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Necessary and sufficient conditions for oscillation of fourth order dynamic equations on time scales

Yong Zhou[1,2*], Jia Wei He[1], Bashir Ahmad[2] and Ahmed Alsaedi[2]

*Correspondence:
yzhou@xtu.edu.cn
[1]Faculty of Mathematics and Computational Science, Xiangtan University, Hunan, P.R. China
[2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

Abstract
In this paper, we obtain necessary and sufficient conditions for oscillation of a fourth order dynamic equation on time scales with deviating arguments. We discuss the oscillation behavior of solutions for strongly superlinear and strongly sublinear cases of the dynamic equation at hand. Our results unify and improve some known results for dynamic equations on time scales.

MSC: 34C10; 34N05

Keywords: Dynamic equations; Time scales; Oscillation; Necessary and sufficient conditions

1 Introduction

The topic of oscillation and stability of dynamic equations on time scales has been developed very rapidly in the past two decades. There are some excellent monographs [1–4] and papers [5–10] containing some interesting works in the field.

The oscillatory behavior of solutions for nonlinear fourth order functional differential equations of the form

$$\bigl[r(t)y''(t)\bigr]'' + g\bigl(t, y\bigl(\eta(t)\bigr)\bigr) = 0, \quad t \geq t_0,$$

has been discussed by Onose [11], where g is superlinear (sublinear) and strongly superlinear (strongly sublinear); he has extended and improved some interesting results of Kusano and Naito [12]. Furthermore, Gopalsamy et al. [13] obtained the sufficient and necessary conditions for oscillation of a fourth order differential equation with multiple deviating arguments given by

$$\bigl[r(t)y''(t)\bigr]'' + g\bigl(t, y\bigl(\eta_1(t)\bigr), y\bigl(\eta_2(t)\bigr), \ldots, y\bigl(\eta_n(t)\bigr)\bigr) = 0, \quad t \geq t_0,$$

where g is strongly superlinear and strongly sublinear.

For some more results on oscillation of solutions for different kinds of fourth order equations on time scales, see [14–21] and the references cited therein. However, it has been observed that there is no work in the related literature concerning the sufficient and necessary conditions for oscillation of fourth order dynamic equations on time scales. Motivated by the aforementioned works, in this paper, we consider the following fourth order dynamic equation with deviating arguments:

$$\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta\Delta} + g\bigl(t, y\bigl(\eta(t)\bigr)\bigr) = 0, \quad t \in [t_0, \infty) = \mathbb{T}_0 \subseteq \mathbb{T}, \tag{1}$$

where $y^{\Delta}(t)$ is the delta (or Hilger) derivative of y at t, $r \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, $\eta \in C_{rd}(\mathbb{T}_0, \mathbb{T})$, $g : \mathbb{T}_0 \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function, and $\operatorname{sgn} g(t, y) = \operatorname{sgn} y$ for $t \in \mathbb{T}_0$. In relation to (1), it is also assumed that $\int_{t_0}^{\infty} t/r(t)\Delta t = \infty$.

The paper is organized as follows. In Sect. 2, we recall some basic concepts of dynamic equations on time scales. In Sect. 3, we establish necessary and sufficient criteria for oscillation of (1) when g is strongly superlinear as well as strongly sublinear.

2 Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$ with $\sup \mathbb{T} = \infty$. For example, $\mathbb{R}$, $h\mathbb{Z}$ for $h > 0$ and $q^{\mathbb{N}} := \{q^k, k \in \mathbb{N}\}$ for $q > 1$ are time scales. In the forthcoming analysis, we assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Let the closed interval in $\mathbb{T}$ be defined by $[c, d] := \{t \in \mathbb{T}, c \leq t \leq d\}$. In a similar manner, one can define open intervals and half-open intervals, etc.

Definition 2.1 For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, then t is called *right-scattered*, while if $\rho(t) < t$, it is called *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. The graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Definition 2.2 A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at all right-dense points and its left-sided limit exists (and is finite) at a left-dense point. We denote the set of rd-continuous functions by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 For a function $f : \mathbb{T} \to \mathbb{R}$, let $F^{\Delta}(t)$ represent the Hilger derivative of f at t. Assume that $t_0 \in \mathbb{T}$ and $f \in C_{rd}(\mathbb{T}_0, \mathbb{R})$. If $F^{\Delta}(t) = f(t)$, then we define

$$\int_{t_0}^{A} f(s)\Delta s := F(A) - F(t_0), \qquad \int_{t_0}^{\infty} f(s)\Delta s := \lim_{A \to \infty} \int_{t_0}^{A} f(s)\Delta s.$$

Definition 2.4 We say g is strongly superlinear if there exists a constant $\alpha > 1$ such that

$$\frac{|g(t, u)|}{|u|^{\alpha}} \leq \frac{|g(t, v)|}{|v|^{\alpha}} \quad \text{for } |u| \leq |v|, uv > 0, t \in \mathbb{T}_0,$$

while g is strongly sublinear if there exists a constant $\alpha \in (0, 1)$ such that

$$\frac{|g(t, u)|}{|u|^{\alpha}} \geq \frac{|g(t, v)|}{|v|^{\alpha}} \quad \text{for } |u| \leq |v|, uv > 0, t \in \mathbb{T}_0.$$

3 Main results

In the sequel, we use the following notations:

$$\begin{aligned} &R(t) = \int_{t_0}^{t}\int_{t_0}^{s_1} \frac{s}{r(s)}\Delta s\Delta s_1, \qquad R_T(t) = \int_{T}^{t}\int_{T}^{s_1} \frac{s-T}{r(s)}\Delta s\Delta s_1, \\ &\widetilde{\mathbf{R}}(t) = \int_{t_0}^{t} \frac{(t-s)(\sigma(s)-t_0)}{r(s)}\Delta s, \qquad \widetilde{\mathbf{R}}_u(t) = \int_{u}^{t} \frac{(t-s)(\sigma(s)-u)}{r(s)}\Delta s, t > u > t_0. \end{aligned} \tag{2}$$

Lemma 3.1 *If $y(t)$ is a nonoscillatory solution of* (1), *then there are only four cases for all sufficiently large $t \geq t_0$*:

$$\begin{aligned} &\text{(i)} \quad y(t) > 0, \quad y^{\Delta}(t) > 0, \quad r(t)y^{\Delta\Delta}(t) > 0, \quad \bigl[r(t)y^{\Delta\Delta}\bigr]^{\Delta} > 0; \\ &\text{(ii)} \quad y(t) > 0, \quad y^{\Delta}(t) > 0, \quad r(t)y^{\Delta\Delta}(t) < 0, \quad \bigl[r(t)y^{\Delta\Delta}\bigr]^{\Delta} > 0; \\ &\text{(iii)} \quad y(t) < 0, \quad y^{\Delta}(t) < 0, \quad r(t)y^{\Delta\Delta}(t) < 0, \quad \bigl[r(t)y^{\Delta\Delta}\bigr]^{\Delta} < 0; \\ &\text{(iv)} \quad y(t) < 0, \quad y^{\Delta}(t) < 0, \quad r(t)y^{\Delta\Delta}(t) > 0, \quad \bigl[r(t)y^{\Delta\Delta}\bigr]^{\Delta} < 0. \end{aligned}$$

Proof Without loss of generality, let $y(t)$ be an eventually positive solution of (1), that is, there exists $t_1 \geq t_0$ such that $y(t) > 0$ for $t \geq t_1$. Then $y(\eta(t)) > 0$ for $t \geq t_1$. From (1), it yields that $[r(t)y^{\Delta\Delta}(t)]^{\Delta\Delta} < 0$ for $t \geq t_1$. Therefore, $[r(t)y^{\Delta\Delta}(t)]^{\Delta}$ is eventually of constant sign. Next we suppose that $[r(t)y^{\Delta\Delta}(t)]^{\Delta} < 0$ at some $t = t_2 \geq t_1$. Then, integrating $[r(t)y^{\Delta\Delta}(t)]^{\Delta\Delta} < 0$ twice from t_2 to t, and multiplying the resulting inequality by $1/r(t)$ and integrating again from t_2 to t, we get

$$y^{\Delta}(t) < \bar{a}\int_{t_2}^{t} \frac{s-t_2}{r(s)}\Delta s + \bar{b}\int_{t_2}^{t} \frac{1}{r(s)}\Delta s + \bar{c}, \quad t \geq t_2,$$

where $\bar{a} = [r(t_2)y^{\Delta\Delta}(t_2)]^{\Delta} < 0$, $\bar{b} = r(t_2)y^{\Delta\Delta}(t_2)$, and $\bar{c} = y^{\Delta}(t_2)$. In consequence, it follows from the assumption $\int_{t_0}^{\infty} s/r(s)\Delta s = \infty$ that $\lim_{t\to\infty} y^{\Delta}(t) = -\infty$, which contradicts the positivity of $y(t)$. Therefore, we have $[r(t)y^{\Delta\Delta}(t)]^{\Delta} > 0$ for all $t \geq t_1$. It means that $r(t)y^{\Delta\Delta}(t)$ eventually keeps the same sign. On the other hand, let $r(t)y^{\Delta\Delta}(t) < 0$ for $t \geq t_1$. Then it can easily be shown that $y^{\Delta}(t)$ is eventually positive. This completes the proof of (i). If there exists $t_2 \geq t_1$ such that $r(t)y^{\Delta\Delta}(t) > 0$ for $t \geq t_2$, then $r(t)y^{\Delta\Delta}(t) \geq c$ for $t \geq t_2$, where $c = r(t_2)y^{\Delta}(t_2)$. Multiplying this inequality by $t/r(t)$ and integrating from t_2 to t, by using the integration by parts formula on time scales, we get

$$ty^{\Delta}(t) - y\bigl(\sigma(t)\bigr) - t_2y^{\Delta}(t_2) + y\bigl(\sigma(t_2)\bigr) \geq c\int_{t_2}^{t} \frac{s}{r(s)}\Delta s, \quad t \geq t_2,$$

which, together with $\int_{t_0}^{\infty} s/r(s)\Delta s = \infty$, implies that $\lim_{t\to\infty} ty^{\Delta}(t) = \infty$. Thus $y^{\Delta}(t) > 0$ for all large $t \geq t_0$. The proof is completed. □

Lemma 3.2 *If $t_1 \geq t_0$ and $t > u$, then $\lim_{t\to\infty} \frac{R_{t_1}(t)}{R(t)} = 1$ and $\widetilde{\mathbf{R}}_u(t)$ is nonincreasing for u, where $R_{t_1}(t)$, $R(t)$, $\widetilde{\mathbf{R}}_u(t)$ are given in* (2).

Proof By applying L'Hôpital's rule [1, Theorem 1.120], we find that

$$\lim_{t\to\infty}\frac{R_{t_1}(t)}{R(t)}=\lim_{t\to\infty}\frac{\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{s-t_1}{r(s)}\Delta s\Delta s_1}{\int_{t_0}^{t}\int_{t_0}^{s_1}\frac{s}{r(s)}\Delta s\Delta s_1}=\lim_{t\to\infty}\frac{\int_{t_1}^{t}\frac{s-t_1}{r(s)}\Delta s}{\int_{t_0}^{t}\frac{s}{r(s)}\Delta s}=\lim_{t\to\infty}\frac{\frac{t-t_1}{r(t)}}{\frac{t}{r(t)}}=1.$$

On the other hand, let

$$f(u,s)=\frac{(t-s)(\sigma(s)-u)}{r(s)},\quad t_0<u<s<t.$$

Then, using [1, Theorem 1.117] for all u, we obtain

$$\bigl[\widetilde{\mathbf{R}}_u(t)\bigr]^{\Delta}=\int_u^t\bigl[f(u,s)\bigr]^{\Delta}\Delta s-f\bigl(\sigma(u),u\bigr)=-\int_u^t\frac{t-s}{r(s)}\Delta s\le 0.$$

This completes the proof. □

Lemma 3.3 *If $y(t)$ is a nonoscillatory solution of (1), then there exist $T>t_0$ and a constant $\tilde{c}>0$ such that*

$$\frac{1}{2}\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}R(t)\le\bigl|y(t)\bigr|\le\tilde{c}R(t)\quad\textit{for } t\ge T.$$

Proof Without loss of generality, we suppose that $y(t)$ is eventually positive. Then, in view of Lemma 3.1, there exists $t_1\ge t_0$ such that

$$y(t)>0,\qquad y^{\Delta}(t)>0,\qquad \bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}>0,\qquad \bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta\Delta}<0\quad\text{for } t\ge t_1. \tag{3}$$

Integrating $[r(t)y^{\Delta\Delta}(t)]^{\Delta\Delta}<0$ twice from t_1 to t, we have

$$r(t)y^{\Delta\Delta}(t)<a_0t+a_1,\quad\text{where } a_0=\bigl[r(t_1)y^{\Delta\Delta}(t_1)\bigr]^{\Delta}>0, a_1=r(t_1)y^{\Delta\Delta}(t_1). \tag{4}$$

Multiplying (4) by $1/r(t)$ and integrating twice from t_1 to t yields

$$y(t)<a_0\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{s}{r(s)}\Delta s\Delta s_1+a_1\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{1}{r(s)}\Delta s\Delta s_1+a_2t+a_3,$$

where $a_2=y^{\Delta}(t_1)$ and $a_3=y(t_1)$ are constants. Noting that $\int_{t_0}^{\infty}s/r(s)\Delta s=\infty$, we deduce that there exist $t_2\ge t_1$ and $\tilde{c}>0$ such that $y(t)<\tilde{c}R(t)$ for $t\ge t_2$.

Now let us prove the left-sided inequality in the lemma. In view of (3), observe that $[r(t)y^{\Delta\Delta}(t)]^{\Delta}$ is nonincreasing, and hence

$$\begin{aligned}
y(t)&\ge\int_{t_1}^{t}y^{\Delta}(s)\Delta s\ge\int_{t_1}^{t}\int_{t_1}^{s_1}y^{\Delta\Delta}(s)\Delta s\Delta s_1=\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{1}{r(s)}r(s)y^{\Delta\Delta}(s)\Delta s\Delta s_1\\
&\ge\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{1}{r(s)}\int_{t_1}^{s}\bigl[r(u)y^{\Delta\Delta}(u)\bigr]^{\Delta}\Delta u\Delta s\Delta s_1\\
&\ge\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}\int_{t_1}^{t}\int_{t_1}^{s_1}\frac{1}{r(s)}\int_{t_1}^{s}\Delta u\Delta s\Delta s_1
\end{aligned}$$

$$= \left[r(t)y^{\Delta\Delta}(t)\right]^{\Delta} \int_{t_1}^{t} \int_{t_1}^{s_1} \frac{s - t_1}{r(s)} \Delta s \Delta s_1 = \left[r(t)y^{\Delta\Delta}(t)\right]^{\Delta} R_{t_1}(t).$$

From Lemma 3.2, there exists $t_3 \geq t_1$ such that $R_{t_1}(t) \geq \frac{1}{2}R(t)$ for $t \geq t_3$, and hence

$$y(t) \geq \frac{1}{2}\left[r(t)y^{\Delta\Delta}(t)\right]^{\Delta} R(t).$$

Letting $T = \max(t_2, t_3)$, the proof is complete. □

Lemma 3.4 *If $y(t)$ is a nonoscillatory solution of* (1), *then there exists $t^* \geq t_0$ such that, for any $T \geq t^*$,*

$$y(t) \geq \int_{t_1}^{t} \widetilde{\mathbf{R}}_{t_1}\big(\sigma(s)\big) g\big(s, y\big(\eta(s)\big)\big) \Delta s, \quad t \geq T. \tag{5}$$

Proof Without loss of generality, we suppose that $y(t)$ is eventually positive. Firstly, if $y(t)$ is a solution of type-(i), then there exists $t_1 \geq t_0$ such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \qquad r(t)y^{\Delta\Delta}(t) > 0, \qquad \left[r(t)y^{\Delta\Delta}(t)\right]^{\Delta} > 0 \quad \text{for } t \geq t_1.$$

Let

$$h(t) = y(t) - \int_{t_1}^{t} \int_{s_3}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \int_{s_1}^{\infty} g\big(s, y\big(\eta(s)\big)\big) \Delta s \Delta s_1 \Delta s_2.$$

Obviously, $h^{\Delta\Delta}(t) > 0$ for $t \geq t_1$. Indeed, differentiating the above equation twice, we get

$$h^{\Delta\Delta}(t) = y^{\Delta\Delta}(t) + \frac{1}{r(t)} \int_{t}^{\infty} \int_{s_1}^{\infty} g\big(s, y\big(\eta(s)\big)\big) \Delta s \Delta s_1 \Delta s_2 > 0.$$

In view of (1), we obtain $[r(t)h^{\Delta\Delta}(t)]^{\Delta\Delta} = 0$ and hence $[r(t)h^{\Delta\Delta}(t)]^{\Delta} = c$. Integrating (1) from t to T, we have

$$\left[r(t)y^{\Delta\Delta}(t)\right]^{\Delta} - \int_{t}^{T} g\big(s, y\big(\eta(s)\big)\big) \Delta s = \left[r(T)y^{\Delta\Delta}(T)\right]^{\Delta} > 0.$$

In the limit $T \to \infty$, we note that $[r(t)h^{\Delta\Delta}(t)]^{\Delta} > 0$ for $t \geq t_1$. Then there exists $c > 0$ such that $h^{\Delta\Delta}(t) = ct/r(t) > 0$ for $t \geq t_1$, which, on integrating from t_1 to t, yields

$$h^{\Delta}(t) = h^{\Delta}(t_1) + c \int_{t_1}^{t} s/r(s) \Delta s. \tag{6}$$

Taking the limit $t \to \infty$ and using the assumption $\int_{t_0}^{\infty} s/r(s)\Delta s = \infty$ in (6), we get $h^{\Delta}(t) > 0$ for all large t. Therefore, there exists $t_2 \geq t_1$ such that $h^{\Delta}(t) > c\int_{t_1}^{t} s/r(s)\Delta s$ for $t \geq t_2$. Next, integrating (6) from t_1 to t, we get

$$h(t) = h(t_1) + c \int_{t_1}^{t} \int_{t_1}^{s_1} s/r(s) \Delta s \Delta s_1, \quad t \geq t_2,$$

which implies that $h(t) > 0$ for large values of t (i.e., $t \to \infty$). Thus, there exists $T \geq t_2$ such that

$$y(t) \geq \int_{t_1}^{t} \int_{s_3}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \int_{s_1}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2 \Delta s_3, \quad t \geq T.$$

Now, by interchanging the order of integration, we get

$$\begin{aligned} y(t) &\geq \int_{t_1}^{t} \int_{t_1}^{\sigma(s_2)} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s) - s_2\bigr) g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_3 \Delta s_2 \\ &\quad + \int_{t}^{\infty} \int_{T}^{t} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s) - s_2\bigr) g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_3 \Delta s_2 \\ &\geq \int_{t_1}^{t} \frac{\sigma(s_2) - t_1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s) - s_2\bigr) g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_2 \\ &= \int_{t_1}^{t} \int_{t_1}^{\sigma(s)} \frac{(\sigma(s) - s_2)(\sigma(s_2) - t_1)}{r(s_2)} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s_2 \Delta s \\ &\quad + \int_{t}^{\infty} \int_{t_1}^{t} \frac{(\sigma(s) - s_2)(\sigma(s_2) - t_1)}{r(s_2)} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s_2 \Delta s \\ &\geq \int_{t_1}^{t} \widetilde{\mathbf{R}}_{t_1}\bigl(\sigma(s)\bigr) g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s. \end{aligned} \tag{7}$$

On the other hand, if $y(t)$ is a solution of type-(ii), then there exists $t_3 \geq t_0$ such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \qquad r(t) y^{\Delta\Delta}(t) < 0, \qquad \bigl[r(t) y^{\Delta\Delta}(t)\bigr]^{\Delta} > 0 \quad \text{for } t \geq t_3.$$

Integrating (1) from t to T, we get

$$\bigl[r(T) y^{\Delta\Delta}(T)\bigr]^{\Delta} - \bigl[r(t) y^{\Delta\Delta}(t)\bigr]^{\Delta} + \int_{t}^{T} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s = 0,$$

which takes the following form after taking the limit $T \to \infty$:

$$\bigl[r(t) y^{\Delta\Delta}(t)\bigr]^{\Delta} \geq \int_{t}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s, \quad t \geq t_3.$$

Integrating the above inequality from t to T, we have

$$r(T) y^{\Delta\Delta}(T) - r(t) y^{\Delta\Delta}(t) \geq \int_{t}^{T} \int_{s_1}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1.$$

Multiplying the above inequality by $1/r(t)$, and then integrating from t to T, we get

$$y^{\Delta}(t) \geq y^{\Delta}(T) + r(T) y^{\Delta\Delta}(T)(t - T) + \int_{t}^{T} \frac{1}{r(s_2)} \int_{s_2}^{T} \int_{s_1}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2,$$

which, on taking the limit $T \to \infty$, yields

$$y^{\Delta}(t) \geq \int_{t}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \int_{s_1}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2, \quad t \geq t_3.$$

Integrating the above inequality from t_1 to t, we get

$$y(t) \geq \int_{t_1}^{t} \int_{s_3}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \int_{s_1}^{\infty} g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2, \quad t \geq t_3.$$

Using the argument employed in (7) and defining $t^* = \max(t_2, t_3)$, we deduce that the conclusion of the lemma holds. □

Lemma 3.5 *Let f and g be Δ-differentiable on $\mathbb{T}$. Assume that $g(t)$, $g^{\Delta}(t)$ are not equal to zero for all $t \in \mathbb{T}$ and have the same sign. Then*

$$\lim_{t\to\infty} \frac{f(\sigma(t))}{g(\sigma(t))} = \xi,$$

if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\infty} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} = \xi \in \mathbb{R}.$$

Proof Since $g(t)$ and $g^{\Delta}(t)$ are not equal to zero for all $t \in \mathbb{T}$, it follows from the identity $g(\sigma(t)) = \mu(t)g^{\Delta}(t) + g(t)$ that $g(\sigma(t))$ is not equal to zero for all $t \in \mathbb{T}$. Hence, for any $\varepsilon > 0$, there exists $T > t_0$ such that, for $t \geq T$, we have

$$\left|\frac{f(t)}{g(t)} - \xi\right| \leq \varepsilon \quad \text{and} \quad \left|\frac{f^{\Delta}(t)}{g^{\Delta}(t)} - \xi\right| \leq \varepsilon.$$

If $\operatorname{sgn} g(t) = \operatorname{sgn} g^{\Delta}(t) > 0$, then

$$\xi g(t) - \varepsilon g(t) \leq f(t) \leq \xi g(t) + \varepsilon g(t) \quad \text{and} \quad \xi g^{\Delta}(t) - \varepsilon g^{\Delta}(t) \leq f^{\Delta}(t) \leq \xi g^{\Delta}(t) + \varepsilon g^{\Delta}(t).$$

Hence, noticing that $f(\sigma(t)) = \mu(t)f^{\Delta}(t) + f(t)$ and $g(\sigma(t)) > 0$, we get

$$\xi g\bigl(\sigma(t)\bigr) - \varepsilon g\bigl(\sigma(t)\bigr) \leq f\bigl(\sigma(t)\bigr) \leq \xi g\bigl(\sigma(t)\bigr) + \varepsilon g\bigl(\sigma(t)\bigr), \quad t \geq T.$$

Consequently, we have

$$\left|\frac{f(\sigma(t))}{g(\sigma(t))} - \xi\right| \leq \varepsilon, \quad t \geq T.$$

Also one can observe that the above expression holds for $\operatorname{sgn} g(t) = \operatorname{sgn} g^{\Delta}(t) < 0$. Thus, in view of the arbitrariness of ε, we obtain the desired result. □

Theorem 3.6 *Assume that g is strongly superlinear and $\eta(t) \geq \sigma(t)$. Then every solution of (1) is oscillatory if and only if*

$$\int_{t_0}^{\infty} \widetilde{\mathbf{R}}\bigl(\sigma(t)\bigr)\bigl|g(t,c)\bigr|\Delta t = \infty \quad \textit{for all } c \neq 0. \tag{8}$$

Proof We first prove the necessity by contradiction. Let us suppose that condition (8) does not hold true. Then there exists a positive constant c such that

$$\int_{t_0}^{\infty} \widetilde{\mathbf{R}}(\sigma(t))\big|g(t,c)\big|\Delta t < \infty. \tag{9}$$

So we can choose a sufficiently large $T > t_0$ such that

$$\int_{T}^{\infty} \widetilde{\mathbf{R}}(\sigma(t))\big|g(t,c)\big|\Delta t < \frac{c}{4}.$$

Let

$$U = \Big\{y \Big| y \in C_{rd}(\mathbb{T}_0,\mathbb{R}), \sup_{t\in\mathbb{T}_0}\big|y(t)\big| < \infty\Big\}.$$

It is clear that U is a Banach space with the norm $\|y\| = \sup_{t\in\mathbb{T}_0}|y(t)|$. Let us introduce a closed, bounded, and convex subset of U defined by

$$\Omega = \Big\{y = y(t) : y \in U, \frac{c}{2} \le \big|y(t)\big| \le c, t \in \mathbb{T}_0\Big\}.$$

Define a map $\mathcal{P}$ on Ω as follows:

$$(\mathcal{P}y)(t) = \begin{cases} \frac{c}{2} + \int_T^t \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty g(s,y(\eta(s)))\Delta s\Delta s_1\Delta s_2\Delta s_3, & t \ge T, \\ (\mathcal{P}y)(T), & t_0 \le t \le T. \end{cases}$$

In the sequel, we will show that $\mathcal{P}$ has a fixed point in Ω.

Step I. $\mathcal{P}$ maps Ω into Ω. Let $y \in \Omega$. Then $c/2 \le |y(t)| \le c$ for $t \ge t_0$. In view of Lemma 3.2, we have that $\widetilde{\mathbf{R}}_T(t) \le \widetilde{\mathbf{R}}(t)$ for $T \ge t_0$. Then, for $t \ge T$, we obtain

$$\begin{aligned} \big|(\mathcal{P}y)(t)\big| &\le \frac{c}{2} + \int_T^t \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty \big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s\Delta s_1\Delta s_2\Delta s_3 \\ &= \frac{c}{2} + \int_T^t \int_T^{\sigma(s_2)} \frac{1}{r(s_2)} \int_{s_2}^\infty \big(\sigma(s)-s_2\big)\big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s\Delta s_3\Delta s_2 \\ &\quad + \int_t^\infty \int_T^t \frac{1}{r(s_2)} \int_{s_2}^\infty \big(\sigma(s)-s_2\big)\big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s\Delta s_3\Delta s_2 \\ &\le \frac{c}{2} + 2\int_T^\infty \frac{\sigma(s_2)-T}{r(s_2)} \int_{s_2}^\infty \big(\sigma(s)-s_2\big)\big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s\Delta s_2 \\ &= \frac{c}{2} + 2\int_T^\infty \int_T^{\sigma(s)} \frac{\sigma(s_2)-T}{r(s_2)}\big(\sigma(s)-s_2\big)g\big(s,y\big(\eta(s)\big)\big)\Delta s_2\Delta s \\ &\le \frac{c}{2} + 2\int_T^\infty \widetilde{\mathbf{R}}\big(\sigma(s)\big)\big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s. \end{aligned}$$

Then it follows from the strong superlinearity of g that

$$\frac{c}{2} \le \big|(\mathcal{P}y)(t)\big| \le \frac{c}{2} + 2\int_T^\infty \widetilde{\mathbf{R}}\big(\sigma(s)\big)\big|g\big(s,y\big(\eta(s)\big)\big)\big|\Delta s < \frac{c}{2} + 2\int_T^\infty \widetilde{\mathbf{R}}\big(\sigma(s)\big)\big|g(s,c)\big|\Delta s \le c,$$

which implies that $c/2 \le (\mathcal{P}y)(t) \le c$ for $t \in \mathbb{T}_0$. This shows that $\mathcal{P}\Omega \subseteq \Omega$.

Step II. $\mathcal{P}$ is completely continuous.

We first show that $\mathcal{P}$ is continuous. Let $y_n \in \Omega$ $(n = 1,2,\ldots)$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. Hence we get $y \in \Omega$ since Ω is a closed set. Then

$$\begin{aligned}
&\bigl|(\mathcal{P}y_n)(t) - (\mathcal{P}y)(t)\bigr| \\
&\quad\le \int_T^t \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \Delta s \Delta s_1 \Delta s_2 \Delta s_3 \\
&\quad\le 2\int_T^\infty \frac{\sigma(s_2) - T}{r(s_2)} \int_{s_2}^\infty \bigl(\sigma(s) - s_2\bigr) \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \Delta s \Delta s_2 \\
&\quad\le 2\int_T^\infty \widetilde{\mathbf{R}}\bigl(\sigma(s)\bigr) \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \Delta s,
\end{aligned}$$

which, by the strong superlinearity of g, yields

$$\bigl|g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \le \bigl|g(s,c)\bigr|, \quad \text{and} \quad \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr)\bigr| \le \bigl|g(s,c)\bigr|, \quad n = 1, 2 \ldots.$$

In consequence, we get

$$\bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \le 2\bigl|g(s,c)\bigr|.$$

Since $|g(s, y_n(\eta(s))) - g(s, y(\eta(s)))| \to 0$ as $n \to \infty$, the Lebesgue dominated convergence theorem implies that $\lim_{n\to\infty} \|\mathcal{P}y_n - \mathcal{P}y\| = 0$, and hence we obtain that $\mathcal{P}$ is continuous in Ω.

Next, we show that $\mathcal{P}\Omega$ is relatively compact. According to the Arzela–Ascoli theorem on time scales (see [6]), we just need to verify that the family of functions $\{\mathcal{P}y : y \in \Omega\}$ is bounded and uniformly Cauchy, and $\{\mathcal{P}y : y \in \Omega\}$ is equi-continuous on $[t_0, T_1]$ for any $T_1 \in \mathbb{T}_0$. Firstly, the boundedness is obvious. Secondly, in view of (9), for any $\varepsilon > 0$, we can choose a sufficiently large number $T^* \ge T$ so that

$$\int_{T^*}^\infty \widetilde{\mathbf{R}}\bigl(\sigma(s)\bigr) \bigl|g(s,c)\bigr| \Delta s < \frac{\varepsilon}{4}.$$

Hence, for $y \in \Omega$, $t_2 > t_1 \ge T^*$, we get

$$\begin{aligned}
\bigl|(\mathcal{P}y)(t_2) - (\mathcal{P}y)(t_1)\bigr| &= \biggl| \int_T^{t_2} \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2 \Delta s_3 \\
&\quad - \int_T^{t_1} \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2 \Delta s_3 \biggr| \\
&\le 4\int_{T^*}^\infty \widetilde{\mathbf{R}}\bigl(\sigma(s)\bigr) \bigl|g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \Delta s \\
&\le 4\int_{T^*}^\infty \widetilde{\mathbf{R}}\bigl(\sigma(s)\bigr) \bigl|g(s,c)\bigr| \Delta s < \varepsilon,
\end{aligned}$$

which implies that $\{\mathcal{P}y : y \in \Omega\}$ is uniformly Cauchy.

For any $T_1 \in \mathbb{T}_0$ and $y \in \Omega$, if $T \le t_1 < t_2 \le T_1$, then

$$\bigl|(\mathcal{P}y)(t_2) - (\mathcal{P}y)(t_1)\bigr| \le \biggl| \int_{t_1}^{t_2} \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty g\bigl(s, y\bigl(\eta(s)\bigr)\bigr) \Delta s \Delta s_1 \Delta s_2 \Delta s_3 \biggr|$$

$$\begin{aligned}
&\le \int_{t_1}^{t_2} \frac{\sigma(s_2)-t_1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s)-s_2\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\Delta s_2 \\
&\quad + \int_{t_2}^{\infty} \frac{t_2-t_1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s)-s_2\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\Delta s_2 \\
&\le (t_2-t_1)\biggl[\int_{t_1}^{t_2} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s)-s_2\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\Delta s_2 \\
&\quad + \int_{t_2}^{\infty} \frac{1}{r(s_2)} \int_{s_2}^{\infty} \bigl(\sigma(s)-s_2\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\Delta s_2\biggr] \\
&= (t_2-t_1)\int_{t_1}^{\infty} \overline{R}_{t_1}\bigl(\sigma(s)\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s,
\end{aligned}$$

where

$$\overline{R}_{t_1}(t) = \int_{t_1}^{t} \frac{t-s}{r(s)}\Delta s = \int_{t_1}^{t}\int_{t_1}^{\sigma(s_1)} \frac{1}{r(s)}\Delta s\Delta s_1.$$

On the other hand, by L'Hôpital's rule, we get

$$\lim_{t\to\infty} \frac{K_{t_1}(t)}{\widetilde{\mathbf{K}}_{t_1}(t)} = \lim_{t\to\infty} \frac{1}{\sigma(t)-t_1} = 0,$$

which, in view of Lemma 3.5, implies that $\lim_{t\to\infty} \frac{K_{t_1}(\sigma(t))}{\widetilde{\mathbf{K}}_{t_1}(\sigma(t))} = 0$, where

$$K_{t_1}(t) = \int_{t_1}^{t} \frac{1}{r(s)}\Delta s \quad \text{and} \quad \widetilde{\mathbf{K}}_{t_1}(t) = \int_{t_1}^{t} \frac{\sigma(s)-t_1}{r(s)}\Delta s.$$

Furthermore, by using the earlier argument, we find that $\lim_{t\to\infty} \frac{\overline{R}_{t_1}(\sigma(t))}{\widetilde{\mathbf{R}}_{t_1}(\sigma(t))} = 0$. Hence, for any $\epsilon > 0$, there exists $T_1^* \ge t_1$ such that $\overline{R}_{t_1}(\sigma(t)) < \epsilon\widetilde{\mathbf{R}}_{t_1}(\sigma(t))$ for $t \ge T_1^*$. This means that

$$\begin{aligned}
\bigl|(\mathcal{P}y)(t_2)-(\mathcal{P}y)(t_1)\bigr| &\le (t_2-t_1)\epsilon \int_{T_1^*}^{\infty} \widetilde{\mathbf{R}}_{t_1}\bigl(\sigma(s)\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \\
&\quad + (t_2-t_1)\int_{t_1}^{T_1^*} \overline{R}_{t_1}\bigl(\sigma(s)\bigr)\bigl|g\bigl(s,y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s.
\end{aligned}$$

Therefore, there exists $\delta > 0$ such that

$$\bigl|(\mathcal{P}y)(t_2)-(\mathcal{P}y)(t_1)\bigr| < \varepsilon, \quad \text{if } |t_2-t_1| < \delta.$$

Moreover, we have

$$\bigl|(\mathcal{P}y)(t_2)-(\mathcal{P}y)(t_1)\bigr| = 0 < \varepsilon, \quad \text{if } t_0 \le t_1 < t_2 \le T.$$

From the preceding arguments, we conclude that $\{\mathcal{P}y : y \in \Omega\}$ is equi-continuous on $[t_0, T_1]$. Hence, $\mathcal{P}\Omega$ is relatively compact. Thus, $\mathcal{P}$ is completely continuous. Hence, by Schauder's fixed point theorem, $\mathcal{P}$ has a fixed point $y_0 \in \Omega$, which is a nonoscillatory solution of (1). This is a contradiction.

We next prove the sufficiency by contradiction. Without loss of generality, let $y(t)$ be an eventually positive solution of (1). Then, from Lemma 3.4, there exists $t_1 \geq t_0$ such that, for any $T \geq t_1$, we have

$$y(\sigma(t)) \geq \int_{t_1}^{\sigma(t)} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) g(s, y(\eta(s))) \Delta s, \quad t \geq T.$$

It follows from $y^{\Delta}(t) > 0$ for $t \geq t_1$ that there exists a constant $c_1 > 0$ such that $y(\eta(t)) \geq c_1$ for $t \geq t_1$. Then, by the strong superlinearity of g, we have

$$g(t, y(\eta(t))) \geq c_1^{-\alpha} y^{\alpha}(\eta(t)) g(t, c_1) \geq c_1^{-\alpha} y^{\alpha}(\sigma(t)) g(t, c_1).$$

Hence

$$y(\sigma(t)) \geq c_1^{-\alpha} \int_{t_1}^{\sigma(t)} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) y^{\alpha}(\sigma(s)) g(s, c_1) \Delta s, \quad t \geq T,$$

that is,

$$\bigl(y(\sigma(t))\bigr)^{-\alpha} \leq c_1^{\alpha^2} \left(\int_{t_1}^{\sigma(t)} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) y^{\alpha}(\sigma(s)) g(s, c_1) \Delta s \right)^{-\alpha}, \quad t \geq T. \tag{10}$$

Notice that there exists $\zeta \in [s, \sigma(s)]$ such that

$$\begin{aligned}
&\left[\left(\int_{t_1}^{s} \widetilde{\mathbf{R}}_{t_1}(\sigma(\theta)) y^{\alpha}(\sigma(\theta)) g(\theta, c_1) \Delta\theta \right)^{1-\alpha} \right]^{\Delta} \\
&\quad = (1-\alpha) \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) y^{\alpha}(\sigma(s)) g(s, c_1) \\
&\qquad \times \left(\int_{t_1}^{\zeta} \widetilde{\mathbf{R}}_{t_1}(\sigma(\theta)) y^{\alpha}(\sigma(\theta)) g(\theta, c_1) \Delta\theta \right)^{-\alpha}.
\end{aligned}$$

Multiplying (10) by $\widetilde{\mathbf{R}}_{t_1}(\sigma(t)) y^{\alpha}(\sigma(t)) g(t, c_1)$ and then integrating from t_2 ($t_2 > T$) to t, we get

$$\begin{aligned}
&\int_{t_2}^{t} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) g(s, c_1) \Delta s \\
&\quad \leq c_1^{\alpha^2} \int_{t_2}^{t} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) y^{\alpha}(\sigma(s)) g(s, c_1) \left(\int_{t_1}^{\zeta} \widetilde{\mathbf{R}}_{t_1}(\sigma(\theta)) y^{\alpha}(\theta) g(\theta, c_1) \Delta\theta \right)^{-\alpha} \Delta s \\
&\quad = \frac{c_1^{\alpha^2}}{(\alpha-1)} \left(\int_{t_1}^{s} \widetilde{\mathbf{R}}_{t_1}(\sigma(\theta)) y^{\alpha}(\sigma(\theta)) g(\theta, c_1) \Delta\theta \right)^{1-\alpha} \Bigg|_{t}^{t_2}.
\end{aligned}$$

This means that

$$\int_{t_2}^{t} \widetilde{\mathbf{R}}_{t_1}(\sigma(s)) g(s, c_1) \Delta s < \infty.$$

On the other hand, we have

$$\widetilde{\mathbf{R}}(t) = \int_{t_0}^{t} \frac{(t-s)(\sigma(s)-t_0)}{r(s)} \Delta s = \int_{t_0}^{t} \int_{t_0}^{\sigma(s_1)} \frac{\sigma(s)-t_0}{r(s)} \Delta s \Delta s_1.$$

Now, we show that $\lim_{t\to\infty}\frac{\widetilde{\mathbf{R}}_{t_1}(\sigma(t))}{\widetilde{\mathbf{R}}(\sigma(t))}=1$. In fact, from L'Hôpital's rule and Lemma 3.5, we need to prove that

$$\lim_{t\to\infty}\frac{\widetilde{\mathbf{R}}_{t_1}(t)}{\widetilde{\mathbf{R}}(t)}=\lim_{t\to\infty}\frac{\widetilde{\mathbf{R}}^{\Delta}_{t_1}(t)}{\widetilde{\mathbf{R}}^{\Delta}(t)}=1,$$

i.e., it is sufficient to show that

$$\lim_{t\to\infty}\frac{\widetilde{\mathbf{K}}_{t_1}(\sigma(t))}{\widetilde{\mathbf{K}}_{t_0}(\sigma(t))}=1. \tag{11}$$

Furthermore, one can see that expression (11) is true. Indeed, from L'Hôpital's rule and Lemma 3.5 again, we just need to show that

$$\lim_{t\to\infty}\frac{\widetilde{\mathbf{K}}_{t_1}(t)}{\widetilde{\mathbf{K}}_{t_0}(t)}=\lim_{t\to\infty}\frac{\widetilde{\mathbf{K}}^{\Delta}_{t_1}(t)}{\widetilde{\mathbf{K}}^{\Delta}_{t_0}(t)}=1. \tag{12}$$

Obviously, (12) is satisfied. Thus, there exists $t_3\geq T$ such that $\widetilde{\mathbf{R}}_{t_1}(\sigma(t))>\frac{1}{2}\widetilde{\mathbf{R}}(\sigma(t))$ for $t\geq t_3$. It means that

$$\int_{t_2}^{\infty}\widetilde{\mathbf{R}}\bigl(\sigma(s)\bigr)g(s,c_1)\Delta s<\infty,$$

which contradicts (3). The proof is complete. □

Theorem 3.7 *Assume that g is strongly sublinear and $\eta(t)\leq t$. Then every solution of* (1) *is oscillatory if and only if*

$$\int_{t_0}^{\infty}\bigl|g\bigl(t,cR\bigl(\eta(t)\bigr)\bigr)\bigr|\Delta t=\infty\quad \textit{for all } c\neq 0. \tag{13}$$

Proof We first prove the necessity by contradiction. Suppose that condition (13) does not hold true. Then there exists $c>0$ such that

$$\int_{t_0}^{\infty}\bigl|g\bigl(t,cR\bigl(\eta(t)\bigr)\bigr)\bigr|\Delta t<\infty.$$

Let $T>t_0$ be so large that

$$\int_{T}^{\infty}\bigl|g\bigl(t,cR\bigl(\eta(t)\bigr)\bigr)\bigr|\Delta t<\frac{c}{2}.$$

Let

$$U=\left\{y\,\middle|\,y\in C_{rd}(\mathbb{T}_0,\mathbb{R}),\sup_{t\in\mathbb{T}_0}\frac{|y(t)|}{R^2(t)}<\infty\right\}.$$

Obviously, U is a Banach space with the norm $\|y\|=\sup_{t\in\mathbb{T}_0}\frac{|y(t)|}{R^2(t)}$. Define a closed and convex subset of U as follows:

$$\Omega=\bigl\{y=y(t):y\in U,cR(t)\leq\bigl|y(t)\bigr|\leq 2cR(t),t\in\mathbb{T}_0\bigr\}.$$

We define a map $\mathcal{P}$ on Ω by

$$(\mathcal{P}y)(t) = \begin{cases} cR(t) + \int_T^t \int_T^{s_3} \frac{1}{r(s_2)} \int_T^{s_2} \int_{s_1}^{\infty} g(s, y(\eta(s)))\Delta s \Delta s_1 \Delta s_2 \Delta s_3, & t \geq T, \\ (\mathcal{P}y)(T), & t_0 \leq t \leq T. \end{cases}$$

In order to show that $\mathcal{P}$ has a fixed point in Ω, we proceed as follows.

Step I. $\mathcal{P}$ maps Ω into Ω. Let $y \in \Omega$. Then $cR(t) \leq |y(t)| \leq 2cR(t)$ for $t \geq t_0$. Furthermore, we have

$$\begin{aligned} \bigl|(\mathcal{P}y)(t)\bigr| &\leq cR(t) + \int_T^t \int_T^{s_3} \frac{1}{r(s_2)} \biggl[\int_T^{s_2} \bigl(\sigma(s) - T\bigr)\bigl|g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \\ &\quad + \int_{s_2}^{\infty} (s_2 - T)\bigl|g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\biggr]\Delta s_2 \Delta s_3 \\ &\leq cR(t) + \int_T^t \int_T^{s_3} \frac{s_2 - T}{r(s_2)} \biggl[\frac{c}{2} + \frac{c}{2}\biggr]\Delta s_2 \Delta s_3 \\ &\leq cR(t) + cR(t) = 2cR(t), \end{aligned}$$

which implies that $cR(t) \leq |(\mathcal{P}y)(t)| \leq 2cR(t)$ for $t \in \mathbb{T}_0$. This shows that $\mathcal{P}\Omega \subseteq \Omega$.

Step II. $\mathcal{P}$ is completely continuous.

Firstly, we show that $\mathcal{P}$ is continuous. Set $y_n \in \Omega$ and $\|y_n - y\| \to 0$ as $n \to \infty$. Hence, we have $y \in \Omega$ since Ω is a closed set. Then

$$\begin{aligned} &\bigl|(\mathcal{P}y_n)(t) - (\mathcal{P}y)(t)\bigr| \\ &\quad\leq \int_T^t \int_T^{s_3} \frac{1}{r(s_2)} \int_T^{s_2} \int_{s_1}^{\infty} \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \Delta s_1 \Delta s_2 \Delta s_3 \\ &\quad\leq \int_T^t \int_T^{s_3} \frac{1}{r(s_2)} \biggl[\int_T^{s_2} \bigl(\sigma(s) - T\bigr)\bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \\ &\quad\quad + \int_{s_2}^{\infty} (s_2 - T)\bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s\biggr]\Delta s_2 \Delta s_3 \\ &\quad\leq \int_T^t \int_T^{s_3} \frac{(s_2 - T)}{r(s_2)}\Delta s_2 \Delta s_3 \int_T^{\infty} \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \\ &\quad\leq \int_T^{\infty} \bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr|\Delta s \times R(t). \end{aligned}$$

By the strong sublinearity of g, we have

$$\begin{aligned} &\bigl|g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \leq 2^{\alpha}\bigl|g\bigl(s, cR\bigl(\bigl(\eta(s)\bigr)\bigr)\bigr|, \quad \text{and} \\ &\bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr)\bigr| \leq 2^{\alpha}\bigl|g\bigl(s, cR\bigl(\eta(s)\bigr)\bigr)\bigr|, \quad n = 1, 2 \ldots. \end{aligned}$$

Then

$$\bigl|g\bigl(s, y_n\bigl(\eta(s)\bigr)\bigr) - g\bigl(s, y\bigl(\eta(s)\bigr)\bigr)\bigr| \leq 2^{\alpha+1}\bigl|g\bigl(s, cR\bigl(\bigl(\eta(s)\bigr)\bigr)\bigr)\bigr|.$$

Since $|g(s, y_n(\eta(s))) - g(s, y(\eta(s)))| \to 0$ as $n \to \infty$, the Lebesgue dominated convergence theorem implies that $\lim_{n\to\infty} \|\mathcal{P}y_n - \mathcal{P}y\| = 0$, and thus $\mathcal{P}$ is continuous in Ω.

We next show that $\mathcal{P}\Omega$ is relatively compact. The boundedness is obvious. For any $\varepsilon > 0$, let $T^* \geq T$ be so large that

$$\left|\frac{1}{R(t)}\right| < \frac{\varepsilon}{4c} \quad \text{for } t \geq T^*.$$

Hence, for $y \in \Omega$, $t_2 > t_1 \geq T^*$,

$$\begin{aligned}
&\left|\left(R^{-2}\mathcal{P}y\right)(t_2) - \left(R^{-2}\mathcal{P}y\right)(t_1)\right| \\
&\quad\leq \frac{c}{R(t_2)} + \frac{1}{R^2(t_2)}\int_T^{t_2}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}\left|g\left(s, y\left(\eta(s)\right)\right)\right|\Delta s\Delta s_1\Delta s_2\Delta s_3 \\
&\qquad + \frac{c}{R(t_1)} + \frac{1}{R^2(t_1)}\int_T^{t_1}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}\left|g\left(s, y\left(\eta(s)\right)\right)\right|\Delta s\Delta s_1\Delta s_2\Delta s_3 \\
&\quad\leq \frac{c}{R(t_2)} + \frac{c}{R(t_1)} + \frac{c}{R^2(t_2)}\int_T^{t_2}\int_T^{s_3}\frac{s_2 - T}{r(s_2)}\Delta s_2\Delta s_3 + \frac{c}{R^2(t_1)}\int_T^{t_1}\int_T^{s_3}\frac{s_2 - T}{r(s_2)}\Delta s_2\Delta s_3 \\
&\quad\leq \frac{c}{R(t_2)} + \frac{c}{R(t_1)} + \frac{c}{R(t_2)} + \frac{c}{R(t_1)} < \varepsilon,
\end{aligned}$$

which implies that $\{\mathcal{P}y : y \in \Omega\}$ is uniformly Cauchy. Furthermore, for any $y \in \Omega$ and $T_1 \in \mathbb{T}_0$, if $T \leq t_1 < t_2 \leq T_1$, then

$$\begin{aligned}
&\left|\left(R^{-2}\mathcal{P}y\right)(t_2) - \left(R^{-2}\mathcal{P}y\right)(t_1)\right| \\
&\quad\leq \left|\frac{c}{R(t_2)} - \frac{c}{R(t_1)}\right| \\
&\qquad + \left|\frac{1}{R^2(t_2)}\int_T^{t_2}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}g\left(s, y\left(\eta(s)\right)\right)\Delta s\Delta s_1\Delta s_2\Delta s_3\right. \\
&\qquad \left. - \frac{1}{R^2(t_1)}\int_T^{t_1}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}g\left(s, y\left(\eta(s)\right)\right)\Delta s\Delta s_1\Delta s_2\Delta s_3\right| \\
&\quad\leq \left|\frac{c}{R(t_2)} - \frac{c}{R(t_1)}\right| \\
&\qquad + \left|\frac{1}{R^2(t_2)} - \frac{1}{R^2(t_1)}\right|\int_T^{t_2}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}\left|g\left(s, y\left(\eta(s)\right)\right)\right|\Delta s\Delta s_1\Delta s_2\Delta s_3 \\
&\qquad + \frac{1}{R^2(t_1)}\int_{t_1}^{t_2}\int_T^{s_3}\frac{1}{r(s_2)}\int_T^{s_2}\int_{s_1}^{\infty}\left|g\left(s, y\left(\eta(s)\right)\right)\right|\Delta s\Delta s_1\Delta s_2\Delta s_3 \\
&\quad\leq \left|\frac{c}{R(t_2)} - \frac{c}{R(t_1)}\right| + \left|\frac{1}{R^2(t_2)} - \frac{1}{R^2(t_1)}\right| \times cR(t_2) \\
&\qquad + \frac{c}{R^2(t_1)}\int_{t_1}^{t_2}\int_T^{s_3}\frac{s_2}{r(s_2)}\Delta s_2\Delta s_3.
\end{aligned}$$

Hence, there exists $\delta > 0$ such that

$$\left|\left(R^{-2}\mathcal{P}y\right)(t_2) - \left(R^{-2}\mathcal{P}y\right)(t_1)\right| < \varepsilon, \quad \text{if } |t_2 - t_1| < \delta.$$

Moreover, we have

$$\left|\left(R^{-2}\mathcal{P}y\right)(t_2) - \left(R^{-2}\mathcal{P}y\right)(t_1)\right| = 0 < \varepsilon, \quad \text{if } t_0 \leq t_1 < t_2 \leq T.$$

In consequence, $\{\mathcal{P}y : y \in \Omega\}$ is equi-continuous on $[t_0, T_1]$. According to the Arzela–Ascoli theorem on time scales, we know that $\mathcal{P}$ is a compact operator. Hence $\mathcal{P}$ is completely continuous. Therefore, $\mathcal{P}$ has a fixed point $y_0 \in \Omega$ according to Schauder's fixed point theorem, which is a nonoscillatory solution of (1). This is a contradiction.

Now, we prove the sufficiency by contradiction. Without loss of generality, let $y(t)$ be an eventually positive solution of (1). From Lemmas 3.1 and 3.3, there exist $t_1 \ge t_0$ and a positive constant c_1 such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \quad \text{and} \quad \bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta} > 0, \quad t \ge t_1,$$

and

$$\frac{1}{2}\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}R(t) \le y(t) \le c_1 R(t), \quad t \ge t_1. \tag{14}$$

Noting that $[r(t)y^{\Delta\Delta}(t)]^{\Delta\Delta} < 0$, we have

$$y\bigl(\eta(t)\bigr) \ge \frac{1}{2}\bigl[r(s)y^{\Delta\Delta}(s)\bigr]^{\Delta}\Big|_{s=\eta(t)} R\bigl(\eta(t)\bigr) \ge \frac{1}{2}\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}R\bigl(\eta(t)\bigr). \tag{15}$$

From (14), (15), and the strong sublinearity of g, there exists $\zeta \in [t, \sigma(t)]$ such that

$$\begin{aligned}
\Bigl(-\bigl(\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}\bigr)^{1-\alpha}\Bigr)^{\Delta} &= -(1-\alpha)\bigl(\bigl[r(\zeta)y^{\Delta\Delta}(\zeta)\bigr]^{\Delta}\bigr)^{-\alpha}\bigl[r(\zeta)y^{\Delta\Delta}(\zeta)\bigr]^{\Delta\Delta} \\
&= (1-\alpha)\bigl(\bigl[r(\zeta)y^{\Delta\Delta}(\zeta)\bigr]^{\Delta}\bigr)^{-\alpha} g\bigl(t, y\bigl(\eta(t)\bigr)\bigr) \\
&\ge (1-\alpha)\bigl(\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}\bigr)^{-\alpha} \frac{(y(\eta(t)))^{\alpha}}{(c_1 R(\eta(t)))^{\alpha}} g\bigl(t, c_1 R\bigl(\eta(t)\bigr)\bigr) \\
&\ge (1-\alpha)\frac{1}{(2c_1)^{\alpha}} g\bigl(t, c_1 R\bigl(\eta(t)\bigr)\bigr).
\end{aligned}$$

Integrating the inequalities above from t_2 to t, we get

$$(1-\alpha)\frac{1}{(2c_1)^{\alpha}}\int_{t_2}^{t} g\bigl(s, c_1 R\bigl(\eta(s)\bigr)\bigr)\Delta s \le \bigl(\bigl[r(t_2)y^{\Delta\Delta}(t_2)\bigr]^{\Delta}\bigr)^{1-\alpha} - \bigl(\bigl[r(t)y^{\Delta\Delta}(t)\bigr]^{\Delta}\bigr)^{1-\alpha} < \infty,$$

and so

$$\int_{t_2}^{\infty} g\bigl(s, c_1 R\bigl(\eta(s)\bigr)\bigr)\Delta s < \infty,$$

which contradicts (13). This completes the proof. □

Remark 3.8 It is noteworthy that the results given in the aforementioned theorems are the same as those in [13] where the dynamic equation on time scales is reduced to a differential equation when $\mathbb{T} = \mathbb{R}_{+}$, $\sigma(t) = t$, and $x^{\Delta} = x'$. If further we set $r(t) = 1$ and $\eta(t) = \sin t$ in (1), we conclude that $R(t) = \widetilde{\mathbf{R}}(t) = t^3/6$, and every solution of (1) is oscillatory in view of Theorem 3.7 whenever (13) holds under a suitable strongly sublinear function g. However, for the case of $\mathbb{T} = \mathbb{N}$, we know that $\sigma(t) = t + 1$ and $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$, there is no work concerning the sufficient and necessary conditions for the following corresponding difference equation:

$$\Delta^2\big[r(t)\Delta^2 x(t)\big] + g\big(t, x\big(\eta(t)\big)\big) = 0. \tag{16}$$

Example 3.9 Let $\mathbb{T} = \{t : t \in \mathbb{N}\}$, $r(t) = \frac{t+1}{2t}$, $\eta(t) = \lambda t$ for $\lambda > 2$, and $g(t, x(t)) = x^3(t)$ in (16). It is easy to see that $\sum_{t=1}^{\infty} t/r(t) = \infty$ and $\sum_{t=1}^{\infty} \widetilde{\mathbf{R}}(t+1)|c^3| = \infty$ for $c \neq 0$, and then all the conditions of Theorem 3.6 are satisfied. Thus every solution of (16) is oscillatory.

Example 3.10 Consider the fourth order dynamic equation

$$\big[ty^{\Delta\Delta}(t)\big]^{\Delta\Delta} + \frac{1}{t}y^5(4t) = 0, \quad t \in 2^{\mathbb{N}}, t \geq t_0 = 2. \tag{17}$$

Here, $\mathbb{T} = 2^{\mathbb{N}}$, $\eta(t) = 4t$ and $g(t, y(t)) = \frac{1}{t}y^5(t)$. Hence we have $\sigma(t) = 2t$. In this case $r(t) = t$, one can check that $\int_2^{\infty} t/r(t)\Delta t = \infty$ and $\widetilde{\mathbf{R}}(t) = \frac{4}{3}t^2 - \frac{4}{3} - 2t\frac{\ln t}{\ln 2}$. Hence $\int_2^{\infty} \widetilde{\mathbf{R}}(2t)|g(t,c)|\Delta t = \infty$. It means that all the conditions of Theorem 3.6 are satisfied. Then every solution of (17) is oscillatory.

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