Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

An asymmetric information non-zero sum differential game of mean-field backward stochastic differential equation with applications

Pengyan Huang[1], Guangchen Wang[1*] and Huanjun Zhang[1]

*Correspondence:
wguangchen@sdu.edu.cn
[1]School of Control Science and Engineering, Shandong University, Jinan, China

Abstract

This paper is concerned with a kind of non-zero sum differential game driven by mean-field backward stochastic differential equation (MF-BSDE) with asymmetric information, whose novel feature is that both the state equation and the cost functional are of mean-field type. A necessary condition and a sufficient condition for Nash equilibrium point of the above problem are established. As applications, a mean-field linear-quadratic (MF-LQ) problem and a financial problem are studied.

Keywords: Mean-field backward stochastic differential equation; Asymmetric information; Nash equilibrium point; Maximum principle; Filter

1 Introduction

Mean-field theory has been an active research field in recent years, which has attracted a lot of researchers to investigate this theory. Mean-field theory was independently proposed by Lasry and Lions [1] and Huang et al. [2], respectively. Since then, research on related topics and their applications has become popular among scholars. For instance, Bensoussan et al. [3] investigated the existence and uniqueness of equilibrium strategies of LQ mean-field games; Øksendal and Sulem [4] researched optimal control of predictive mean-field equation and applied the theoretical results to solve optimal portfolio and consumption rate problems; Wu and Liu [5] derived the maximum principle for mean-field zero-sum stochastic differential game with partial information and applied the results to study a portfolio game problem; Hafayed et al. [6] studied the mean-field stochastic control problem under partial information and derived a necessary condition and a sufficient condition for optimal control; Huang and Wang [7] investigated a partial information linear-quadratic-Gauss game of larger-population system and obtained the decentralized strategy and approximate Nash equilibrium by studying the related mean-field game. We emphasize that the systems introduced in [3–7] are governed by forward stochastic differential equations (SDEs).

Nonlinear BSDE was introduced by Pardoux and Peng [8]. From then on, the theory of BSDE has made a rapid development due to its wide applications. Shen and Jiang [9] proved the existence and uniqueness of BSDE driven by time-changed Lévy noise, where the generator is monotonic and general growth with respect to variable y, and Mu and Wu [10] provided an existence result of a coupled Markovian BSDE system. Hamadène and Lepeltier [11] discussed a stochastic zero-sum differential game of BSDE and obtained the existence of saddle point under the bounded case and Isaacs' condition. Wang and Yu [12] established a necessary condition and a verification theorem for open-loop Nash equilibrium point of non-zero sum differential game of BSDE under partial information. Furthermore, Wang et al. [13] discussed asymmetric information LQ non-zero sum differential game of BSDE and gave the feedback Nash equilibrium points. See also Wang and Yu [14], Li and Yu [15] for more information.

In 2009, Buckdahn et al. [16] firstly introduced a new kind of BSDE by investigating a special mean-field problem, which is called MF-BSDE. Then, Ma and Liu [17] studied a partial information optimal control of infinite horizon MF-BSDE with delay, and a necessary condition and a sufficient condition for optimal control were derived; Li et al. [18] solved an LQ control problem of MF-BSDE, and the optimal control was represented by two Riccati-type equations and a mean-field SDE. Besides, Wu and Liu [19] studied an optimal control problem for mean-field zero-sum stochastic differential game under partial information. Recently, Lin et al. [20] discussed an open-loop LQ leader-follower of mean-field stochastic differential game and solved the corresponding optimal control problems for the follower and the leader; Du and Wu [21] considered a new kind of Stackelberg differential game of MF-BSDE, and they obtained the open-loop Stackelberg equilibrium, which admits a state feedback representation. What is more, Zhang [22] investigated an optimal control problem for terminal constraint mean-field SDE under partial information, which was solved by the backward separation method with a decomposition technique. To our best knowledge, the mean-field backward stochastic differential game has important applications in economic and financial fields, and the corresponding result is quite lacking in literature, then we highly desire to study such a topic. Note that the system in [17, 18] only contains one control process, [19] discussed the zero-sum game for a forward system, and [20, 21] investigated game problems under full information. Due to this, Problem (MFBNZ) is distinguished from the above literature.

The rest of this paper is organized as follows. In Sect. 2, we introduce some basic notations and formulate the asymmetric information non-zero sum differential game of MF-BSDE. In Sect. 3, we establish a necessary condition and a sufficient condition for Nash equilibrium point of Problem (MFBNZ). In Sect. 4, we investigate the well-posedness of initial coupled mean-field forward and backward stochastic differential equation (MF-FBSDE), which plays an important role in Sects. 5 and 6. In Sect. 5, we use the theoretical results to study an MF-LQ problem with asymmetric information and give an explicit form of Nash equilibrium point. In Sect. 6, we put a financial problem into the framework of Problem (MFBNZ) and obtain a feedback optimal investment strategy. In Sect. 7, we give some concluding remarks.

2 Notations and problem formulation

Throughout this paper, we denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, by $\mathbb{R}^{k\times l}$ the collection of $k \times l$ matrices, by A^τ the transpose of A, by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the inner product and the norm in Euclidean space, respectively. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtered probability space with a natural filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by an $\mathcal{F}_t$-adapted, l-dimensional standard Brownian motion $\{W(t), t \geq 0\}$. Also, we denote by $\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ the space of all

$\mathbb{R}^n$-valued, $\mathcal{F}_t$-adapted processes such that $\mathbb{E}\int_0^T |x(t)|^2\,dt < +\infty$, and by $\mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ the space of all $\mathbb{R}^n$-valued, $\mathcal{F}_T$-measurable random variables such that $\mathbb{E}|\xi|^2 < +\infty$.

In this paper, we study a kind of asymmetric information non-zero sum differential game of MF-BSDE. We only consider the case of two players. Similarly, we can study the case of n players. Consider the MF-BSDE

$$\begin{cases} -dy^v(t) = f(t, y^v(t), z^v(t), \mathbb{E}y^v(t), \mathbb{E}z^v(t), v_1(t), v_2(t))\,dt - z^v(t)\,dW(t), \\ y^v(T) = \xi, \end{cases} \tag{1}$$

where we adopt the notation $v(\cdot) = (v_1(\cdot), v_2(\cdot))$ for simplicity; $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \longrightarrow \mathbb{R}^n$ is a given continuous function; $\xi \in \mathcal{L}^2_{\mathcal{F}}(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$; $v_1(\cdot)$ and $v_2(\cdot)$ are the control processes of Player 1 and Player 2, respectively. MF-BSDE (1) characters that the two players cooperate to reach a terminal goal ξ at T.

Let U_i $(i = 1, 2)$ be a nonempty convex subset of $\mathbb{R}^{k_i}$, and $\mathcal{F}^i_t \subseteq \mathcal{F}_t$ be a given sub-filtration which is the information available to Player i at time $t \in [0, T]$. Introduce the admissible control set

$$\mathcal{U}_i = \left\{ v_i(\cdot) \in \mathcal{L}^2_{\mathcal{F}^i_t}\left(0, T; \mathbb{R}^{k_i}\right) | v_i(t) \in U_i, t \in [0, T] \right\} \quad (i = 1, 2),$$

which is called an open-loop admissible control set for Player i.

Hypothesis 1 *The function f is continuously differentiable in $(y, z, \bar{y}, \bar{z}, v_1, v_2)$. Moreover, its partial derivatives $f_y, f_z, f_{\bar{y}}, f_{\bar{z}}, f_{v_1}, f_{v_2}$ are uniformly bounded.*

Suppose that $v_1(\cdot)$ and $v_2(\cdot)$ are admissible controls and Hypothesis 1 holds. With the assumptions, MF-BSDE (1) has a unique solution $(y(\cdot), z(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n) \times \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^{n\times l})$ (see Buckdahn et al. [16]). Ensuring achievement of the goal ξ, the two players have their own benefits described by the cost functional

$$J_i\big(v_1(\cdot), v_2(\cdot)\big) = \mathbb{E}\left[\int_0^T l_i\big(t, y^v(t), z^v(t), \mathbb{E}y^v(t), \mathbb{E}z^v(t), v_1(t), v_2(t)\big)\,dt + \Phi_i\big(y^v(0)\big)\right] \tag{2}$$

$(i = 1, 2)$, where $l_i : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}$ and $\Phi_i : \mathbb{R}^n \to \mathbb{R}$ are continuous, and l_i satisfies

$$\mathbb{E}\int_0^T \left| l_i\big(t, y^v(t), z^v(t), \mathbb{E}y^v(t), \mathbb{E}z^v(t), v_1(t), v_2(t)\big)\right| dt < +\infty$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$.

Hypothesis 2 *l_i and Φ_i $(i = 1, 2)$ are continuously differentiable with respect to $(y, z, \bar{y}, \bar{z}, v_1, v_2)$ and y, respectively. Besides, there exists a constant C such that the partial derivatives $l_{iy}, l_{iz}, l_{i\bar{y}}, l_{i\bar{z}}, l_{iv_1}, l_{iv_2}$ $(i = 1, 2)$ are bounded by $C(1 + |y| + |z| + |\bar{y}| + |\bar{z}| + |v_1| + |v_2|)$.*

Problem (MFBNZ) Find a pair of $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)), \end{cases} \tag{3}$$

subject to state equation (1) and cost functional (2). We call the above problem a mean-field backward non-zero sum stochastic differential game with asymmetric information. If $(u_1(\cdot), u_2(\cdot))$ satisfies (3), we call it a Nash equilibrium point of Problem (MFBNZ).

3 Maximum principle

In this section, we will give a necessary condition and a sufficient condition for Nash equilibrium point of Problem (MFBNZ).

Define the Hamiltonian function H_i: $\Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^n \to \mathbb{R}$ by $H_i(t,y,z,\bar{y},\bar{z},v_1,v_2,p_i) = \langle p_i, -f(t,y,z,\bar{y},\bar{z},v_1,v_2)\rangle + l_i(t,y,z,\bar{y},\bar{z},v_1,v_2)$ $(i = 1,2)$, where $p_i(\cdot)$ satisfies

$$\begin{cases} dp_i(t) = -[H_{iy}(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_i(t)) \\ \qquad\qquad + \mathbb{E}H_{i\bar{y}}(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_i(t))]\,dt \\ \qquad\qquad - [H_{iz}(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_i(t)) \\ \qquad\qquad + \mathbb{E}H_{i\bar{z}}(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_i(t))]\,dW(t), \\ p_i(0) = -\Phi_{iy}(y(0)) \end{cases} \tag{4}$$

with $H_{iy}, H_{i\bar{y}}, H_{iz}$, and $H_{i\bar{z}}$ being the partial derivatives of H_i with respect to $y, \bar{y}, z$, and $\bar{z}$, respectively.

3.1 The necessary condition

Assume that $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point of Problem (MFBNZ). For fixed $u_1(\cdot)$, to minimize the aforementioned cost functional $J_2(u_1(\cdot), v_2(\cdot))$, subject to state equation (1) over $\mathcal{U}_2$ is a "non-Markovian" optimal control problem of MF-BSDE. Similarly, for the case corresponding to fixed $u_2(\cdot)$. Using the method introduced in Peng [23], we can analyze the differential game problem. Here we only present the main result for saving space.

Theorem 3.1 *Let Hypotheses 1–2 hold. If $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point of Problem (MFBNZ) and $(y(\cdot), z(\cdot))$ is the corresponding state trajectory, then we have*

$$\mathbb{E}\big[\big\langle H_{iv_i}\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_i(t)\big), v_i - u_i(t)\big\rangle \big| \mathcal{F}_t^i\big] \geq 0 \quad (i = 1,2)$$

holds for any $(v_1, v_2) \in U_1 \times U_2$, where $p_i(\cdot)$ satisfies (4).

3.2 The sufficient condition

Now we weaken Hypothesis 2 to

Hypothesis 3 *For each $(v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, l_i and Φ_i $(i = 1,2)$ are differentiable with respect to $(y^v, z^v, \bar{y}^v, \bar{z}^v, v_1, v_2)$ and y, respectively; besides, $l_i(t,y,z,\bar{y},\bar{z},v_1,v_2) \in \mathcal{L}^1_{\mathcal{F}}(0,T;\mathbb{R})$.*

Theorem 3.2 *Let Hypothesis 1 and Hypothesis 3 hold. Suppose that l_i $(i = 1,2)$ is continuously differentiable in v_i. Assume that adjoint equation (4) is uniquely solvable. Suppose that $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ is given such that $l_{iy}(\mathcal{X}(\cdot))$, $l_{iz}(\mathcal{X}(\cdot))$, $l_{i\bar{y}}(\mathcal{X}(\cdot))$, $l_{i\bar{z}}(\mathcal{X}(\cdot))$, $l_{iv_i}(\mathcal{X}(\cdot)) \in \mathcal{L}^2_{\mathcal{F}}(0,T)$ $(i = 1,2)$, where*

$$\mathcal{X}(\cdot) = \big(\cdot, y(\cdot), z(\cdot), \mathbb{E}y(\cdot), \mathbb{E}z(\cdot), u_1(\cdot), u_2(\cdot)\big).$$

In addition, for any $(t,v_i) \in [0,T] \times U_i$ $(i=1,2)$, $l_{iv_i}(\cdot,y(\cdot),z(\cdot),\mathbb{E}y(\cdot),\mathbb{E}z(\cdot),v_i,u_{3-i}(\cdot)) \in \mathcal{L}^1(\Omega,\mathcal{F},P)$. *Assume that* $H_i(t,y,z,\mathbb{E}y,\mathbb{E}z,v_i,u_{3-i}(t),p_i(t))$ $(i=1,2)$ *and* $\Phi_i(y)$ *are convex in* $(y,z,\bar{y},\bar{z},v_i)$ *and* y, *respectively. Assume that*

$$\begin{aligned}
&\mathbb{E}\big[H_1\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_1(t)\big)|\mathcal{F}_t^1\big] \\
&\quad = \min_{v_1\in\mathcal{U}_1} \mathbb{E}\big[H_1\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),v_1,u_2(t),p_1(t)\big)|\mathcal{F}_t^1\big], \\
&\mathbb{E}\big[H_2\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_2(t)\big)|\mathcal{F}_t^2\big] \\
&\quad = \min_{v_2\in\mathcal{U}_2} \mathbb{E}\big[H_2\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),v_2,p_2(t)\big)|\mathcal{F}_t^2\big].
\end{aligned}$$

What is more, assume that $\mathbb{E}[H_{iv_i}(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),v_i,u_{3-i}(t),p_i(t))|\mathcal{F}_t^i]$ $(i=1,2)$ *is continuous at* $v_i = u_i(t)$ *for all* $t \in [0,T]$. *Then* $(u_1(\cdot),u_2(\cdot))$ *is a Nash equilibrium point of Problem* (MFBNZ).

Proof For any $v_1(\cdot) \in \mathcal{U}_1$, we consider

$$\begin{aligned}
&J_1\big(u_1(t),u_2(t)\big) - J_1\big(v_1(t),u_2(t)\big) \\
&\quad = \mathbb{E}\int_0^T \big[l_1\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t)\big) \\
&\qquad - l_1\big(t,y^{v_1}(t),z^{v_1}(t),\mathbb{E}y^{v_1}(t),\mathbb{E}z^{v_1}(t),v_1(t),u_2(t)\big)\big]\,dt \\
&\qquad + \mathbb{E}\big[\Phi_1\big(y(0)\big) - \Phi_1\big(y^{v_1}(0)\big)\big],
\end{aligned}$$

where $(y(\cdot),z(\cdot),\mathbb{E}y(\cdot),\mathbb{E}z(\cdot))$ and $(y^{v_1}(\cdot),z^{v_1}(\cdot),\mathbb{E}y^{v_1}(\cdot),\mathbb{E}z^{v_1}(\cdot))$ are the state trajectories corresponding to $(u_1(\cdot),u_2(\cdot))$ and $(v_1(\cdot),u_2(\cdot))$, respectively.

Let

$$\begin{aligned}
A_1 &= \mathbb{E}\int_0^T \big[l_1\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t)\big) \\
&\quad - l_1\big(t,y^{v_1}(t),z^{v_1}(t),\mathbb{E}y^{v_1}(t),\mathbb{E}z^{v_1}(t),v_1(t),u_2(t)\big)\big]\,dt, \\
A_2 &= \mathbb{E}\big[\Phi_1\big(y(0)\big) - \Phi_1\big(y^{v_1}(0)\big)\big].
\end{aligned}$$

Then we have

$$J_1\big(u_1(t),u_2(t)\big) - J_1\big(v_1(t),u_2(t)\big) = A_1 + A_2.$$

The integration A_1 is written as

$$\begin{aligned}
A_1 &= \mathbb{E}\int_0^T \big[H_1\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t),p_1(t)\big) \\
&\quad - H_1\big(t,y^{v_1}(t),z^{v_1}(t),\mathbb{E}y^{v_1}(t),\mathbb{E}z^{v_1}(t),v_1(t),u_2(t),p_1(t)\big)\big]\,dt \\
&\quad + \mathbb{E}\int_0^T \big\langle p_1(t), f\big(t,y(t),z(t),\mathbb{E}y(t),\mathbb{E}z(t),u_1(t),u_2(t)\big) \\
&\quad - f\big(t,y^{v_1}(t),z^{v_1}(t),\mathbb{E}y^{v_1}(t),\mathbb{E}z^{v_1}(t),v_1(t),u_2(t)\big)\big\rangle\,dt. \qquad (5)
\end{aligned}$$

Since $\Phi_1(y)$ is convex on y,

$$A_2 \le \mathbb{E}\big[-\Phi_{1y}^{\tau}\big(y(0)\big)\big(y^{v_1}(0) - y(0)\big)\big] = \mathbb{E}\big\langle p_1(0), y^{v_1}(0) - y(0)\big\rangle.$$

Applying Itô's formula to $\langle p_1(\cdot), y(\cdot) - y^{v_1}(\cdot)\rangle$, we get

$$\begin{aligned}
&\big\langle p_1(0), y^{v_1}(0) - y(0)\big\rangle \\
&\quad = -\int_0^T \big\langle p_1(t), f\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t)\big) \\
&\qquad - f\big(t, y^{v_1}(t), z^{v_1}(t), \mathbb{E}y^{v_1}(t), \mathbb{E}z^{v_1}(t), v_1(t), u_2(t)\big)\big\rangle\,dt \\
&\qquad - \int_0^T \big\langle y(t) - y^{v_1}(t), H_{1y}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\qquad + \mathbb{E}H_{1\bar{y}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\,dt \\
&\qquad - \int_0^T \big\langle z(t) - z^{v_1}(t), H_{1z}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\qquad + \mathbb{E}H_{1\bar{z}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\,dt \\
&\qquad - \int_0^T \big\langle y(t) - y^{v_1}(t), \big(H_{1z}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\qquad + \mathbb{E}H_{1\bar{z}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big)\,dW(t)\big\rangle \\
&\qquad + \int_0^T \big\langle p_1(t), \big(z(t) - z^{v_1}(t)\big)\,dW(t)\big\rangle.
\end{aligned}$$

Then,

$$\begin{aligned}
A_2 &\le \mathbb{E}\big[p_1^{\tau}(0)\big(y^{v_1}(0) - y(0)\big)\big] \\
&= -\mathbb{E}\int_0^T \big\langle p_1(t), f\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t)\big) \\
&\quad - f\big(t, y^{v_1}(t), z^{v_1}(t), \mathbb{E}y^{v_1}(t), \mathbb{E}z^{v_1}(t), v_1(t), u_2(t)\big)\big\rangle\,dt \\
&\quad - \mathbb{E}\int_0^T \big\langle y(t) - y^{v_1}(t), H_{1y}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\quad + \mathbb{E}H_{1\bar{y}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\,dt \\
&\quad - \mathbb{E}\int_0^T \big\langle z(t) - z^{v_1}(t), H_{1z}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\quad + \mathbb{E}H_{1\bar{z}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\,dt.
\end{aligned} \tag{6}$$

Combining (5) with (6), we have

$$\begin{aligned}
&J_1\big(u_1(t), u_2(t)\big) - J_1\big(v_1(t), u_2(t)\big) \\
&\quad \le \mathbb{E}\int_0^T \big[H_1\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&\qquad - H_1\big(t, y^{v_1}(t), z^{v_1}(t), \mathbb{E}y^{v_1}(t), \mathbb{E}z^{v_1}(t), v_1(t), u_2(t), p_1(t)\big)\big]\,dt
\end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}\int_0^T \big\langle y(t) - y^{v_1}(t), H_{1y}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&+ \mathbb{E}H_{1\bar{y}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\, dt \\
&- \mathbb{E}\int_0^T \big\langle z(t) - z^{v_1}(t), H_{1z}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big) \\
&+ \mathbb{E}H_{1\bar{z}}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big)\big\rangle\, dt. \qquad (7)
\end{aligned}
$$

Since $H_1(t, y, z, \mathbb{E}y, \mathbb{E}z, v_1, u_2(t), p_1(t))$ is convex in $(y, z, \bar{y}, \bar{z}, v_1)$, then (7) becomes

$$
\begin{aligned}
&J_1\big(u_1(t), u_2(t)\big) - J_1\big(v_1(t), u_2(t)\big) \\
&\quad \le \mathbb{E}\int_0^T \big[\big\langle H_{1v_1}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big), u_1(t) - v_1(t)\big\rangle\big]\, dt \\
&\quad \le \mathbb{E}\int_0^T \big[\mathbb{E}\big\langle H_{1v_1}\big(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), u_1(t), u_2(t), p_1(t)\big), u_1(t) - v_1(t)\big\rangle \big| \mathcal{F}_t^1\big]\, dt.
\end{aligned}
$$

Noticing that $v_1 \to \mathbb{E}[H_1(t, y(t), z(t), \mathbb{E}y(t), \mathbb{E}z(t), v_1, u_2(t), p_1(t)) | \mathcal{F}_t^1]$ can be the minimum at $v_1 = u_1(t)$, we have $J_1(u_1(t), u_2(t)) \le J_1(v_1(t), u_2(t))$ for any $v_1(\cdot) \in \mathcal{U}_1$. Then it implies that

$$
J_1\big(u_1(t), u_2(t)\big) = \min_{v_1(\cdot)\in\mathcal{U}_1} J_1\big(v_1(t), u_2(t)\big).
$$

Similarly, we have $J_2(u_1(t), u_2(t)) = \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(t), v_2(t))$.

Thus, we draw the desired conclusion. □

4 Mean-field FBSDE

In this section, we study the existence and uniqueness of solution to an initial coupled MF-FBSDE, which will be used in the rest of this paper.

Consider the MF-FBSDE

$$
\begin{cases}
dx(t) = f(\Pi(t))\, dt + \sigma_1(\Pi(t))\, dW_1(t) + \sigma_2(\Pi(t))\, dW_2(t), \\
-dy(t) = g(\Pi(t))\, dt - z_1(t)\, dW_1(t) - z_2(t)\, dW_2(t), \\
x(0) = \Psi(y(0)), \qquad y(T) = \xi,
\end{cases} \tag{8}
$$

where $\Pi(\cdot) = (\cdot, x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot), \mathbb{E}x(\cdot), \mathbb{E}y(\cdot), \mathbb{E}z_1(\cdot), \mathbb{E}z_2(\cdot))$; x, y, z_1, z_2 take values in $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m\times d}$, and $\mathbb{R}^{m\times d}$, respectively; $f, \sigma_1, \sigma_2, g, \Psi$ are functions with appropriate dimensions.

Let G be an $m \times n$ full-rank matrix and use the notations

$$
\lambda = (x, y, z_1, z_2)^\tau, \qquad \breve{\lambda} = (\breve{x}, \breve{y}, \breve{z}_1, \breve{z}_2)^\tau, \qquad A(t, \lambda, \breve{\lambda}) = \big(-G^\tau g, Gf, G\sigma_1, G\sigma_2\big)^\tau (t, \lambda, \breve{\lambda}).
$$

Definition 4.1 $(x, y, z_1, z_2) : \Omega \times [0, T] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$ is called an adapted solution of (8) if $(x, y, z_1, z_2) \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d})$ and satisfies (8).

Hypothesis 4

(1) *$A(t, \lambda, \breve{\lambda})$ is uniformly Lipschitz with respect to λ, $\breve{\lambda}$, and for each $\lambda, \breve{\lambda}$, $A(t, \lambda, \breve{\lambda})$ is in $\mathcal{L}^2_{\mathcal{F}}(0, T)$; $\Psi(x)$ is uniformly Lipschitz with respect to x.*

(2)

$$\begin{cases} \mathbb{E}\langle A(t,\lambda,\breve{\lambda}) - A(t,\bar{\lambda},\bar{\breve{\lambda}}), \lambda - \bar{\lambda}\rangle \\ \quad \le -\beta_1(\mathbb{E}|G\underline{x}|^2) - \beta_2(\mathbb{E}|G^\tau \underline{y}|^2 + \mathbb{E}|G^\tau \underline{z_1}|^2 + \mathbb{E}|G^\tau \underline{z_2}|^2), \\ \mathbb{E}\langle G(\Psi(y) - \Psi(\bar{y})), y - \bar{y}\rangle \le -\mu_2 \mathbb{E}|G^\tau \underline{y}|^2 \end{cases} \tag{9}$$

for all $\lambda = (x,y,z_1,z_2)$, $\bar{\lambda} = (\bar{x},\bar{y},\bar{z}_1,\bar{z}_2)$, $\breve{\lambda} = (\breve{x},\breve{y},\breve{z}_1,\breve{z}_2)$, $\bar{\breve{\lambda}} = (\bar{\breve{x}},\bar{\breve{y}},\bar{\breve{z}}_1,\bar{\breve{z}}_2)$, $(\underline{x},\underline{y},\underline{z_1},\underline{z_2}) = (x-\bar{x}, y-\bar{y}, z_1-\bar{z}_1, z_2-\bar{z}_2)$, *where* β_1,β_2,μ_2 *are given non-negative constants with* $\beta_1+\beta_2>0, \beta_1+\mu_2>0$. *What is more, we have* $\beta_1>0$ (*respectively,* $\beta_2>0,\mu_2>0$) *when* $m>n$ (*respectively,* $m<n$).

Theorem 4.1 *Assume that Hypothesis 4 holds. MF-FBSDE (8) admits a unique solution* (x,y,z_1,z_2).

Proof Similar to Yu and Ji [24] and Bensoussan et al. [25], we can prove this result. We omit the details for saving space. □

5 An MF-LQ problem

This section focuses on solving an LQ case of Problem (MFBNZ). Applying Theorem 3.1 and Theorem 3.2, we obtain an explicit form of Nash equilibrium point by optimal filters and Riccati equations. For simplicity, here we only deal with the case of one-dimensional Brownian motion.

Consider the linear MF-BSDE

$$\begin{cases} -dy^v(t) = [A(t)y^v(t) + C_2(t)z_2^v(t) + \bar{A}(t)\mathbb{E}y^v(t) \\ \qquad\qquad + \bar{C}_2(t)\mathbb{E}z_2^v(t) + B_1(t)v_1(t) + B_2(t)v_2(t)]\,dt \\ \qquad\qquad - z_1^v(t)\,dW_1(t) - z_2^v(t)\,dW_2(t), \\ y^v(T) = \xi, \end{cases} \tag{10}$$

and the cost functional

$$\begin{aligned} &J_i\big(v_1(\cdot),v_2(\cdot)\big) \\ &\quad = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[M_i(t)v_i^2(t) + N_i(t)\big(y^v(t)\big)^2 + \bar{N}_i(t)\big(\mathbb{E}y^v(t)\big)^2\right]dt + \gamma_{i0}\big(y^v(0)\big)^2\right\} \end{aligned} \tag{11}$$

$(i=1,2)$, where $v(\cdot) = (v_1(\cdot),v_2(\cdot))$; $A(\cdot),C_2(\cdot),\bar{A}(\cdot),\bar{C}_2(\cdot),B_i(\cdot)$ $(i=1,2)$ are deterministic, uniformly bounded functions; $\xi \in \mathcal{L}^2_{\mathcal{F}}(\Omega,\mathcal{F}_T,P;\mathbb{R})$; $v_1(\cdot)$ and $v_2(\cdot)$ are the control processes; $\bar{N}_i(\cdot)$ $(i=1,2)$ is deterministic, non-negative, and uniformly bounded function; $M_i(\cdot), N_i(\cdot)$, and γ_{i0} $(i=1,2)$ are deterministic, positive and uniformly bounded functions. Here, we require $N_i(\cdot)$ and γ_{i0} $(i=1,2)$ to be positive, which guarantees $m_i \neq 0$ $(i=1,2)$ and $k_3 \neq 0$ given by (13)–(14) and (42), respectively.

To what follows, we want to get an explicit form of Nash equilibrium point. Due to the fact that $\mathcal{F}_t^i$ available to Player i $(i=1,2)$ is only an abstract sub-filtration of $\mathcal{F}_t$, it is impossible to obtain a feedback Nash equilibrium point in general. So we mainly study three special information structures as follows: (1) $\mathcal{F}_t^1 = \mathcal{F}_t^2 = \sigma\{W_2(s); 0 \le s \le t\} = \mathcal{F}_t^{W_2}$, i.e., Player

1 and Player 2 have the same observation information; (2) $\mathcal{F}_t^1 = \sigma\{W_1(s), W_2(s); 0 \le s \le t\} = \mathcal{F}_t, \mathcal{F}_t^2 = \mathcal{F}_t^{W_2}$, i.e., Player 1 has more information than Player 2; (3) $\mathcal{F}_t^1 = \sigma\{W_1(s); 0 \le s \le t\} = \mathcal{F}_t^{W_1}, \mathcal{F}_t^2 = \mathcal{F}_t^{W_2}$, i.e., Player 1 and Player 2 possess the mutually independent information.

According to Theorem 3.1 and Theorem 3.2, $(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point of the MF-LQ problem if and only if

$$u_1(t) = M_1^{-1}(t)B_1(t)\mathbb{E}\big[p_1(t)|\mathcal{F}_t^1\big], \qquad u_2(t) = M_2^{-1}(t)B_2(t)\mathbb{E}\big[p_2(t)|\mathcal{F}_t^2\big],$$

where (y, z_1, z_2, p_1, p_2) satisfies

$$\begin{cases} -dy = (Ay + C_2 z_2 + \bar{A}\mathbb{E}y + \bar{C}_2\mathbb{E}z_2 + B_1^2 M_1^{-1}\mathbb{E}[p_1|\mathcal{F}_t^1] \\ \qquad\quad + B_2^2 M_2^{-1}\mathbb{E}[p_2|\mathcal{F}_t^2])\,dt - z_1\,dW_1 - z_2\,dW_2, \\ dp_1 = (Ap_1 + \bar{A}\mathbb{E}p_1 - N_1 y - \bar{N}_1\mathbb{E}y)\,dt + (C_2 p_1 + \bar{C}_2\mathbb{E}p_1)\,dW_2, \\ dp_2 = (Ap_2 + \bar{A}\mathbb{E}p_2 - N_2 y - \bar{N}_2\mathbb{E}y)\,dt + (C_2 p_2 + \bar{C}_2\mathbb{E}p_2)\,dW_2, \\ y(T) = \xi, \qquad p_1(0) = -\gamma_{10}y(0), \qquad p_2(0) = -\gamma_{20}y(0). \end{cases} \tag{12}$$

For the sake of simplicity, here we omit the time variable t in (12). Similar convention will be taken for the subsequent equations except for the initial or terminal conditions. In addition, we give Hypothesis 5 throughout Sect. 5.

Hypothesis 5 *$B_1^2(t)M_1^{-1}(t) = B_2^2(t)M_2^{-1}(t)$, and $B_i^2(t)M_i^{-1}(t)$ $(i = 1, 2)$ is independent of the time variable t.*

5.1 Symmetric information: $\mathcal{F}_t^1 = \mathcal{F}_t^2 = \mathcal{F}_t^{W_2}$

In this case, we denote $\mathbb{E}[p_1(t)|\mathcal{F}_t^1] = \mathbb{E}[p_1(t)|\mathcal{F}_t^{W_2}] = \hat{p}_1(t)$ and $\mathbb{E}[p_2(t)|\mathcal{F}_t^2] = \mathbb{E}[p_2(t)|\mathcal{F}_t^{W_2}] = \hat{p}_2(t)$.

To derive an explicit form of Nash equilibrium point, we first introduce two sets of ordinary differential equations (ODEs):

$$\begin{cases} \dot{\lambda}_1 - [B_1^2 M_1^{-1} + (C_2 + \bar{C}_2)^2 m_1^{-1}]\lambda_1^2 - B_2^2 M_2^{-1}\lambda_1\lambda_2 - [2(A + \bar{A}) + 2B_1^2 M_1^{-1} m_1 \\ \quad + B_2^2 M_2^{-1} m_2 + 2(C_2 + \bar{C}_2)^2]\lambda_1 - B_2^2 M_2^{-1} m_1 \lambda_2 \\ \quad + [\bar{N}_1 - (\bar{C}_2(2C_2 + \bar{C}_2) + 2\bar{A})m_1] = 0, \\ \dot{m}_1 - B_1^2 M_1^{-1} m_1^2 - B_2^2 M_2^{-1} m_1 m_2 - (C_2^2 + 2A)m_1 + N_1 = 0, \\ \dot{n}_1 - [B_1^2 M_1^{-1}(\lambda_1 + m_1) + (A + \bar{A}) + (C_2 + \bar{C}_2)^2(m_1^{-1}\lambda_1 + 1)]n_1 \\ \quad - B_2^2 M_2^{-1}(\lambda_1 + m_1)n_2 = 0, \\ \lambda_1(0) = 0, \qquad m_1(0) = -\gamma_{10}, \qquad n_1(0) = 0 \end{cases} \tag{13}$$

and

$$\begin{cases} \dot{\lambda}_2 - [B_2^2M_2^{-1} + (C_2+\bar{C}_2)^2 m_2^{-1}]\lambda_2^2 - B_1^2M_1^{-1}\lambda_1\lambda_2 - [2(A+\bar{A}) + 2B_2^2M_2^{-1}m_2 \\ \quad + B_1^2M_1^{-1}m_1 + 2(C_2+\bar{C}_2)^2]\lambda_2 - B_1^2M_1^{-1}m_2\lambda_1 \\ \quad + [\bar{N}_2 - (\bar{C}_2(2C_2+\bar{C}_2) + 2\bar{A})m_2] = 0, \\ \dot{m}_2 - B_2^2M_2^{-1}m_2^2 - B_1^2M_1^{-1}m_1m_2 - (C_2^2+2A)m_2 + N_2 = 0, \\ \dot{n}_2 - [B_2^2M_2^{-1}(\lambda_2+m_2) + (A+\bar{A}) + (C_2+\bar{C}_2)^2(m_2^{-1}\lambda_2+1)]n_2 \\ \quad - B_1^2M_1^{-1}(\lambda_2+m_2)n_1 = 0, \\ \lambda_2(0)=0, \qquad m_2(0) = -\gamma_{20}, \qquad n_2(0)=0. \end{cases} \tag{14}$$

Since (13) and (14) are coupled with each other, it is difficult to prove their existence and uniqueness except for some special cases. For example,

Hypothesis 6 $C_2(t) = -\bar{C}_2(t)$.

Lemma 5.1 *Under Hypotheses 5–6, there exists a unique solution* $(\lambda_1, m_1, n_1, \lambda_2, m_2, n_2)$ *to* (13) *and* (14).

Proof Noticing (13) and (14), we introduce

$$\begin{cases} \dot{m}_1 - B_1^2M_1^{-1}m_1^2 - B_2^2M_2^{-1}m_1m_2 - (C_2^2+2A)m_1 + N_1 = 0, \\ \dot{m}_2 - B_2^2M_2^{-1}m_2^2 - B_1^2M_1^{-1}m_1m_2 - (C_2^2+2A)m_2 + N_2 = 0, \\ m_1(0) = -\gamma_{10}, \qquad m_2(0) = -\gamma_{20}. \end{cases} \tag{15}$$

In what follows, we prove that (15) is uniquely solvable. Let $m = m_1 + m_2$. Under Hypothesis 5, we have

$$\dot{m} - B_1^2M_1^{-1}m^2 - \big(C_2^2+2A\big)m + (N_1+N_2) = 0, \qquad m(0) = -\gamma_{10} - \gamma_{20}. \tag{16}$$

Obviously, (16) is a standard Riccati equation, so it admits a unique solution $m(\cdot)$. Introduce two new ODEs:

$$\dot{\widetilde{m}}_1 - \big(C_2^2 + 2A + B_1^2M_1^{-1}m\big)\widetilde{m}_1 + N_1 = 0, \quad \widetilde{m}_1(0) = -\gamma_{10}, \tag{17}$$

$$\dot{\widetilde{m}}_2 - \big(C_2^2 + 2A + B_1^2M_1^{-1}m\big)\widetilde{m}_2 + N_2 = 0, \quad \widetilde{m}_2(0) = -\gamma_{20}, \tag{18}$$

where $m(\cdot)$ is the solution to (16). It is easy to see that (17) and (18) have unique solutions $\widetilde{m}_1$ and $\widetilde{m}_2$, respectively. Besides, we check that m_1 and m_2 in (15) are the solutions to (17) and (18), respectively. According to the existence and uniqueness of solutions to (17) and (18), we have

$$m_1 = \widetilde{m}_1, \qquad m_2 = \widetilde{m}_2, \tag{19}$$

which implies that (15) admits a unique solution (m_1, m_2).

Similarly, we introduce

$$\begin{cases} \dot{\lambda}_1 - B_1^2 M_1^{-1}\lambda_1^2 - B_2^2 M_2^{-1}\lambda_1\lambda_2 - [2(A+\bar{A}) + 2B_1^2 M_1^{-1} m_1 + B_2^2 M_2^{-1} m_2]\lambda_1 \\ \quad - B_2^2 M_2^{-1} m_1 \lambda_2 + [\bar{N}_1 - (C_2\bar{C}_2 + 2\bar{A})m_1] = 0, \\ \dot{\lambda}_2 - B_2^2 M_2^{-1}\lambda_2^2 - B_1^2 M_1^{-1}\lambda_1\lambda_2 - [2(A+\bar{A}) + 2B_2^2 M_2^{-1} m_2 + B_1^2 M_1^{-1} m_1]\lambda_2 \\ \quad - B_1^2 M_1^{-1} m_2 \lambda_1 + [\bar{N}_2 - (C_2\bar{C}_2 + 2\bar{A})m_2] = 0, \\ \lambda_1(0) = 0, \qquad \lambda_2(0) = 0 \end{cases} \tag{20}$$

with Hypothesis 5, where (m_1, m_2) is the solution to (15). Let $\lambda = \lambda_1 + \lambda_2$. Then we have

$$\begin{cases} \dot{\lambda} - B_1^2 M_1^{-1}\lambda^2 - 2[(A+\bar{A}) + B_1^2 M_1^{-1}(m_1 + m_2)]\lambda + [\bar{N}_1 - (C_2\bar{C}_2 + 2\bar{A})m_1] \\ \quad + [\bar{N}_2 - (C_2\bar{C}_2 + 2\bar{A})m_2] = 0, \\ \lambda(0) = 0. \end{cases} \tag{21}$$

Similar to (16), we can prove that (21) has a unique solution $\lambda(\cdot)$. Introduce two other ODEs:

$$\begin{cases} \dot{\tilde{\lambda}}_1 - [B_1^2 M_1^{-1}\lambda + 2(A+\bar{A}) + B_1^2 M_1^{-1}(m_1 + m_2)]\tilde{\lambda}_1 \\ \quad - B_1^2 M_1^{-1} m_1 \lambda + [\bar{N}_1 - (C_2\bar{C}_2 + 2\bar{A})m_1] = 0, \\ \tilde{\lambda}_1(0) = 0, \end{cases}$$

$$\begin{cases} \dot{\tilde{\lambda}}_2 - [B_1^2 M_1^{-1}\lambda + 2(A+\bar{A}) + B_1^2 M_1^{-1}(m_1 + m_2)]\tilde{\lambda}_2 \\ \quad - B_1^2 M_1^{-1} m_2 \lambda + [\bar{N}_2 - (C_2\bar{C}_2 + 2\bar{A})m_2] = 0, \\ \tilde{\lambda}_2(0) = 0, \end{cases}$$

where $\lambda(\cdot)$ is the solution to (21). Similar to (15), (20) admits a unique solution (λ_1, λ_2) satisfying

$$\lambda_1 = \tilde{\lambda}_1, \qquad \lambda_2 = \tilde{\lambda}_2.$$

Finally, we introduce

$$\begin{cases} \dot{n}_1 - [B_1^2 M_1^{-1}(\lambda_1 + m_1) + (A+\bar{A})]n_1 - B_2^2 M_2^{-1}(\lambda_1 + m_1)n_2 = 0, \\ \dot{n}_2 - [B_2^2 M_2^{-1}(\lambda_2 + m_2) + (A+\bar{A})]n_2 - B_1^2 M_1^{-1}(\lambda_2 + m_2)n_1 = 0, \\ n_1(0) = 0, \qquad n_2(0) = 0, \end{cases} \tag{22}$$

where λ_i, m_i $(i = 1, 2)$ are the solutions to (20) and (15), respectively. Besides, similar to (20) and (15), we know that (22) is uniquely solvable. For simplicity, we let $n = n_1 + n_2$.

Based on the arguments above, we get that (13) and (14) have unique solutions (λ_1, m_1, n_1) and (λ_2, m_2, n_2), respectively. □

In the following, we will use five steps to give the explicit form of Nash equilibrium point.

Step 1: The unexplicit form of Nash equilibrium point.

$(u_1(\cdot), u_2(\cdot))$ is the Nash equilibrium point of the MF-LQ problem if and only if it is uniquely determined by

$$u_1(t) = M_1^{-1}(t)B_1(t)\hat{p}_1(t), \qquad u_2(t) = M_2^{-1}(t)B_2(t)\hat{p}_2(t),$$

where (y, z_1, z_2, p_1, p_2) is the solution of

$$\begin{cases} -dy = (Ay + C_2 z_2 + \bar{A}\mathbb{E}y + \bar{C}_2\mathbb{E}z_2 + B_1^2 M_1^{-1}\hat{p}_1 + B_2^2 M_2^{-1}\hat{p}_2)\,dt \\ \qquad - z_1\,dW_1 - z_2\,dW_2, \\ dp_1 = (Ap_1 + \bar{A}\mathbb{E}p_1 - N_1 y - \bar{N}_1\mathbb{E}y)\,dt + (C_2 p_1 + \bar{C}_2\mathbb{E}p_1)\,dW_2, \\ dp_2 = (Ap_2 + \bar{A}\mathbb{E}p_2 - N_2 y - \bar{N}_2\mathbb{E}y)\,dt + (C_2 p_2 + \bar{C}_2\mathbb{E}p_2)\,dW_2, \\ y(T) = \xi, \qquad p_1(0) = -\gamma_{10}y(0), \qquad p_2(0) = -\gamma_{20}y(0). \end{cases} \tag{23}$$

Noticing that since (23) contains the conditional expectation of $p_i(\cdot)$ $(i = 1, 2)$ with respect to $\mathcal{F}_t^{W_2}$, we call it a conditional MF-FBSDE.

Step 2: Filtering equation.

Since (23) contains $\hat{p}_i(\cdot)$ $(i = 1, 2)$, we need to compute the optimal filter $(\hat{y}, \hat{z}_2, \hat{p}_1, \hat{p}_2)$ of (y, z_2, p_1, p_2) with respect to $\mathcal{F}_t^{W_2}$. Applying Lemma 5.4 in Xiong [26], we have

$$\begin{cases} -d\hat{y} = (A\hat{y} + C_2\hat{z}_2 + \bar{A}\mathbb{E}y + \bar{C}_2\mathbb{E}z_2 + B_1^2 M_1^{-1}\hat{p}_1 + B_2^2 M_2^{-1}\hat{p}_2)\,dt - \hat{z}_2\,dW_2, \\ d\hat{p}_1 = (A\hat{p}_1 + \bar{A}\mathbb{E}p_1 - N_1\hat{y} - \bar{N}_1\mathbb{E}y)\,dt + (C_2\hat{p}_1 + \bar{C}_2\mathbb{E}p_1)\,dW_2, \\ d\hat{p}_2 = (A\hat{p}_2 + \bar{A}\mathbb{E}p_2 - N_2\hat{y} - \bar{N}_2\mathbb{E}y)\,dt + (C_2\hat{p}_2 + \bar{C}_2\mathbb{E}p_2)\,dW_2, \\ \hat{y}(T) = \hat{\xi}, \qquad \hat{p}_1(0) = -\gamma_{10}\hat{y}(0), \qquad \hat{p}_2(0) = -\gamma_{20}\hat{y}(0). \end{cases} \tag{24}$$

Note that this filtering equation is different from the case introduced in Chap. 2 of Wang et al. [27], whose existence and uniqueness need to be proved below.

Step 3: Existence and uniqueness of (24).

Introduce a new MF-FBSDE

$$\begin{cases} -dY = (AY + C_2 Z_2 + \bar{A}\mathbb{E}Y + \bar{C}_2\mathbb{E}Z_2 + P)\,dt - Z_2\,dW_2, \\ dP = [AP + \bar{A}\mathbb{E}P - (B_1^2 M_1^{-1}N_1 + B_2^2 M_2^{-1}N_2)Y \\ \qquad - (B_1^2 M_1^{-1}\bar{N}_1 + B_2^2 M_2^{-1}\bar{N}_2)\mathbb{E}Y]\,dt \\ \qquad + (C_2 P + \bar{C}_2\mathbb{E}P)\,dW_2, \\ Y(T) = \hat{\xi}, \qquad P(0) = -[B_1^2(0)M_1^{-1}(0)\gamma_{10} + B_2^2(0)M_2^{-1}(0)\gamma_{20}]Y(0). \end{cases} \tag{25}$$

With the help of Hypothesis 5, it is easy to check that Hypothesis 4 holds. Then Theorem 4.1 implies that (25) is uniquely solvable.

Now we intend to prove that the existence and uniqueness of (25) are equivalent to those of (24). On the one hand, we prove that the solution of (24) is the solution of (25). In fact, the conclusion is easily drawn with the assumption

$$Y = \hat{y}, \qquad Z_2 = \hat{z}_2, \qquad P = B_1^2 M_1^{-1}\hat{p}_1 + B_2^2 M_2^{-1}\hat{p}_2.$$

On the other hand, we prove that the solution of (25) is the solution of (24). Let (Y, Z_2, P) be a solution of (25), and set

$$\hat{y} = Y, \qquad \hat{z}_2 = Z_2. \tag{26}$$

It follows from the existence and uniqueness of mean-field stochastic differential equation that $\hat{p}_i$ satisfies

$$\begin{cases} d\hat{p}_i = (A\hat{p}_i + \bar{A}\mathbb{E}p_i - N_iY - \bar{N}_i\mathbb{E}Y)\,dt + (C_2\hat{p}_i + \bar{C}_2\mathbb{E}p_i)\,dW_2, \\ \hat{p}_i(0) = -\gamma_{i0}Y(0) \quad (i = 1,2). \end{cases}$$

In order to say $(Y, Z_2, \hat{p}_i)$ $(i = 1,2)$ is a solution of (24), we only check

$$P = B_1^2M_1^{-1}\hat{p}_1 + B_2^2M_2^{-1}\hat{p}_2. \tag{27}$$

Letting $\bar{P} = B_1^2M_1^{-1}\hat{p}_1 + B_2^2M_2^{-1}\hat{p}_2$, we have

$$\begin{cases} d\bar{P} = [A\bar{P} + \bar{A}\mathbb{E}\bar{P} - (B_1^2M_1^{-1}N_1 + B_2^2M_2^{-1}N_2)Y - (B_1^2M_1^{-1}\bar{N}_1 + B_2^2M_2^{-1}\bar{N}_2)\mathbb{E}Y]\,dt \\ \qquad + (C_2\bar{P} + \bar{C}_2\mathbb{E}\bar{P})\,dW_2, \\ \bar{P}(0) = -[B_1^2(0)M_1^{-1}(0)\gamma_{10} + B_2^2(0)M_2^{-1}(0)\gamma_{20}]Y(0). \end{cases}$$

Fixing Y, we derive $P = \bar{P}$, and then (27) holds indeed. Hence, the existence and uniqueness of (25) are equivalent to those of (24).

Step 4: Existence and uniqueness of (23).

Fixing $\hat{p}_1$ and $\hat{p}_2$ in (23), we can easily prove that (23) admits a unique solution (y, z_1, z_2, p_1, p_2).

Step 5: The feedback Nash equilibrium point.

According to the first equation of (23) together with the terminal condition in (23), we set

$$p_i = \lambda_i\mathbb{E}y + m_iy + n_i, \qquad \lambda_i(0) = 0, \qquad m_i(0) = -\gamma_{i0}, \qquad n_i(0) = 0 \quad (i = 1,2). \tag{28}$$

Applying Itô's formula to p_1 in (28), we have

$$\begin{aligned} dp_1 = {} & \big\{\big[\dot{\lambda}_1 - (A + \bar{A})\lambda_1 - B_1^2M_1^{-1}\big(\lambda_1^2 + 2\lambda_1m_1\big) - B_2^2M_2^{-1}(\lambda_1\lambda_2 + \lambda_1m_2) \\ & - \bar{A}m_1 - B_2^2M_2^{-1}m_1\lambda_2\big]\mathbb{E}y \\ & + (\dot{m}_1 - Am_1)y - \big(B_1^2M_1^{-1}m_1^2 + B_2^2M_2^{-1}m_1m_2\big)\hat{y} + \big[\dot{n}_1 - \big(B_1^2M_1^{-1}n_1 + B_2^2M_2^{-1}n_2\big)\lambda_1 \\ & - \big(B_1^2M_1^{-1}n_1 + B_2^2M_2^{-1}n_2\big)m_1\big] - \big[(C_2 + \bar{C}_2)\lambda_1 + \bar{C}_2m_1\big]\mathbb{E}z_2 - C_2m_1z_2\big\}\,dt \\ & + m_1z_1\,dW_1 + m_1z_2\,dW_2. \end{aligned} \tag{29}$$

Putting (28) into the second equation of (23), and comparing the coefficients of (29) and (23), we get

$$m_1z_2 = (C_2\lambda_1 + \bar{C}_2\lambda_1 + \bar{C}_2m_1)\mathbb{E}y + C_2m_1y + (C_2 + \bar{C}_2)n_1, \tag{30}$$

$$\begin{aligned}
&\Big\{\big[\dot{\lambda}_1 - (A+\bar{A})\lambda_1 - B_1^2M_1^{-1}\big(\lambda_1^2 + 2\lambda_1 m_1\big) - B_2^2M_2^{-1}(\lambda_1\lambda_2 + \lambda_1 m_2) \\
&\quad - \bar{A}m_1 - B_2^2M_2^{-1}m_1\lambda_2\big]\mathbb{E}y \\
&\quad + (\dot{m}_1 - Am_1)y - \big(B_1^2M_1^{-1}m_1^2 + B_2^2M_2^{-1}m_1m_2\big)\hat{y} + \big[\dot{n}_1 - \big(B_1^2M_1^{-1}n_1 + B_2^2M_2^{-1}n_2\big)\lambda_1 \\
&\quad - \big(B_1^2M_1^{-1}n_1 + B_2^2M_2^{-1}n_2\big)m_1\big] - \big[(C_2+\bar{C}_2)\lambda_1 + \bar{C}_2m_1\big]\mathbb{E}z_2 - C_2m_1z_2\Big\} \\
&= \big[A\lambda_1 + \bar{A}(\lambda_1+m_1) - \bar{N}_1\big]\mathbb{E}y + (Am_1 - N_1)y + (A+\bar{A})n_1.
\end{aligned} \tag{31}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^{W_2}]$ on both sides of (28) and (30), we have

$$\hat{p}_i = \lambda_i\mathbb{E}y + m_i\hat{y} + n_i \quad (i=1,2), \tag{32}$$

$$m_1\hat{z}_2 = (C_2\lambda_1 + \bar{C}_2\lambda_1 + \bar{C}_2m_1)\mathbb{E}y + C_2m_1\hat{y} + (C_2+\bar{C}_2)n_1. \tag{33}$$

Here we assume $m_1 \neq 0$. Substituting (30) into (31) and taking $\mathbb{E}[\cdot|\mathcal{F}_t^{W_2}]$ on both sides of (31), it becomes

$$\begin{aligned}
&\Big\{\dot{\lambda}_1 - \big[B_1^2M_1^{-1} + (C_2+\bar{C}_2)^2m_1^{-1}\big]\lambda_1^2 - B_2^2M_2^{-1}\lambda_1\lambda_2 - \big[2(A+\bar{A}) + 2B_1^2M_1^{-1}m_1 \\
&\quad + B_2^2M_2^{-1}m_2 + 2(C_2+\bar{C}_2)^2\big]\lambda_1 - B_2^2M_2^{-1}m_1\lambda_2 + \big[\bar{N}_1 - \big(\bar{C}_2(2C_2+\bar{C}_2) + 2\bar{A}\big)m_1\big]\Big\}\mathbb{E}y \\
&\quad + \big[\dot{m}_1 - B_1^2M_1^{-1}m_1^2 - B_2^2M_2^{-1}m_1m_2 - \big(C_2^2 + 2A\big)m_1 + N_1\big]\hat{y} + \Big\{\dot{n}_1 - \big[B_1^2M_1^{-1}(\lambda_1+m_1) \\
&\quad + (A+\bar{A}) + (C_2+\bar{C}_2)^2\big(m_1^{-1}\lambda_1 + 1\big)\big]n_1 - B_2^2M_2^{-1}(\lambda_1+m_1)n_2\Big\} = 0.
\end{aligned} \tag{34}$$

From (34), we derive (13). Similarly, (14) is derived by applying Itô's formula to p_2.

To close this subsection, we give the explicit form of $\hat{y}(t)$. Putting (30) into the first equation of (23) and taking $\mathbb{E}[\cdot]$, we have

$$\begin{cases} \frac{d\mathbb{E}y(t)}{dt} + q_1(t)\mathbb{E}y(t) = -q_2(t), \\ \mathbb{E}y(T) = \mathbb{E}\xi \end{cases} \tag{35}$$

with

$$q_1(t) = A(t) + \bar{A}(t) + B_1^2(t)M_1^{-1}(t)\big(\lambda(t) + m(t)\big), \qquad q_2(t) = B_1^2(t)M_1^{-1}(t)n(t), \tag{36}$$

where m, λ and $n = n_1 + n_2$ are represented by (16), (21), and (22), respectively. Solving (35), we get its unique solution

$$\mathbb{E}y(t) = e^{\int_t^T q_1(r)\,dr}\mathbb{E}\xi + \int_t^T q_2(s)e^{\int_t^s q_1(r)\,dr}\,ds. \tag{37}$$

Set

$$q_3(t) = A(t) + B_1^2(t)M_1^{-1}(t)m(t), \qquad q_4(t) = \big[\bar{A}(t) + B_1^2(t)M_1^{-1}(t)\lambda(t)\big]\mathbb{E}y(t) + q_2(t). \tag{38}$$

Then the first equation of (24) is written as

$$\begin{cases} -d\hat{y}(t) = [q_3(t)\hat{y}(t) + C_2(t)\hat{z}_2(t) + q_4(t)]\,dt - \hat{z}_2(t)\,dW_2(t), \\ \hat{y}(T) = \hat{\xi}, \end{cases} \tag{39}$$

whose unique solution is

$$\hat{y}(t) = \mathbb{E}\left[\hat{\xi} X_T + \int_t^T q_4(s) X_s \, ds \middle| \mathcal{F}_t^{W_2}\right], \tag{40}$$

where $X_s = \exp\{\int_t^s [q_3(r) - \frac{1}{2}C_2^2(r)]\, dr + \int_t^s C_2(r)\, dW_2(r)\}$.

Theorem 5.1 *Under Hypotheses 5–6, the feedback Nash equilibrium point $(u_1(t), u_2(t))$ of the MF-LQ problem is uniquely denoted by*

$$\begin{cases} u_1(t) = M_1^{-1}(t)B_1(t)(\lambda_1(t)\mathbb{E}y(t) + m_1(t)\hat{y}(t) + n_1(t)), \\ u_2(t) = M_2^{-1}(t)B_2(t)(\lambda_2(t)\mathbb{E}y(t) + m_2(t)\hat{y}(t) + n_2(t)), \end{cases}$$

where λ_i, m_i, n_i $(i = 1, 2)$, $\mathbb{E}y$ and $\hat{y}$ are given by (13), (14), (37), and (40), respectively.

5.2 Asymmetric information

Here we solve two asymmetric information structures introduced above. The corresponding derivation procedures are similar to those of Sect. 5.1, so we omit the nonessential details and only give the main results.

5.2.1 $\mathcal{F}_t^1 = \mathcal{F}_t$ and $\mathcal{F}_t^2 = \mathcal{F}_t^{W_2}$

In this case, $\mathbb{E}[p_1(t)|\mathcal{F}_t^1] = \mathbb{E}[p_1(t)|\mathcal{F}_t] = p_1(t)$, $\mathbb{E}[p_1(t)|\mathcal{F}_t^2] = \mathbb{E}[p_1(t)|\mathcal{F}_t^{W_2}] = \hat{p}_2(t)$.

Theorem 5.2 *Let Hypotheses 5–6 hold. Then the feedback Nash equilibrium point $(u_1(t), u_2(t))$ of the MF-LQ problem is uniquely determined by*

$$\begin{cases} u_1(t) = M_1^{-1}(t)B_1(t)(k_1(t)\mathbb{E}y(t) + k_2(t)\hat{y}(t) + k_3(t)y(t) + k_4(t)), \\ u_2(t) = M_2^{-1}(t)B_2(t)(\lambda_2(t)\mathbb{E}y(t) + m_2(t)\hat{y}(t) + n_2(t)), \end{cases} \tag{41}$$

where λ_2, m_2, n_2, $\mathbb{E}y$, and $\hat{y}$ are the unique solutions to (14), (37), and (40), respectively; k_i $(i = 1, 2, 3, 4)$ and y satisfy

$$\begin{cases} \dot{k}_1 - (A + \bar{A} + q_1 + B_1^2 M_1^{-1} k_3)k_1 - (2\bar{A} + C_2\bar{C}_2 + B_1^2 M_1^{-1}\lambda)k_2 \\ \quad - (2\bar{A} + B_2^2 M_2^{-1}\lambda_2 + C_2\bar{C}_2)k_3 + \bar{N}_1 = 0, \\ \dot{k}_2 - (q_3 + B_1^2 M_1^{-1} k_3 + C_2^2 + A)k_2 - B_2^2 M_2^{-1} m_2 k_3 = 0, \\ \dot{k}_3 - (2A + C_2^2)k_3 - B_1^2 M_1^{-1} k_3^2 + N_1 = 0, \\ \dot{k}_4 - (B_1^2 M_1^{-1} k_3 + A + \bar{A})k_4 - q_2(k_1 + k_2) - B_2^2 M_2^{-1} n_2 k_3 = 0, \\ k_1(0) = 0, \qquad k_2(0) = 0, \qquad k_3(0) = -\gamma_{10}, \qquad k_4(0) = 0 \end{cases} \tag{42}$$

and

$$y(t) = \mathbb{E}\left[\xi \Sigma_T + \int_t^T q_6(s) \Sigma_s \, ds \middle| \mathcal{F}_t\right]$$

with q_1, q_2, q_3 *are represented by* (36) *and* (38), *respectively, and*

$$\begin{cases} \Sigma_s = \exp\{\int_t^s [q_5(r) - \frac{1}{2}C_2^2(r)]\,dr + \int_t^s C_2(r)\,dW_2(r)\}, \\ q_5(t) = A(t) + B_1^2(t)M_1^{-1}(t)k_3(t), \\ q_6(t) = [\bar{A}(t) + B_1^2(t)M_1^{-1}(t)k_1(t) + B_2^2(t)M_2^{-1}(t)\lambda_2(t)]\mathbb{E}y(t) \\ \qquad + [B_1^2(t)M_1^{-1}(t)k_2(t) + B_2^2(t)M_2^{-1}(t)m_2(t)]\hat{y}(t) \\ \qquad + B_1^2(t)M_1^{-1}(t)k_4(t) + B_2^2(t)M_2^{-1}(t)n_2(t), \end{cases}$$

respectively.

Feedback Nash equilibrium point (41) shows that although Player 1 observes full information, the control strategy of Player 1 is heavily influenced by the information available to Player 2 via $\mathbb{E}y$ and $\hat{y}$. This is an interesting phenomenon.

5.2.2 $\mathcal{F}_t^1 = \mathcal{F}_t^{W_1}$ and $\mathcal{F}_t^2 = \mathcal{F}_t^{W_2}$

Hypothesis 7 $C_2 = \bar{C}_2 = 0$.

Hypothesis 7 guarantees that the filtering equation of (12) with respect to $\mathcal{F}_t^{W_1}$ is uniquely solvable.

Theorem 5.3 *Assume that Hypothesis 5 and Hypothesis 7 hold. Then the MF-LQ problem has a unique Nash equilibrium point* $(u_1(t), u_2(t))$ *represented by*

$$\begin{cases} u_1(t) = M_1^{-1}(t)B_1(t)(w_1(t)\mathbb{E}y(t) + w_2(t)\tilde{y}(t) + w_3(t)), \\ u_2(t) = M_2^{-1}(t)B_2(t)(\rho_1(t)\mathbb{E}y(t) + \rho_2(t)\hat{y}(t) + \rho_3(t)), \end{cases}$$

where $\tilde{y}(t) = \mathbb{E}[y(t)|\mathcal{F}_t^{W_1}]$, w_i, ρ_i $(i = 1, 2, 3)$ *are given by*

$$\begin{cases} \dot{w}_1 - [2A + \bar{A} + B_1^2M_1^{-1}(d_1 + d_2) + B_1^2M_1^{-1}w_2]w_1 \\ \quad + [\bar{N}_1 - (\bar{A} + B_2^2M_2^{-1}d_2)w_2 - \bar{A}d_1] = 0, \\ \dot{w}_2 - B_1^2M_1^{-1}w_2^2 - 2Aw_2 + N_1 = 0, \\ \dot{w}_3 - (B_1^2M_1^{-1}w_2 + A)w_3 - [B_1^2M_1^{-1}(l_1 + l_2)w_1 + B_2^2M_2^{-1}l_2w_2 + \bar{A}l_1] = 0, \\ w_1(0) = 0, \qquad w_2(0) = -\gamma_{10}, \qquad w_3(0) = 0 \end{cases}$$

and

$$\begin{cases} \dot{\rho}_1 - [2A + \bar{A} + B_1^2M_1^{-1}(d_1 + d_2) + B_2^2M_2^{-1}\rho_2]\rho_1 + [\bar{N}_2 - (\bar{A} + B_2^2M_2^{-1}d_1)\rho_2 - \bar{A}d_2] = 0, \\ \dot{\rho}_2 - B_2^2M_2^{-1}\rho_2^2 - 2A\rho_2 + N_2 = 0, \\ \dot{\rho}_3 - (B_2^2M_2^{-1}\rho_2 + A)\rho_3 - [B_1^2M_1^{-1}(l_1 + l_2)\rho_1 + B_1^2M_1^{-1}l_1\rho_2 + \bar{A}l_2] = 0, \\ \rho_1(0) = 0, \qquad \rho_2(0) = -\gamma_{20}, \qquad \rho_3(0) = 0 \end{cases}$$

with d_i, l_i $(i = 1, 2)$ *satisfying*

$$\begin{cases} \dot{d}_1 - B_1^2 M_1^{-1} d_1^2 - B_2^2 M_2^{-1} d_1 d_2 - 2(A + \bar{A}) d_1 + (N_1 + \bar{N}_1) = 0, \\ \dot{l}_1 - B_1^2 M_1^{-1} d_1 l_1 - B_2^2 M_2^{-1} d_1 l_2 - (A + \bar{A}) l_1 = 0, \\ d_1(0) = -\gamma_{10}, \qquad l_1(0) = 0 \end{cases}$$

and

$$\begin{cases} \dot{d}_2 - B_2^2 M_2^{-1} d_2^2 - B_1^2 M_1^{-1} d_1 d_2 - 2(A + \bar{A}) d_2 + (N_2 + \bar{N}_2) = 0, \\ \dot{l}_2 - B_1^2 M_1^{-1} d_2 l_1 - B_2^2 M_2^{-1} d_2 l_2 - (A + \bar{A}) l_2 = 0, \\ d_2(0) = -\gamma_{20}, \qquad l_2(0) = 0, \end{cases}$$

respectively; $\mathbb{E}y$, $\tilde{y}$ *and* $\hat{y}$ *are given by*

$$\begin{aligned} \mathbb{E}y(t) &= e^{\int_t^T q_8(r)\,dr} \mathbb{E}\xi + \int_t^T q_7(s) e^{\int_t^s q_8(r)\,dr}\, ds, \\ \tilde{y}(t) &= \Xi_T \mathbb{E}\bigl[\xi \,|\, \mathcal{F}_t^{W_1}\bigr] + \int_t^T q_9(s) \Xi_s \, ds, \\ \hat{y}(t) &= \Theta_T \mathbb{E}\bigl[\xi \,|\, \mathcal{F}_t^{W_2}\bigr] + \int_t^T q_{10}(s) \Theta_s \, ds \end{aligned}$$

with

$$\begin{aligned} q_7(t) &= B_1^2(t) M_1^{-1}(t) \bigl(l_1(t) + l_2(t)\bigr), \\ q_8(t) &= A(t) + \bar{A}(t) + B_1^2(t) M_1^{-1}(t) \bigl(d_1(t) + d_2(t)\bigr), \\ \Xi_s &= \exp\biggl\{ \int_t^s \bigl[A(r) + B_1^2(r) M_1^{-1}(r) w_2(r)\bigr]\, dr \biggr\}, \\ q_9(t) &= \bigl[\bar{A}(t) + B_1^2(t) M_1^{-1}(t) w_1(t) + B_2^2(t) M_2^{-1}(t)\, d_2(t)\bigr] \mathbb{E}y(t) \\ &\quad + B_1^2(t) M_1^{-1}(t) w_3(t) + B_2^2(t) M_2^{-1}(t) l_2(t), \\ \Theta_s &= \exp\biggl\{ \int_t^s \bigl[A(r) + B_2^2(r) M_2^{-1}(r) \rho_2(r)\bigr]\, dr \biggr\}, \\ q_{10}(t) &= \bigl[\bar{A}(t) + B_1^2(t) M_1^{-1}(t)\, d_1(t) + B_2^2(t) M_2^{-1}(t) \rho_1(t)\bigr] \mathbb{E}y(t) \\ &\quad + B_1^2(t) M_1^{-1}(t) l_1(t) + B_2^2(t) M_2^{-1}(t) \rho_3(t), \end{aligned}$$

respectively.

6 Application in a financial problem

In this section, we consider an investment problem in a financial market. With the help of Theorem 3.1 and Theorem 3.2, an explicit form of optimal investment strategy is obtained.

We begin with a typical setup for the financial market, in which a bond and two stocks are continuously traded, and their prices satisfy

$$\begin{cases} dB(t) = r(t)B(t)\,dt, \\ dS_1(t) = S_1(t)[\mu_1(t)\,dt + \sigma_1(t)\,dW_1(t)], \\ dS_2(t) = S_2(t)[\mu_2(t)\,dt + \sigma_2(t)\,dW_2(t)], \end{cases}$$

where $r(\cdot)$ is called the interest rate of the bond; $\mu_i(\cdot),\sigma_i(\cdot)$ $(i = 1,2)$ are called the appreciation rate of return and volatility coefficient of the ith stock.

Hypothesis 8 *The market coefficients* $r(\cdot)$, $\mu_i(\cdot),\sigma_i(\cdot)$ $(i = 1,2)$ *are deterministic and bounded processes. What is more,* $\sigma_i^{-1}(\cdot)$ $(i = 1,2)$ *is also bounded.*

Suppose that there are two investors cooperating to invest a bond and two stocks, whose decision cannot influence the prices in the financial market. Furthermore, we assume that Investor 1 only cares about the price of the first stock, i.e., $\mathcal{F}_t^1 = \sigma\{S_1(s); 0 \le s \le t\} = \mathcal{F}_t^{S_1}$; however, Investor 2 cares about the prices of these two stocks, i.e., $\mathcal{F}_t^2 = \sigma\{S_1(s), S_2(s); 0 \le s \le t\} = \mathcal{F}_t^{S_1,S_2}$. Clearly, $\mathcal{F}_t^{S_1} = \sigma\{W_1(s); 0 \le s \le t\}$ and $\mathcal{F}_t^{S_1,S_2} = \sigma\{W_1(s), W_2(s); 0 \le s \le t\}$.

Assume that these two investors want to obtain a terminal wealth ξ at T, which is an $\mathcal{F}_T$-measurable, non-negative, and square-integrable random variable. Meanwhile, both of them hope to minimize their own risks, described by J_i $(i = 1,2)$. In detail, we denote by $\pi_i(\cdot)$ the amount that the investors invest in the ith $(i = 1,2)$ stock and by $y(\cdot)$ the wealth of the two investors. Then, $y(\cdot)$ is modeled by

$$\begin{cases} dy(t) = [r(t)y(t) + b_1(t)z_1(t) + b_2(t)z_2(t) + \nu_1(t) + \nu_2(t)]\,dt \\ \qquad\qquad + z_1(t)\,dW_1(t) + z_2(t)\,dW_2(t), \\ y(T) = \xi, \end{cases} \tag{43}$$

where $b_i(\cdot) = (\mu_i(\cdot) - r(\cdot))\sigma_i^{-1}(\cdot)$ $(i = 1,2)$; $z_i(\cdot) = \pi_i(\cdot)\sigma_i(\cdot)$ $(i = 1,2)$; $\nu_1(\cdot)$ and $\nu_2(\cdot)$ are certain economic factors, which can be interpreted as capital injection or withdrawal.

Define the associated performance functional for each investor as follows:

$$J_i\big(\nu_1(\cdot),\nu_2(\cdot)\big) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[\big(y(t) - \mathbb{E}y(t)\big)^2 + \big(\nu_i(t) - BM_i(t)\big)^2\right] dt + \Phi_i y^2(0)\right\} \tag{44}$$

$(i = 1,2)$, where $BM_i(\cdot)$ is a deterministic benchmark; Φ_i $(i = 1,2)$ is a positive constant. In the performance functional, the first term measures the variance of the wealth $y(\cdot)$; the second term measures the difference between the economic factor $\nu_i(\cdot)$ and benchmark $BM_i(\cdot)$ $(i = 1,2)$ decided by the two investors; the last term in the performance functional characterizes the initial wealth $y(\cdot)$ at time 0.

Let U be a nonempty and convex subset of $\mathbb{R}$. Define the admissible control set

$$\mathcal{U}_i = \left\{\nu_i(\cdot) \in \mathcal{L}^2_{\mathcal{F}_t^i}(0,T;\mathbb{R}) | \nu_i(t) \in U, t \in [0,T]\right\} \quad (i = 1,2).$$

Then the investment problem is stated as follows.

Problem (F) Find a pair of $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)), \end{cases}$$

subject to (43) and (44). If such a pair of (u_1, u_2) exists, we call it an optimal investment strategy of Problem (F).

Clearly, Problem (F) can be regarded as a non-zero sum mean-field backward stochastic differential game with asymmetric information, which is a special case of Problem (MFBNZ).

Here, we point out that Problem (F) is different from the MF-LQ problem, due to two distinguishing features below: firstly, the generator of the first equation of (47) has an additional term $b_2\check{z}_2$, which leads to a difficulty of proving the existence and uniqueness of solution to it; secondly, (44) contains the first terms of $y(\cdot)$ and $v_i(\cdot)$, and then Theorem 5.2 cannot be used to solve Problem (F).

We firstly study a special case that $\mathcal{F}_t^1 = \mathcal{F}_t^2 = \mathcal{F}_t^{S_1}$, which plays an important role in solving the asymmetric information case.

6.1 Symmetric information: $\mathcal{F}_t^1 = \mathcal{F}_t^2 = \mathcal{F}_t^{S_1}$

Let $\check{g}(\cdot) = \mathbb{E}[g(\cdot)|\mathcal{F}_t^{S_1}]$. We have $\mathbb{E}[p_1(t)|\mathcal{F}_t^1] = \mathbb{E}[p_1(t)|\mathcal{F}_t^{S_1}] = \check{p}_1(t)$ and $\mathbb{E}[p_2(t)|\mathcal{F}_t^2] = \mathbb{E}[p_1(t)|\mathcal{F}_t^{S_1}] = \check{p}_2(t)$. In the following, we use four steps to solve this case.

Step 1: Optimal investment strategy.

The optimal investment strategy (u_1, u_2) of Problem (F) has the form of

$$u_1(t) = BM_1(t) - \check{p}_1(t), \qquad u_2(t) = BM_2(t) - \check{p}_2(t), \tag{45}$$

where (y, z_1, z_2, p_1, p_2) satisfies

$$\begin{cases} dy = (ry + b_1z_1 + b_2z_2 + u_1 + u_2)\,dt + z_1\,dW_1 + z_2\,dW_2, \\ dp_1 = (\mathbb{E}y - y - rp_1)\,dt - b_1p_1\,dW_1 - b_2p_1\,dW_2, \\ dp_2 = (\mathbb{E}y - y - rp_2)\,dt - b_1p_2\,dW_1 - b_2p_2\,dW_2, \\ y(T) = \xi, \qquad p_1(0) = -\Phi_1 y(0), \qquad p_2(0) = -\Phi_2 y(0). \end{cases} \tag{46}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^{S_1}]$ on both sides of each equation of (46) yields

$$\begin{cases} d\check{y} = (r\check{y} + b_1\check{z}_1 + b_2\check{z}_2 + u_1 + u_2)\,dt + \check{z}_1\,dW_1, \\ d\check{p}_1 = (\mathbb{E}y - \check{y} - r\check{p}_1)\,dt - b_1\check{p}_1\,dW_1, \\ d\check{p}_2 = (\mathbb{E}y - \check{y} - r\check{p}_2)\,dt - b_1\check{p}_2\,dW_1, \\ \check{y}(T) = \check{\xi}, \qquad \check{p}_1(0) = -\Phi_1\check{y}(0), \qquad \check{p}_2(0) = -\Phi_2\check{y}(0). \end{cases} \tag{47}$$

Step 2: Existence and uniqueness of (50).

Introduce

$$\begin{cases} \dot{\gamma}_1 - \gamma_1^2 + (2r - b_1^2 - b_2^2)\gamma_1 - \gamma_1\gamma_2 = 0, \\ \dot{\eta}_1 + (r - b_1^2 - b_2^2 - \gamma_1)\eta_1 - \gamma_1\eta_2 + (BM_1 + BM_2)\gamma_1 = 0, \\ \gamma_1(0) = -\Phi_1, \qquad \eta_1(0) = 0, \end{cases} \tag{48}$$

$$\begin{cases} \dot{\gamma}_2 - \gamma_2^2 + (2r - b_1^2 - b_2^2)\gamma_2 - \gamma_1\gamma_2 = 0, \\ \dot{\eta}_2 + (r - b_1^2 - b_2^2 - \gamma_2)\eta_2 - \gamma_2\eta_1 + (BM_1 + BM_2)\gamma_2 = 0, \\ \gamma_2(0) = -\Phi_2, \qquad \eta_2(0) = 0. \end{cases} \tag{49}$$

Similar to Lemma 5.1, we can prove that (48) and (49) are uniquely solvable. What is more, it follows from $\Phi_1 > 0$ that the solution $\gamma_1(\cdot) < 0$. For convenience, let $\gamma = \gamma_1 + \gamma_2$, $\eta = \eta_1 + \eta_2$.

Introduce an auxiliary MF-FBSDE

$$\begin{cases} d\check{y} = [r\check{y} + b_1\check{z}_1 + BM_1 + BM_2 - (1 + \gamma_1^{-1}b_2^2)\check{p}_1 - \check{p}_2]\,dt + \check{z}_1\,dW_1, \\ d\check{p}_1 = (\mathbb{E}y - \check{y} - r\check{p}_1)\,dt - b_1\check{p}_1\,dW_1, \\ d\check{p}_2 = (\mathbb{E}y - \check{y} - r\check{p}_2)\,dt - b_1\check{p}_2\,dW_1, \\ \check{y}(T) = \check{\xi}, \qquad \check{p}_1(0) = -\Phi_1\check{y}(0), \qquad \check{p}_2(0) = -\Phi_2\check{y}(0), \end{cases} \tag{50}$$

which is subject to an additional hypothesis as follows.

Hypothesis 9 $1 + \gamma_1^{-1}(t)b_2^2(t) \geq 0$.

According to Hypothesis 4, (50) has a unique solution $(\check{y}, \check{z}_1, \check{p}_1, \check{p}_2)$.

Step 3: The equivalence between (47) *and* (50) *with* (45).

We first prove that the solution $(\check{y}, \check{z}_1, \check{p}_1, \check{p}_2)$ of (50) satisfies (47). If $u_i(t) = BM_i(t) - \check{p}_i(t)$ $(i = 1, 2)$, then (46) is uniquely solvable. Set

$$p_i = \gamma_i y + \eta_i, \qquad \gamma_i(0) = -\Phi_i, \qquad \eta_i(0) = 0 \quad (i = 1, 2). \tag{51}$$

Applying Itô's formula to p_1 in (51), we get

$$\begin{aligned} dp_1 = {} & \big[(\dot{\gamma}_1 + r\gamma_1)y + \gamma_1(b_1z_1 + b_2z_2) - \gamma_1(\gamma_1 + \gamma_2)\check{y} + \dot{\eta}_1 \\ & + \gamma_1(BM_1 + BM_2 - \eta_1 - \eta_2)\big]\,dt \\ & + \gamma_1z_1\,dW_1 + \gamma_1z_2\,dW_2. \end{aligned} \tag{52}$$

Comparing (52) with the second equation of (46), we have

$$\gamma_1 z_i = -b_i p_1 \quad (i = 1, 2), \tag{53}$$

$$\begin{aligned} & \big[(\dot{\gamma}_1 + r\gamma_1)y + \gamma_1(b_1z_1 + b_2z_2) - \gamma_1(\gamma_1 + \gamma_2)\check{y} + \dot{\eta}_1 + \gamma_1(BM_1 + BM_2 - \eta_1 - \eta_2)\big] \\ & \quad = \big[\mathbb{E}y - (1 + r\gamma_1)y - r\eta_1\big]. \end{aligned} \tag{54}$$

Putting (53) into (54) subject to (51), and taking $\mathbb{E}[\cdot]$ on both sides of (54), we arrive at

$$\big[\dot{\gamma}_1 - \gamma_1^2 + \big(2r - b_1^2 - b_2^2\big)\gamma_1 - \gamma_1\gamma_2\big]\mathbb{E}y + \big[\dot{\eta}_1 + \big(r - b_1^2 - b_2^2 - \gamma_1\big)\eta_1 - \gamma_1\eta_2$$

$$+ (BM_1 + BM_2)\gamma_1\big] = 0, \tag{55}$$

which implies (48). Similarly, applying Itô's formula to p_2 in (51), we derive (49).

In addition, taking $\mathbb{E}[\cdot|\mathcal{F}_t^{S_1}]$ on both sides of (51) and (53), we get

$$\check{p}_i = \gamma_i \check{y} + \eta_i, \tag{56}$$

$$\check{z}_i = -\gamma_1^{-1} b_i \check{p}_1, \tag{57}$$

$(i = 1, 2)$. Putting (57) into (50), it is easy to see that $(\check{y}, \check{z}_1, \check{z}_2, \check{p}_1, \check{p}_2)$ solves (47).

Next, with u_1 and u_2 fixed, we prove that the solution $(\check{y}, \check{z}_1, \check{z}_2, \check{p}_1, \check{p}_2)$ of (47) is a solution of (50). Take $u_i = BM_i - \check{p}_i$ $(i = 1, 2)$. Then (y, z_1, z_2, p_1, p_2) is the unique solution to (46). Substituting $\check{z}_i = -\gamma_1^{-1} b_i \check{p}_1$ and $u_i = BM_i - \check{p}_i$ $(i = 1, 2)$ into (47), we arrive at (50), which implies that $(\check{y}, \check{z}_1, \check{p}_1, \check{p}_2)$ is a solution of (50).

Based on the analysis above, we know that the existence and uniqueness of (47) are equivalent to those of (50).

Step 4: The explicit form of optimal investment strategy.

Due to (56) and (57), the first equation of (47) is written as

$$\begin{cases} d\check{y}(t) = [f_1(t)\check{y}(t) + b_1(t)\check{z}_1(t) + f_2(t)]\,dt + \check{z}_1(t)\,dW_1(t), \\ \check{y}(T) = \check{\xi}, \end{cases} \tag{58}$$

where

$$\begin{cases} f_1(t) = r(t) - \gamma(t) - b_2^2(t), \\ f_2(t) = BM_1(t) + BM_2(t) - \eta(t) - \gamma_1^{-1}(t) b_2^2(t)\eta_1(t). \end{cases} \tag{59}$$

Solving (58), we get

$$\check{y}(t) = \mathbb{E}\left[\check{\xi}\Pi_T - \int_t^T f_2(s)\Pi_s\,ds \middle| \mathcal{F}_t^{S_1}\right] \tag{60}$$

with

$$\Pi_s = \exp\left\{\int_t^s -\left[f_1(r) + \frac{1}{2}b_1^2(r)\right]dr - \int_t^s b_1(r)\,dW_1(r)\right\}.$$

What is more, the expectation of $\check{y}(t)$ represented by (60) is

$$\mathbb{E}y(t) = e^{\int_t^T [b_1^2(r) - f_1(r)]\,dr}\mathbb{E}\xi + \int_t^T \big[\gamma_1^{-1}(t)b_1^2(t)\eta_1(t) - f_2(t)\big] e^{\int_t^s [b_1^2(r) - f_1(r)]\,dr}\,ds. \tag{61}$$

Theorem 6.1 *Under Hypotheses 8–9, the optimal investment strategy of Problem (F) is denoted by*

$$\begin{cases} u_1(t) = BM_1(t) - (\gamma_1(t)\check{y}(t) + \eta_1(t)), \\ u_2(t) = BM_2(t) - (\gamma_2(t)\check{y}(t) + \eta_2(t)), \end{cases}$$

where γ_i, η_i $(i = 1, 2)$ and $\check{y}$ are given by (48), (49), and (60), respectively.

6.2 Asymmetric information: $\mathcal{F}_t^1 = \mathcal{F}_t^{S_1}, \mathcal{F}_t^2 = \mathcal{F}_t^{S_1,S_2}$

In this case, we have $\mathbb{E}[p_1(t)|\mathcal{F}_t^1] = \mathbb{E}[p_1(t)|\mathcal{F}_t^{S_1}] = \check{p}_1(t)$ and $\mathbb{E}[p_2(t)|\mathcal{F}_t^2] = \mathbb{E}[p_2(t)|\mathcal{F}_t^{S_1,S_2}] = p_2(t)$. Based on Theorem 6.1, we start to solve Problem (F) by three steps.

Step 1: Optimal investment strategy.

The optimal investment strategy of Problem (F) is

$$u_1(t) = BM_1(t) - \check{p}_1(t), \qquad u_2(t) = BM_2(t) - p_2(t),$$

where (y, z_1, z_2, p_1, p_2) is the unique solution of

$$\begin{cases} dy = (ry + b_1 z_1 + b_2 z_2 + BM_1 + BM_2 - \check{p}_1 - p_2)\,dt + z_1\,dW_1 + z_2\,dW_2, \\ dp_1 = (\mathbb{E}y - y - rp_1)\,dt - b_1 p_1\,dW_1 - b_2 p_1\,dW_2, \\ dp_2 = (\mathbb{E}y - y - rp_2)\,dt - b_1 p_2\,dW_1 - b_2 p_2\,dW_2, \\ y(T) = \xi, \qquad p_1(0) = -\Phi_1 y(0), \qquad p_2(0) = -\Phi_2 y(0). \end{cases} \tag{62}$$

It is easy to check that the optimal filter $(\check{y}, \check{z}_1, \check{z}_2, \check{p}_1, \check{p}_2)$ of (y, z_1, z_2, p_1, p_2) in (62) still satisfies (47), and then $\check{y}$ and $\check{p}_1$ are represented by (60) and (56), respectively.

Step 2: Existence and uniqueness of (63).

The first equation and the third equation of (62) are written as

$$\begin{cases} dy = (ry + b_1 z_1 + b_2 z_2 + BM_1 + BM_2 - \gamma_1 \check{y} - \eta_1 - p_2)\,dt + z_1\,dW_1 + z_2\,dW_2, \\ dp_2 = (\mathbb{E}y - y - rp_2)\,dt - b_1 p_2\,dW_1 - b_2 p_2\,dW_2, \\ y(T) = \xi, \qquad p_2(0) = -\Phi_2 y(0). \end{cases} \tag{63}$$

It follows from Theorem 4.1 that (63) is uniquely solvable.

Step 3: The explicit form of optimal investment strategy.

In order to obtain the feedback optimal investment strategy, we have to establish the relationship between p_2 and $y, \check{y}$. Noticing the first equation of (63) together with the terminal condition in (63), we set

$$p_2 = \delta_1 y + \delta_2 \check{y} + \delta_3, \qquad \delta_1(0) = -\Phi_2, \qquad \delta_2(0) = \delta_3(0) = 0. \tag{64}$$

Applying Itô's formula to p_2 in (64), we have

$$\begin{aligned} dp_2 = {} & \big[\big(\dot{\delta}_1 + r\delta_1 - \delta_1^2\big)y + \big(\dot{\delta}_2 - \gamma_1\delta_1 - \delta_1\delta_2 + \big(f_1 - b_1^2\big)\delta_2\big)\check{y} + \delta_1(b_1 z_1 + b_2 z_2) \\ & + \dot{\delta}_3 + (BM_1 + BM_2 - \eta_1)\delta_1 - \delta_1\delta_3 + \big(f_2 - \gamma_1^{-1}\eta_1 b_1^2\big)\delta_2\big]\,dt \\ & + \big(\delta_1 z_1 - \big(\check{y} + \gamma_1^{-1}\eta_1\big)b_1\delta_2\big)\,dW_1 + \delta_1 z_2\,dW_2, \end{aligned} \tag{65}$$

where f_1 and f_2 are given by (59). Comparing (65) with the second equation of (63), we obtain

$$\begin{cases} \delta_1 z_1 - b_1\delta_2\check{y} - b_1\gamma_1^{-1}\delta_2\eta_1 = -b_1(\delta_1 y + \delta_2\check{y} + \delta_3), \\ \delta_1 z_2 = -b_2(\delta_1 y + \delta_2\check{y} + \delta_3), \end{cases} \tag{66}$$

$$\big[\big(\dot{\delta}_1 + r\delta_1 - \delta_1^2\big)y + \big(\dot{\delta}_2 - \gamma_1\delta_1 - \delta_1\delta_2 + \big(f_1 - b_1^2\big)\delta_2\big)\check{y} + \delta_1(b_1 z_1 + b_2 z_2) + \dot{\delta}_3$$

$$
\begin{aligned}
&+ (BM_1 + BM_2 - \eta_1)\delta_1 - \delta_1\delta_3 + \bigl(f_2 - \gamma_1^{-1}\eta_1 b_1^2\bigr)\delta_2\bigr] \\
&= \bigl[\mathbb{E}y - (1 + r\delta_1)y - r\delta_2\check{y} - r\delta_3\bigr].
\end{aligned} \tag{67}
$$

Substituting (66) into (67), we obtain

$$
\begin{aligned}
&\bigl[\dot{\delta}_1 + \bigl(2r - b_1^2 - b_2^2\bigr)\delta_1 - \delta_1^2 + 1\bigr]y + \bigl[\dot{\delta}_2 + \bigl(r + f_1 - \delta_1 - b_1^2 - b_2^2\bigr)\delta_2 - \gamma_1\delta_1\bigr]\check{y} \\
&\quad + \bigl[\dot{\delta}_3 + \bigl(r - b_1^2 - b_2^2 - \delta_1\bigr)\delta_3 + (BM_1 + BM_2 - \eta_1)\delta_1 + \delta_2 f_2 - \mathbb{E}y\bigr] = 0,
\end{aligned} \tag{68}
$$

where $\mathbb{E}y$ is given by (61). (68) implies that

$$
\begin{cases}
\dot{\delta}_1 + (2r - b_1^2 - b_2^2)\delta_1 - \delta_1^2 + 1 = 0, \\
\dot{\delta}_2 + (r + f_1 - \delta_1 - b_1^2 - b_2^2)\delta_2 - \gamma_1\delta_1 = 0, \\
\dot{\delta}_3 + (r - b_1^2 - b_2^2 - \delta_1)\delta_3 + (BM_1 + BM_2 - \eta_1)\delta_1 + \delta_2 f_2 - \mathbb{E}y = 0, \\
\delta_1(0) = -\Phi_2, \qquad \delta_2(0) = 0, \qquad \delta_3(0) = 0,
\end{cases} \tag{69}
$$

which has a unique solution $(\delta_1, \delta_2, \delta_3)$.

Due to (64), the first equation of (63) is written as

$$
\begin{cases}
dy(t) = [(r(t) - \delta_1(t))y(t) + b_1(t)z_1(t) + b_2(t)z_2(t) + f_3(t)]\,dt \\
\qquad\qquad + z_1(t)\,dW_1(t) + z_2(t)\,dW_2(t), \\
y(T) = \xi,
\end{cases} \tag{70}
$$

where $f_3(t) = BM_1(t) + BM_2(t) - \eta_1(t) - \delta_3(t) - (\gamma_1(t) + \delta_2(t))\check{y}(t)$. Solving (70), we get its unique solution

$$
y(t) = \mathbb{E}\biggl[\xi\Lambda_T - \int_t^T f_3(s)\Lambda_s\,ds \biggm| \mathcal{F}_t^{S_1,S_2}\biggr], \tag{71}
$$

where $\Lambda_s = \exp\{\int_t^s[-r(\omega) + \delta_1(\omega) - \frac{1}{2}b_1^2(\omega) - \frac{1}{2}b_2^2(\omega)]\,d\omega - \int_t^s b_1(\omega)\,dW_1(\omega) - \int_t^s b_2(\omega)\,dW_2(\omega)\}$.

Theorem 6.2 *Under Hypotheses 8–9, the optimal investment strategy of Problem (F) is uniquely given by*

$$
\begin{cases}
u_1(t) = BM_1(t) - (\gamma_1(t)\check{y}(t) + \eta_1(t)), \\
u_2(t) = BM_2(t) - (\delta_1(t)y(t) + \delta_2(t)\check{y}(t) + \delta_3(t)),
\end{cases}
$$

where γ_1, η_1, and $\check{y}$ are given by (48) *and* (60), *respectively*; $\delta_1, \delta_2, \delta_3$, *and y are given by* (69) *and* (71), *respectively.*

7 Conclusion and outlook

In this paper, a necessary condition and a sufficient condition for Nash equilibrium point of MF-BSDE under asymmetric information are derived, which are used to solve an MF-LQ problem and a financial problem. Some explicit Nash equilibrium points and optimal investment strategies are obtained. The results obtained here extend the first two authors' previous works of [12, 14], and [28].

The results in Sects. 5–6 are based on special information structures, which are generated by Brown motion or its components. The case that the information structures are general is not considered. We hope to return to it in a future work. Besides, we assume that all coefficients of the MF-LQ problem and Problem (F) are deterministic. Otherwise, there is an immediate difficulty to solve the problems with stochastic coefficients. We will come back to the problems in the future.

Acknowledgements
Not applicable.

Funding
This work is supported in part by the NSF of China under Grants 61821004, 61633015, and 11831010, by the Young Chang Jiang Scholars Program of Chinese Education Ministry, and by the Cultivation Program of Distinguished Young Scholars of Shandong University under Grant 2017JQ06.

Abbreviations
MF-BSDE, mean-field backward stochastic differential equation; MF-LQ, mean-field linear-quadratic; SDEs, stochastic differential equations; MF-FBSDE, mean-field forward and backward stochastic differential equation; ODEs, ordinary differential equations.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no conflict of interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Authors' information
Not applicable.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 August 2018 Accepted: 3 June 2019 Published online: 17 June 2019

References

1. Lasry, J.M., Lions, P.L.: Mean-field games. Jpn. J. Math. **2**, 229–260 (2007)
2. Huang, M., Malhamé, R.P., Caines, P.E.: Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. Commun. Inf. Syst. **6**(3), 221–252 (2006)
3. Bensoussan, A., Sung, K.C.J., Yam, S.C.P., Yung, S.P.: Linear-quadratic mean field games. J. Optim. Theory Appl. **169**, 496–529 (2016)
4. Øksendal, B., Sulem, A.: Optimal control of predictive mean-field equations and applications to finance. In: Stochastics of Environmental and Financial Economics. Springer Proceedings in Mathematics and Statistics, vol. 138, pp. 301–320. Springer, Cham (2016)
5. Wu, J., Liu, Z.: Maximum principle for mean-field zero-sum stochastic differential game with partial information and its application to finance. Eur. J. Control **37**, 8–15 (2017)
6. Hafayed, M., Abbas, S., Abba, A.: On mean-field partial information maximum principle of optimal control for stochastic systems with Lévy processes. J. Optim. Theory Appl. **167**, 1051–1069 (2015)
7. Huang, J., Wang, S.: Dynamic optimization of large-population systems with partial information. J. Optim. Theory Appl. **168**, 231–245 (2016)
8. Pardoux, E., Peng, S.: Adapted solution of a backward stochastic differential equation. Syst. Control Lett. **14**, 55–61 (1990)
9. Shen, X., Jiang, L.: BSDEs with monotone generator driven by time-changed Lévy noises. Adv. Differ. Equ., **2017**, 214 (2017)
10. Mu, R., Wu, Z.: One kind of multiple dimensional Markovian BSDEs with stochastic linear growth generators. Adv. Differ. Equ. **2015**, 265 (2015)
11. Hamadène, S., Lepeltier, J.P.: Zero-sum stochastic differential games and backward equations. Syst. Control Lett. **24**, 259–263 (1995)
12. Wang, G., Yu, Z.: A partial information non-zero sum differential game of backward stochastic differential equations with applications. Automatica **48**, 342–352 (2012)
13. Wang, G., Xiao, H., Xiong, J.: A kind of linear quadratic non-zero sum differential game of backward stochastic differential equation with asymmetric information. Automatica **97**, 346–352 (2018)

14. Wang, G., Yu, Z.: A Pontryagin's maximum principle for non-zero sum differential games of BSDEs with applications. IEEE Trans. Autom. Control **55**(7), 1742–1747 (2010)
15. Li, N., Yu, Z.: Recursive stochastic linear-quadratic optimal control and nonzero-sum differential game problems with random jumps. Adv. Differ. Equ. **2015**, 144 (2015)
16. Buckdahn, R., Djehiche, B., Li, J., Peng, S.: Mean-field backward stochastic differential equations: a limit approach. Ann. Probab. **37**(4), 1524–1565 (2009)
17. Ma, H., Liu, B.: Infinite horizon optimal control problem of mean-field backward stochastic delay differential equation under partial information. Eur. J. Control **36**, 43–50 (2017)
18. Li, X., Sun, J., Xiong, J.: Linear quadratic optimal control problems for mean-field backward stochastic differential equations. Appl. Math. Optim. (2017). https://doi.org/10.1007/s00245-017-9464-7
19. Wu, J., Liu, Z.: Maximum principle for mean-field zero-sum stochastic differential game with partial information and its application to finance. Eur. J. Control **37**, 8–15 (2017)
20. Lin, Y., Jiang, X., Zhang, W.: Open-loop Stackelberg strategy for the linear quadratic mean-field stochastic differential game. IEEE Trans. Autom. Control **64**(1), 97–110 (2019)
21. Du, K., Wu, Z.: Linear-quadratic Stackelberg game for mean-field backward stochastic differential system and application. Math. Probl. Eng. (2019). https://doi.org/10.1155/2019/1798585
22. Zhang, H.: An optimal control problem for linear SDE of mean-field type with terminal constraint and partial information. Adv. Differ. Equ. **2019**, 160 (2019)
23. Peng, S.: Backward stochastic differential equations and applications to optimal control. Appl. Math. Optim. **27**, 125–144 (1993)
24. Yu, Z., Ji, S.: Linear-quadratic nonzero-sum differential game of backward stochastic differential equations. In: Proc. 27th Chinese Control Conf, pp. 562–566 (2008)
25. Bensoussan, A., Yam, S.C.P., Zhang, Z.: Well-posedness of mean-field type forward-backward stochastic differential equations. Stoch. Process. Appl. **125**, 3327–3354 (2015)
26. Xiong, J.: An Introduction to Stochastic Filtering Theory. Oxford University Press, London (2008)
27. Wang, G., Wu, Z., Xiong, J.: An Introduction to Optimal Control of FBSDE with Incomplete Information. Springer, Berlin (2018)
28. Huang, P., Wang, G.: A non-zero sum differential game of mean-field backward stochastic differential equation. In: Proc. 2017 Chinese Automation Congress, pp. 4827–4831 (2017)