theta = \frac{1}{2}\delta^2\rho^2, \\ m_2(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{f_2(\rho,\theta) + \frac{1}{2}\left[\frac{\partial g_{21}(\rho,\theta)}{\partial\rho}g_{11}(\rho,\theta) + \frac{\partial g_{21}(\rho,\theta)}{\partial\theta}g_{21}(\rho,\theta)\right]\right\}\mathrm{d}\theta \\ &= -\frac{1}{2}\bigl(B + A^2\bigr) - \frac{1}{8}\rho^2, \\ \sigma_2^2(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\bigl(g_{21}(\rho,\theta)\bigr)^2\,\mathrm{d}\theta = \frac{\delta^2}{2}. \end{aligned}$$

Therefore, system (2.4) can be rewritten as

$$\begin{cases} \mathrm{d}\rho = [\frac{1}{2}(B - A^2 - 1 + \frac{3\delta^2}{2})\rho - \frac{1}{8}\rho^3]\,\mathrm{d}t + (\frac{\delta^2\rho^2}{2})^{\frac{1}{2}}\,\mathrm{d}W_\rho(t), \\ \mathrm{d}\theta = [-\frac{1}{2}(B + A^2) - \frac{1}{8}\rho^2]\,\mathrm{d}t + (\frac{\delta^2}{2})^{\frac{1}{2}}\,\mathrm{d}W_\theta(t). \end{cases} \tag{2.5}$$

Here, we can find that the average amplitude $\rho(t)$ is a one-dimensional Markov diffusion process. Thus, it is efficient to consider the averaging amplitude equation of system (2.5) to obtain the critical point of stochastic stability and bifurcation phenomena of system (1.2). That is, we only need to investigate the following equation:

$$\mathrm{d}\rho = \left[\frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho - \frac{1}{8}\rho^3\right]\mathrm{d}t + \left(\frac{\delta^2\rho^2}{2}\right)^{\frac{1}{2}}\mathrm{d}W_\rho(t). \tag{2.6}$$

3 Stochastic stability

The essential idea of this section is to consider the stability properties of the trivial solution $\rho = 0$ of the stochastic model (2.6), which also addresses the asymptotical stability of the fixed point $E^*(A, \frac{B}{A})$ of system (1.2). Hereafter, we always assume $B < A^2 + 1$.

3.1 Local stochastic stability

In the theory of stochastic stabilization, the largest Lyapunov exponent is fairly prevalent as a measure to judge stability of a stochastic dynamical system. Hence, we will study the change of stability of the averaging amplitude equation (2.6) at the equilibrium point by calculating the largest Lyapunov exponent of the associated linearized system. We obtain the following results.

Theorem 3.1
(i) *When* $\delta^2 < A^2 + 1 - B$, *the trivial solution of the linear Itô stochastic differential equation* (2.6) *is stable in the meaning of probability, i.e., the stochastic system* (1.2) *is stable at the equilibrium point* E^*.
(ii) *When* $\delta^2 > A^2 + 1 - B$, *the trivial solution of the linear Itô stochastic differential equation* (2.6) *is unstable in the meaning of probability, i.e., the stochastic system* (1.2) *is unstable at the equilibrium point* E^*.

Proof To evaluate the largest Lyapunov exponent of averaged Itô equation (2.6), we consider the stability of linear Itô stochastic differential equation at the trivial solution $\rho = 0$ first. Then one can obtain the following linearized Itô differential equation:

$$\mathrm{d}\rho = \frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho\,\mathrm{d}t + \left(\frac{\delta^2}{2}\right)^{\frac{1}{2}}\rho\,\mathrm{d}W_\rho(t), \tag{3.1}$$

of which the solution is

$$\rho(t) = \rho(0)\exp\left(\int_0^t\left[\frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right) - \frac{\delta^2}{4}\right]\mathrm{d}s + \int_0^t\left(\frac{\delta^2}{2}\right)^{\frac{1}{2}}\mathrm{d}W_\rho(t)\right), \tag{3.2}$$

so the associated largest Lyapunov exponent is

$$\lambda = \lim_{t\to+\infty}\frac{\ln\|\rho(t)\|}{t} = \frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right) - \frac{\delta^2}{4} = \frac{B - A^2 - 1 + \delta^2}{2}.$$

Based on Oseledet's multiplicative ergodic theorem [21] (see Chap. 3 in [20] for more details), the necessary and sufficient conditions for asymptotic stability with probability one of the trivial solution of system (1.2) require that the largest Lyapunov exponent in Eq. (2.6) should be negative. Consequently, the trivial solution is locally asymptotic stable if and only if $\lambda < 0$, i.e., $\delta^2 < A^2 + 1 - B$.

This completes the proof. □

3.2 Global stochastic stability

It is worth to mention that the largest Lyapunov exponent based on the multiplicative ergodic theorem is effective to estimate local stability, but it is incapable of global stability. Alternatively, we use the singular boundary theory associated with Eq. (2.6) to examine the global behaviors of the trivial solution of the system.

Generally, the boundaries of diffusion process are singular, and the boundary classification is often determined by the diffusion exponent, the drift exponent and the character value [18].

Theorem 3.2 *If* $\delta^2 < A^2 + 1 - B$, *the trivial solution of system* (2.6) *is globally stable, i.e., the stochastic system* (1.2) *is globally stable at the equilibrium point.*

Proof As $\rho \to 0^+$, it is found that the asymptotic expressions for the drift term $m_1(\rho)$ and the square of the diffusion term $\sigma_1(\rho)$ are of the forms

$$m_1(\rho) = \frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho + O(\rho),$$
$$\sigma_1^2(\rho) = \frac{\delta^2\rho^2}{2}.$$

From the above equations, we know that the diffusion exponent α_l (l denotes the left boundary), drift exponent β_l and character value c_l are, respectively,

$$\alpha_l = 2, \qquad \beta_l = 1, \qquad c_l = \lim_{\rho\to 0^+} \frac{2m_1(\rho)(\rho - 0)^{\alpha_l-\beta_l}}{\sigma_1^2(\rho)} = \frac{2(B - A^2 - 1 + \frac{3\delta^2}{2})}{\delta^2}.$$

Hence, the left boundary $\rho = 0$ of the averaged Itô (2.6) is a trap belong to the singular boundary of the first kind. In addition, according to the classification for singular boundary in Ref. [18] (see Table 2.8-2 in Sect. 2.8.2 for details), we know that the left boundary $\rho = 0$ is repulsively natural if $c_l > 1$, i.e., $\delta^2 > A^2 + 1 - B$, a strictly natural if $c_l = 1$, i.e., $\delta^2 = A^2 + 1 - B$, and an attractively natural if $c_l < 1$, i.e., $\delta^2 < A^2 + 1 - B$.

Similarly, for the right boundary $\rho \to +\infty$, it is found that the asymptotic expressions for the drift term $m_1(\rho)$ and the square of the diffusion term $\sigma_1(\rho)$ are of the forms

$$m_1(\rho) = -\frac{1}{8}\rho^3, \qquad \sigma_1^2(\rho) = \frac{\delta^2\rho^2}{2}.$$

So, it is a singular boundary of the second kind. Accordingly, the diffusion exponent α_r (r denotes the right boundary), the drift exponent β_r and character value c_r for the right boundary $\rho \to +\infty$ are, respectively,

$$\alpha_r = 2, \qquad \beta_r = 3, \qquad c_r = -\lim_{\rho\to +\infty} \frac{2m_1(\rho)\rho^{\alpha_r-\beta_r}}{\sigma_1^2(\rho)} = \frac{1}{2\delta^2} > 0,$$

i.e., for any $\delta > 0$, $c_r > -1$. According to the classification for singular boundary in [18] (see Table 2.8-4 in Sect. 2.8.2 for details), we know the right boundary $\rho = +\infty$ is a repulsively natural. Therefore, if $\delta^2 < A^2 + 1 - B$, the boundary $\rho = 0$ is attractively natural and the boundary $\rho = +\infty$ is exclusively natural. All solution curves enter the inner system from the right boundary and will be attracted by the left boundary, which implies that the trivial solution of system (2.6) is globally stable, i.e., the stochastic system (1.2) is globally stable at the equilibrium point.

This completes the proof. □

Remark 3.1 Note that if $\delta^2 < A^2 + 1 - B$ holds, the stochastic system (1.2) is not only locally but also globally stable at the equilibrium point. And the high-order-term coefficients of the drift term have no effects on the global stability of the trivial solution.

4 Stochastic bifurcation

In this section, we mainly investigate the bifurcation behaviors of some related systems from the dynamical viewpoint and phenomenological approach.

4.1 Stochastic D-bifurcation

In this subsection, similar to the discussions in [20, 22], we will prove that the stochastic system (2.6) undergoes a D-bifurcation when $\delta^2 = A^2 + 1 - B$. One can see Example 4.2.15 in [20] for detailed introductions.

Suppose $v_t = (\frac{1}{8})^{\frac{1}{2}}\rho$, we rewrite system (2.6) as

$$\mathrm{d}v_t = \left[\frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)v_t - v_t^3\right]\mathrm{d}t + \left(\frac{\delta^2}{2}\right)^{\frac{1}{2}} v_t \,\mathrm{d}W_r(t),$$

which is equivalent to the following Stratonovich stochastic differential equation:

$$\mathrm{d}v_t = \left(\alpha v_t - v_t^3\right)\mathrm{d}t + \epsilon v_t \circ \mathrm{d}W(t), \tag{4.1}$$

where $\alpha = \frac{B-A^2-1+\delta^2}{2}$, $\epsilon = (\frac{\delta^2}{2})^{\frac{1}{2}}$.

As is well known, (4.1) is solved by

$$v \to \psi_\alpha(t,\omega)v = \frac{v\exp(\alpha t + \epsilon W_t(\omega))}{(1 + 2v^2\int_0^t \exp[2(\alpha s + \epsilon W_s(\omega))]\,\mathrm{d}s)^{\frac{1}{2}}}, \tag{4.2}$$

where v is the initial value of v_t. We now determine the domain $D_\alpha(t,\omega)$, where $D_\alpha(t,\omega) := \{\phi \in \mathbf{R} | (t,\omega,\phi) \in D\} \subset X (D = \mathbf{R} \times \Omega \times X)$ is the (in general possibly empty) set of initial values $v \in \mathbf{R}$ for which the trajectories still exist at time t and the range $R_\alpha(t,\omega)$ of $\psi_\alpha(t,\omega)$: $D_\alpha(t,\omega) \to R_\alpha(t,\omega)$.

We obtain

$$D_\alpha(t,\omega) = \begin{cases} \mathbf{R}, & t \geq 0; \\ (-d_\alpha(t,\omega), d_\alpha(t,\omega)), & t < 0. \end{cases}$$

Here

$$d_\alpha(t,\omega) = \frac{1}{(2|\int_0^t \exp(2\alpha s + 2\epsilon W_s(\omega))\,\mathrm{d}s|)^{\frac{1}{2}}} > 0$$

and

$$R_\alpha(t,\omega) = D_\alpha\left(-t, \upsilon(t)\omega\right) = \begin{cases} (-r_\alpha(t,\omega), r_\alpha(t,\omega)), & t > 0; \\ \mathbf{R}, & t \leq 0, \end{cases}$$

where

$$r_\alpha(t,\omega) = d_\alpha\left(-t, \upsilon(t)\omega\right) = \frac{\exp(\alpha t + \epsilon W_t(\omega))}{(2|\int_0^t \exp(2(\alpha s + \epsilon W_s(\omega)))\,\mathrm{d}s|)^{\frac{1}{2}}} > 0.$$

We can now determine

$$E_\alpha(\omega) := \bigcap_{t\in\mathbf{R}} D_\alpha(t,\omega)$$

and obtain

$$E_\alpha(\omega) = \begin{cases} (-d_\alpha^-(t,\omega), d_\alpha^-(t,\omega)), & \alpha > 0; \\ \{0\}, & \alpha \le 0, \end{cases}$$

where

$$0 < d_\alpha^-(t,\omega) = \frac{1}{(2|\int_0^\infty \exp(2(\alpha s + \epsilon W_s(\omega)))\,\mathrm{d}s|)^{\frac{1}{2}}} < \infty.$$

The ergodic invariant measures are

- for $\alpha \le 0$, the only invariant measures is $\mu_{1,\omega}^\alpha = \delta_0$,
- for $\alpha > 0$, we have the three invariant forward Markov measures $\mu_{1,\omega}^\alpha = \delta_0$, $\mu_{2,\omega}^\alpha = \delta_{-d_\alpha^-(\omega)}$ and $\mu_{3,\omega}^\alpha = \delta_{d_\alpha^-(\omega)}$; the functions are the solutions of the corresponding Fokker–Planck equations (FPK).

Next, we will calculate the Lyapunov exponent for each of these measures.

The linearized random dynamical system (RDS) $\phi_t = D\psi_\alpha(t,\omega,\nu)\phi$ is as follows:

$$\mathrm{d}\phi_t = \left[\alpha - 3\psi_\alpha^2(t,\omega,\nu)\right]\phi_t\,\mathrm{d}t + \epsilon\phi_t \circ \mathrm{d}W_t.$$

So

$$D\psi_\alpha(t,\omega,\nu)\phi = \phi\exp\left(\alpha t + \epsilon W_t(\omega) - 3\int_0^t \psi_\alpha^2(s,\omega,\nu)\,\mathrm{d}s\right).$$

Hence, if $\mu_\omega = \delta_{\nu_0(\omega)}$ is a ψ_α-invariant measure, its Lyapunov exponent is

$$\lambda(\mu) = \lim_{t\to\infty}\frac{1}{t}\log\left\|D\psi_\alpha(t,\omega,\nu)\phi\right\| = \alpha - 3\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi_\alpha^2(s,\omega,\nu)\,\mathrm{d}s = \alpha - 3\mathbf{E}\nu^2,$$

on the condition that the integrability condition (IC) $\nu^2 \in L^1(\mathcal{P})$ is satisfied.

Therefore, we conclude to the following.

(i) For $\alpha \in \mathbf{R}$, the IC for $\mu_{1,\omega}^\alpha = \delta_0$ is trivially satisfied and we have

$$\lambda\left(\mu_{1,\omega}^\alpha\right) = \alpha.$$

So $\mu_{1,\omega}^\alpha$ is stable for $\alpha < 0$ and unstable for $\alpha > 0$.

(ii) For $\alpha > 0$, $\mu_{2,\omega}^\alpha = \delta_{-d_\alpha^-(\omega)}$ is $\mathcal{F}_{-\infty}^0$ measurable, as a result, the density $p_2(\nu) = \mathbf{E}\mu_{2,\omega}^\alpha$ satisfies the FPK

$$L^* p_2(\nu) = -\left(\left(\alpha\nu + \frac{\epsilon^2}{2}\nu - \nu^3\right)p_2(\nu)\right)' + \frac{\epsilon^2}{2}\left(\nu^2 p_2(\nu)\right)'' = 0,$$

of which the unique probability density solution is

$$p_2(\nu) = N_\alpha \nu^{\frac{2\alpha}{\epsilon^2}-1}\exp\left(-\frac{\nu^2}{\epsilon^2}\right), \quad \nu > 0,$$

where $N_\alpha^{-1} = \Gamma(\frac{\alpha}{\epsilon^2})\epsilon^{\frac{2\alpha}{\epsilon^2}}$.

Since

$$\mathbf{E}_{\mu^{\alpha}_{2,\omega}} v^2 = \mathbf{E}\bigl(-d^{\alpha}_{-}\bigr)^2 = \int_0^{\infty} v^2 p_2(v)\,\mathrm{d}v < \infty,$$

the IC is satisfied. The calculation of the Lyapunov exponent is accomplished by observing that

$$\bigl(d^{-}_{\alpha}(\upsilon_t\omega)\bigr)^2 = \frac{\exp(2\alpha t + 2\epsilon W_t(\omega))}{2\int_{-\infty}^{t}\exp(2\alpha s + 2\epsilon W_s(\omega))\,\mathrm{d}s} = \frac{\Psi'(t)}{2\Psi(t)},$$

where

$$\Psi(t) = \int_{-\infty}^{t} \exp\bigl(2\alpha s + 2\epsilon W_s(\omega)\bigr)\,\mathrm{d}s.$$

Then by the ergodic theorem

$$\mathbf{E}\bigl(d^{-}_{\alpha}\bigr)^2 = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\log\Psi(t) = \alpha,$$

therefore,

$$\lambda\bigl(\mu^{\alpha}_{2,\omega}\bigr) = -2\alpha < 0,$$

which implies the invariant measure $\mu^{\alpha}_{2,\omega}$ is stable for $\alpha > 0$.

(iii) For $\alpha > 0$, $\mu^{\alpha}_{3,\omega} = \delta_{d^{-}_{\alpha}(\omega)}$ is $\mathcal{F}^{0}_{-\infty}$ measurable and because of its probability density $p_3(v)$ is equal to $p_2(v)$,

$$\mathbf{E}\bigl(-d^{-}_{\alpha}\bigr)^2 = \mathbf{E}\bigl(d^{-}_{\alpha}\bigr)^2 = \alpha,$$

then $\lambda(\mu^{\alpha}_{3,\omega}) = \lambda(\mu^{\alpha}_{2,\omega}) = -2\alpha < 0$. Likewise, we conclude that the invariant measure $\mu^{\alpha}_{3,\omega}$ is stable for $\alpha > 0$.

Hence, the following result is obvious.

Theorem 4.1 *If $\alpha < 0$, i.e., $\delta^2 < A^2 + 1 - B$, the RDS $\psi_{\alpha}(t,\omega)$ possesses exactly one invariant measure $\mu^{\alpha}_{1,\omega}$, which is stable. If $\alpha > 0$, i.e., $\delta^2 > A^2 + 1 - B$, the RDS $\psi_{\alpha}(t,\omega)$ possesses three random Dirac measures $\mu^{\alpha}_{1,\omega}$, $\mu^{\alpha}_{2,\omega}$, $\mu^{\alpha}_{2,\omega}$, where $\mu^{\alpha}_{1,\omega}$ is unstable, while $\mu^{\alpha}_{2,\omega}$ and $\mu^{\alpha}_{3,\omega}$ are stable. Therefore, the RDS $\psi_{\alpha}(t,\omega)$ undergoes a D-bifurcation at the point $\alpha_D = 0$, i.e., the stochastic system* (2.6) *undergoes a stochastic pitchfork bifurcation* (*for more details, see* [20]) *when δ^2 passes through $A^2 + 1 - B$.*

Remark 4.1 Note that if $\delta = 0$, $\alpha_D = 0$ is equivalent to $B = A^2 + 1$, which is just the critical condition for the Hopf bifurcation of the determined system (1.1).

4.2 Stochastic P-bifurcation

In this section, we would like to use the phenomenological approach to determining the stochastic bifurcation of system (2.6). In the following, we consider the steady-state probability density function $P_{\mathrm{st}}(\rho)$ of the linear Itô stochastic differential equation. According to

Namachivaya's theory ([20, 23]), the extreme value of $P_{st}(\rho)$ gives essential information on the stationary behavior of the Fokker–Planck equation arising from nonlinear stochastic systems.

More specifically, if $P_{st}(\rho)$ has a maximum value at ρ^*, the sample trajectory will stay for a longer time in the neighborhood of ρ^*, i.e., ρ^* is stable in the meaning of probability (with a bigger probability). If $P_{st}(\rho)$ has a minimum value (zero), it is just the opposite.

Notice that the averaged FPK equation associated with Eq. (2.6) is

$$\frac{\partial P(\rho)}{\partial t} = -\frac{\partial}{\partial \rho}\left\{\left[\frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho - \frac{1}{8}\rho^3\right]P(\rho)\right\} + \frac{1}{2}\frac{\partial^2}{\partial \rho^2}\left[\left(\frac{\delta^2\rho^2}{2}\right)P(\rho)\right],$$

where $P(\rho)$ is the probability density of diffusion process $\rho(t)$ and the stationary probability density $P_{st}(\rho)$ satisfies the following degenerate equation:

$$0 = -\frac{\partial}{\partial \rho}\left\{\left[\frac{1}{2}\left(B - A^2 - 1 + \frac{3\delta^2}{2}\right)\rho - \frac{1}{8}\rho^3\right]P_{st}(\rho)\right\} + \frac{1}{2}\frac{\partial^2}{\partial \rho^2}\left[\left(\frac{\delta^2\rho^2}{2}\right)P_{st}(\rho)\right].$$

For the above equation, its exact stationary solution $P_{st}(\rho)$ can be expressed as follows:

$$P_{st}(\rho) = C\rho^{\frac{2(B-A^2-1)+\delta^2}{\delta^2}}\exp\left(-\frac{\rho^2}{4\delta^2}\right)$$

in which C is a normalization constant and $C^{-1} = \frac{\Gamma(\frac{B-A^2-1+\delta^2}{\delta^2})(2\delta)^{\frac{2(B-A^2-1+\delta^2)}{\delta^2}}}{2}$.

To obtain the extreme value point of the probability density $P_{st}(\rho)$, we need to solve

$$\frac{\mathrm{d}P_{st}(\rho)}{\mathrm{d}\rho} = 0,$$

that is,

$$\rho^{-\frac{2(A^2+1-B)}{\delta^2}}\left[\rho^2 - 2\left(\delta^2 - 2\left(A^2 + 1 - B\right)\right)\right]\exp\left(-\frac{\rho^2}{4\delta^2}\right) = 0. \tag{4.3}$$

For the roots of Eq. (4.3), there are three cases to discuss.

(i) If $\delta^2 < 2(A^2 + 1 - B)$, for Eq. (4.3), there are no real roots. The probability density function $P_{st}(\rho)$ tends to infinite as $\rho \to 0^+$. In this case, the random trajectories of system (2.6) are concentrated in a neighborhood of the point $\rho = 0$.

(ii) When $\delta^2 = 2(A^2 + 1 - B)$, for (4.3), there is only one root, $\rho = 0$, for which

$$\left.\frac{\mathrm{d}^2P_{st}(\rho)}{\mathrm{d}\rho^2}\right|_{\rho=0} = -\left.\frac{(A^2 + 1 - B - \frac{\rho^2}{4})\exp(-\frac{\rho^2}{8(A^2+1-B)})}{4(A^2 + 1 - B)^2}\right|_{\rho=0} = -\frac{1}{4(A^2 + 1 - B)} < 0,$$

i.e., the probability density function $P_{st}(\rho)$ possesses the maximum value at the point $\rho = 0$.

(iii) When $\delta^2 > 2(A^2 + 1 - B)$, for (4.3), there is one positive root, $\rho = \rho^* = \sqrt{2(\delta^2 - 2(A^2 + 1 - B))}$, of the probability density function $P_{st}(\rho)$ and, by calculation,

$$\left.\frac{\mathrm{d}^2P_{st}(\rho)}{\mathrm{d}\rho^2}\right|_{\rho=\rho^*} < 0,$$

i.e., $P_{st}(\rho)$ possesses a maximum value at the point $\rho = \rho^*$. Meantime, $P_{st}(\rho)$ attains the minimum value at the point $\rho = 0$, but the derivative of $P_{st}(\rho)$ does not exist at the point $\rho = 0$. In this case, the random trajectories of system (2.6) are concentrated in a neighborhood of the point $\rho = \rho^*$.

Therefore, we obtain the following result.

Theorem 4.2 *If* $A^2 + 1 - B > 0$, *Eq.* (2.6) *undergoes stochastic P-bifurcations when the parameter* $\delta = \sqrt{2(A^2 + 1 - B)}$, *i.e., system* (1.2) *undergoes phenomenological bifurcations at the critical parameter value* $\delta = \sqrt{2(A^2 + 1 - B)}$.

5 Numerical simulations

In this section, we consider an illustrative numerical example to verify the analytic results obtained above.

Example 5.1 For (2.6), the values of the parameters in simulations are chosen as follows: $A = 1$, $B = 1.88$. In this case, $A^2 + 1 - B = 0.12 > 0$. Following the discussion in Sect. 4.2, we have stationary probability density function

$$P_{st}(\rho) = \frac{\rho^{\frac{\delta^2 - 0.24}{\delta^2}} e^{-\frac{\rho^2}{4\delta^2}}}{\Gamma(\frac{\delta^2 - 0.12}{\delta^2})[(2\delta)^{\frac{2(\delta^2 - 0.12)}{\delta^2}}]},$$

and by solving $\frac{dP_{st}(\rho)}{d\rho} = 0$, we need to solve

$$\rho^{\frac{-0.24}{\delta^2}} \left[\rho^2 - 2\left(\delta^2 - 0.24\right)\right] = 0. \tag{5.1}$$

Hence, we have the following.

Case 1. When $\delta^2 < 0.24$, Eq. (5.1) has no real roots. The probability density function $P_{st}(\rho)$ tends to infinity as $\rho \to 0^+$. In this case, the random trajectories of system (2.6) are concentrated in a neighborhood of the point $\rho = 0$. See Fig. 1.

Figure 1 Portraits of $P_{st}(\rho)$ when $A = 1$, $B = 1.88$, $\delta = 0.3898979486 < 0.4898979486 = \delta_0$

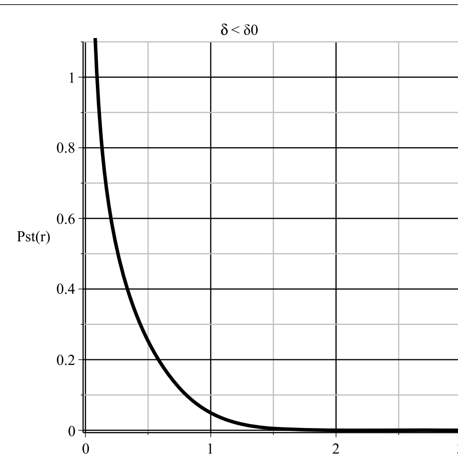


Figure 2 Portraits of $P_{st}(\rho)$ when $A = 1$, $B = 1.88$, $\delta = \delta_0 = 0.4898979486$

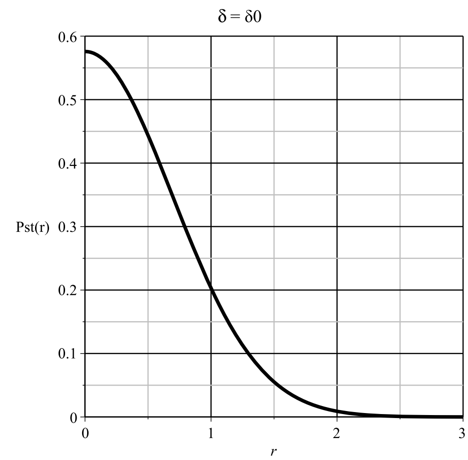
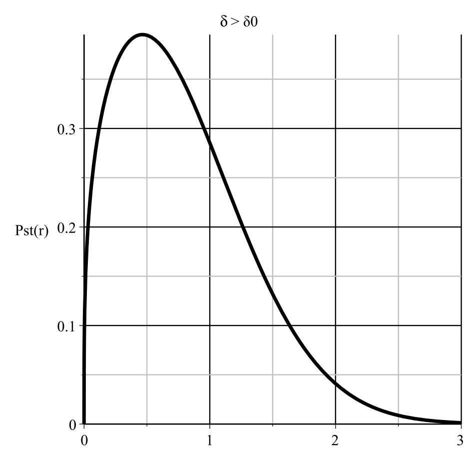


Figure 3 Portraits of $P_{st}(\rho)$ when $A = 1$, $B = 1.88$, $\delta = 0.5898979486 > 0.4898979486 = \delta_0$



Case 2. When $\delta^2 = 0.24$, Eq. (5.1) has only one root $\rho = 0$. And

$$\left.\frac{\mathrm{d}^2 P_{st}(\rho)}{\mathrm{d}\rho^2}\right|_{\rho=0} < 0,$$

the probability density function $P_{st}(\rho)$ becomes smooth and possesses the maximum value at the point $\rho = 0$. See Fig. 2, for details.

Case 3. When $\delta^2 > 0.24$, there is one positive root $\rho = \rho^* = 0.4647140841$ of (5.1). $P_{st}(\rho)$ possesses a maximum value at the point $\rho = \rho^*$. In the meantime, $P_{st}(\rho)$ attains the minimum value at the point $\rho = 0$, but the derivative of $P_{st}(\rho)$ does not exist at the point $\rho = 0$. In this case, the random trajectories of system (2.6) are concentrated in a neighborhood of the point $\rho = \rho^*$. See Fig. 3.

Therefore, we find the critical point

$$\delta_0 = \sqrt{2(A^2 + 1 - B)} = 0.4898979486.$$

As a result, we conclude that Eq. (2.6) undergoes a stochastic phenomenological bifurcation at $\delta = \delta_0$, as shown in Fig. 1, 2, 3.

Finally, we investigate the dynamics of system (1.2). We mainly consider the joint probability density $\varrho(x,y)$ to Cartesian coordinates x and y. By the relation $\varrho(x,y) = |J|\varpi(\rho,\theta)$ and $P(r) = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \varpi(\rho,\theta)\,d\theta$, where the determinant of the Jacobian matrix J of the nonlinear transformation is given by $|J| = \frac{1}{\rho}$ and ϖ is the joint probability density to ρ and θ (see [24]), we obtain

$$\varrho(x,y) = \frac{(x^2+y^2)^{\frac{\delta^2-0.24}{2\delta^2}} e^{-\frac{x^2+y^2}{4\delta^2}}}{\pi\Gamma(\frac{\delta^2-0.12}{\delta^2})[(2\delta)^{\frac{2(\delta^2-0.12)}{\delta^2}}]}.$$

By similar discussions, we can draw the following conclusions.

Case 1. When $\delta^2 < 0.24$, the joint probability density function $\varrho(x,y)$ tends to infinite as $x \to 0$ and $y \to 0$. The result is shown in Fig. 4.

Case 2. When $\delta^2 = 0.24$, a maximum value appears at the point $O(0,0)$. See Fig. 5 for details.

Case 3. When $\delta^2 > 0.24$, a maximum value appears at the points of stable limit cycle $x^2 + y^2 = 0.4319183600$, and a minimum value arises at the point $O(0,0)$. In the meantime, we notice that the partial derivatives of joint probability density $\varrho(x,y)$ are discontinuous at the original point $O(0,0)$. See Fig. 6.

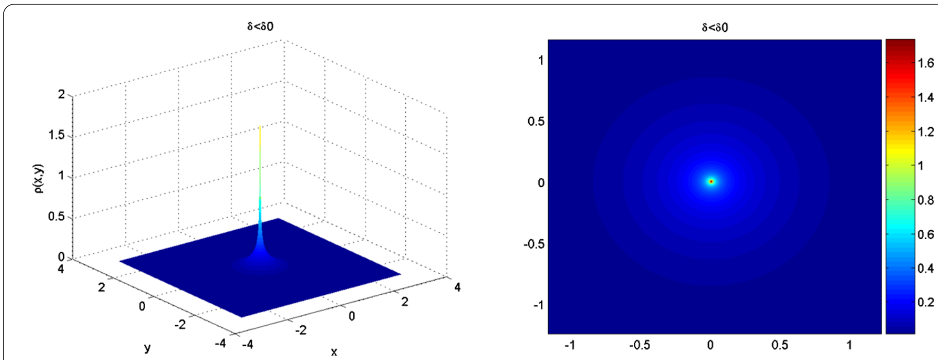


Figure 4 Portraits of $\varrho(x,y)$ (left) and distribution of invariant densities (right) when $A = 1$, $B = 1.88$, $\delta = 0.3898979486 < 0.4898979486 = \delta_0$

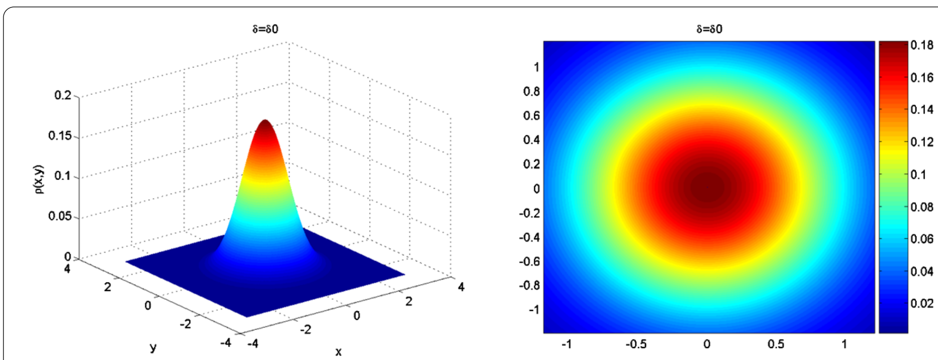


Figure 5 Portraits of $\varrho(x,y)$ (left) and distribution of invariant densities (right) when $A = 1$, $B = 1.88$, $\delta = 0.4898979486 = \delta_0$

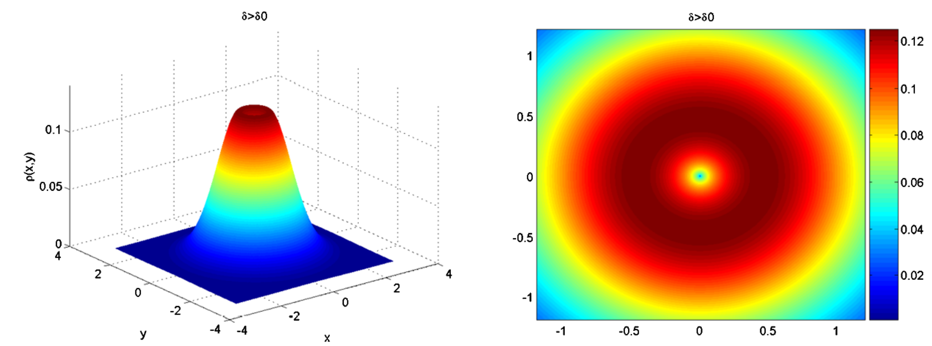


Figure 6 Portraits of $\varrho(x,y)$ (left) and distribution of invariant densities (right) when $A = 1$, $B = 1.88$, $\delta = 0.5898979486 > 0.4898979486 = \delta_0$

Therefore, we find the critical point

$$\delta_0 = \sqrt{2\left(A^2 + 1 - B\right)} = 0.4898979486.$$

As a result, we conclude that Eq. (1.2) undergoes a stochastic phenomenological bifurcation at $\delta = \delta_0$.

6 Conclusions

In this paper, by applying the stochastic average method, singular boundary theory and invariant measure theory, we study the stochastic stability and stochastic bifurcation of the Brusselator system with multiplicative white noise. Finally, an example is given to illustrate the effectiveness of our analyzing procedure.

Throughout the paper, we always assume that $B < A^2 + 1$. In this case, as is well known, the deterministic system (1.1) is stable. But for the stochastic case, system (1.2) undergoes stochastic D-bifurcation when $\delta = \sqrt{A^2 + 1 - B}$ and the stochastic P-bifurcation occurs if $\delta = \sqrt{2(A^2 + 1 - B)}$. Therefore, we conclude that the noise will destabilize the system under some condition.

Furthermore, we find that, for some small noise when $\delta < \sqrt{A^2 + 1 - B}$, system (1.2) is globally stable. We conclude that the small noise cannot affect the dynamical behaviors of system (1.2).

However, to the best of our knowledge, most of the existing results as regards the stochastic bifurcation analysis are mainly reported for the lower-dimensional stochastic models, especially one- or two-dimensional ones (see [24]). There are very few results about the high-dimensional stochastic systems because of the difficulty of the model reduction of high-dimensional ones. Moreover, besides the noise effects, many other factors such as time delays and impulsive input may affect the dynamical behaviors of the models. These factors we will consider in the future.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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References

1. Prigogine, L., Lefever, R.: Symmetry-breaking instabilities in dissipative systems. J. Chem. Phys. **48**, 1695–1700 (1968)
2. Li, B., Wang, M.X.: Diffusion-driven instability and Hopf bifurcation in Brusselator system. Appl. Math. Mech. **29**(6), 825–831 (2008)
3. Zuo, W., Wei, J.: Multiple bifurcations and spatiotemporal patterns for a coupled two-cell Brusselator model. Dyn. Partial Differ. Equ. **8**(4), 363–384 (2011)
4. Brown, K.J., Davidson, F.A.: Global bifurcation in the Brusselator system. Nonlinear Anal. **24**, 1713–1725 (1995)
5. Yu, P., Gumel, A.B.: Bifurcation and stability analysis for a couple Brusselator model. J. Sound Vib. **244**, 795–820 (2001)
6. Li, X.H., Hou, J.Y., Shen, Y.J.: Slow-fast effect and generation mechanism of brusselator based on coordinate transformation. Open Phys. **14**(1), 261–268 (2016)
7. Tu, J.Y., You, Y.C.: Random attractor of stochastic Brusselator system with multiplicative noise. Discrete Contin. Dyn. Syst., Ser. A **36**(5), 2757–2779 (2016)
8. Arnold, L., Bleckert, G., Schenk-Hoppé, K.R.: The stochastic brusselator: parametric noise destroys Hopf bifurcation. In: Stochastic Dynamics, pp. 71–92. Springer, New-York (1999)
9. Xu, Y., Ma, S.J., Zhang, H.Q.: The stability analysis in stochastic Brusselator system with random parameter. Sci. Sin. Phys. Mech. Astron. **41**(10), 1203–1209 (2011)
10. Tambe, S.S., Inamdar, S.R., Kulkarni, B.D.: Diffusive broadening of limit cycle in presence of noise: a case study of reversible Brusselator. Math. Comput. Simul. **39**, 115–124 (1995)
11. Bashkirtseva, I.A., Ryashko, L.B.: Sensitivity analysis of the stochastically and periodically forced Brusselator. Physica A **278**, 126–139 (2000)
12. Ma, S.J.: The stochastic Hopf bifurcation analysis in Brusselator system with random parameter. Appl. Math. Comput. **219**, 306–319 (2012)
13. Bandyopadhyay, M., Chattopadhyay, J.: Ratio-dependent predator–prey model: effect of environmental fluctuation and stability. Nonlinearity **18**, 913–936 (2005)
14. Bandyopadhyay, M., Chakrabarti, C.G.: Deterministic and stochastic analysis of a nonlinear preypredator system. J. Biol. Syst. **11**, 161–172 (2003)
15. Beretta, E., Kolmanovskii, V., Shaikhet, L.: Stability of epidemic model with time delays influenced by stochastic perturbations. Math. Comput. Simul. **45**, 269–277 (1998)
16. Shaikhet, L.: Stability of predator–prey model after effect by stochastic perturbations. Stab. Control: Theory Appl. **1**(1), 3–13 (1998)
17. Saha, T., Chakrabarti, C.: Stochastic analysis of prey–predator model with stage structure for prey. J. Appl. Math. Comput. **35**, 195–209 (2011)
18. Zhu, W.Q.: Nonlinear Stochastic Dynamics and Control: Framework of Hamiltonian Theory. Science Press, Beijing (2003)
19. Lin, Y.K., Cai, G.Q.: Probabilistic Structural Dynamics, Advanced Theory and Applications. McGraw-Hill, New York (2004)
20. Arnold, L.: Random Dynamical Systems. Springer, New York (1998)
21. Oseledets, V.I.: A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. Trans. Mosc. Math. Soc. **19**, 197–231 (1968)
22. Huang, Z.T., Yang, Q.G., Cao, J.F.: Stochastic stability and bifurcation for the chronic state in Marchuk's model with noise. Appl. Math. Model. **35**, 5842–5855 (2011)
23. Sri Namachchivaya, N.: Stochastic bifurcation. Appl. Math. Comput. **38**(2), 101–159 (1990)
24. Luo, C.L., Guo, S.J.: Stability and bifurcation of two-dimensional stochastic differential equations with multiplicative excitations. Bull. Malays. Math. Sci. Soc. **40**(2), 1–23 (2016)