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Local fractional homotopy analysis method for solving non-differentiable problems on Cantor sets

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Abstract

In this paper, we present a semi-analytic method called the local fractional homotopy analysis method (LFHAM) for solving differential equations involving local fractional derivatives based on the local fractional calculus and the homotopy analysis method. The suggested analytical technique always provides a simple way of constructing a series of solutions from the higher-order deformation equation. The LFHAM guarantees the convergence of the series solutions using the nonzero convergence-control parameter. Three examples are provided to illustrate the efficiency and high accuracy of the method.

MSC: 34K50; 34A12; 34A30; 34D20

Keywords: Local fractional homotopy analysis method; Problems involving local fractional derivatives; Numeric and symbolic computations

1 Introduction

Many real-life problems are modeled to linear and nonlinear partial differential equations in most cases. However, solving these equations in closed form is very difficult, more especially the nonlinear models. In recent years, many mathematicians and engineers have devoted considerable time to develop efficient and stable techniques for solving nonlinear models including the non-differentiable problems which arise naturally in mathematical physics and engineering [1–37]. In mathematical literature, the most commonly used fractional derivative operators are the Caputo and Riemann–Liouville fractional derivatives [38, 39]. However, these derivatives have a kernel with singularity which limits their applicability to many real-world problems [40]. To overcome the problem of the singularity of the kernel, a new fractional derivative with exponential kernel was introduced by Caputo and Fabrizio in 2015 [41]. Unfortunately, the kernel of Caputo–Fabrizio fractional derivative was not non-local and the associated integral was not a fractional operator [42]. To solve the problem of the Caputo–Fabrizio fractional derivative, Atangana and Baleanu introduced an efficient fractional derivative operator with non-local and non-singular kernel called the Atangana–Baleanu fractional derivative (AB) in Caputo and Riemann–Liouville sense in 2016 (see [43]). Due to the non-local behavior of the AB fractional derivative, it has been used to develop some powerful mathematical methods for solving real-world problems such as the fractional Laplace decomposition method [44], the fractional homotopy perturbation transform method [45, 46], the fractional Adams-Bashforth method [47], and the fractional homotopy analysis transform method [48–51] to mention a few. Besides, all the recently proposed techniques have nothing to do with the existence of any small/large physical parameter. So, in real-world problems, these methods can be applied much more widely than the well-known analytical method called the perturbation technique which entirely relies on the existence of small/large parameter [52] (perturbation quantity), since not all mathematical models contain the so-called perturbation quantity. Other semi-analytical methods, such as the local fractional Adomian decomposition method [53, 54], the local fractional homotopy perturbation method [55, 56], the local fractional variational iteration method [57–60], the local fractional Sumudu decomposition method [61], the local fractional natural homotopy perturbation method [62], and many more, have been proposed and successfully applied to various linear and nonlinear models. However, most of these existing techniques cannot guarantee the convergence of the series solutions, hence are not suitable for solving highly nonlinear problems. To overcome the limitations of the current methods, a Chinese mathematician Liao proposed a semi-analytic method called the homotopy analysis method (HAM) for solving highly nonlinear models [63]. The homotopy analysis method is a combination of the classical perturbation technique [64–67] and the homotopy, a concept in topology, and does not rely on the small/large parameter. The advantage of the homotopy analysis method over the existing techniques is the excellent freedom of choosing the initial guess and the existence of the so-called nonzero convergence-control parameter. Based on the basic idea of the homotopy analysis method, many numerical and analytical techniques have been proposed. Marinca and Herisa suggested the optimal homotopy analysis method [68] in 2008. In 2009, Niu and Wang introduced a one-step optimal homotopy analysis method [69], and the spectral homotopy analysis method based on the Chebyshev pseudospectral method [70] was proposed by Motsa et al. in 2010. The predictor homotopy analysis method [71] was also suggested in 2010, and recently in 2018 Singh et al. successfully applied the homotopy analysis method and the Sumudu transform method to fractional Drinfeld–Sokolov–Wilson equation [72]. Besides, many authors have discovered that the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), and the variational iteration method (VIM) are all special cases of the homotopy analysis method (HAM) when the nonzero convergence-control parameter $\hbar = -1$ (see [73–77]).

Motivated by the ongoing research in the literature, in this paper we introduce an iterative method called the local fractional homotopy analysis method (LFHAM) for solving non-differentiable problems arising in fractal media. The LFHAM gives a series of solutions which converge rapidly within a few terms with the help of the nonzero convergence-control parameter. Some applications are given to verify the efficiency and stability of the method. In Table 1, some useful results in fractal space are presented.

The remaining sections of this work are organized as follows. In Sect. 2, some background notations of local fractional calculus are presented. In Sect. 3, the local analysis and convergence of local fractional homotopy analysis method are discussed. Applications of LFHAM are shown in Sect. 4. The conclusion of this paper is given in Sect. 5.

Table 1 Some useful identities of the local fractional calculus are give below

	Basic identities
1	$\cos_\alpha(t^\alpha) = \sum_{n=0}^{+\infty}(-1)^n \frac{t^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}, 0 < \alpha \le 1$
2	$\sin_\alpha(t^\alpha) = \sum_{n=0}^{+\infty}(-1)^n \frac{t^{2n\alpha}}{\Gamma(1+(2n+1)\alpha)}, 0 < \alpha \le 1$
3	$E_\alpha(t^\alpha) = \sum_{n=0}^{+\infty} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, 0 < \alpha \le 1$
4	$\frac{d^\alpha}{dt^\alpha} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{t^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$
5	${}_0I_t^{(\alpha)} \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{t^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}$
6	$\frac{d^\alpha}{dt^\alpha} \cos_\alpha(t^\alpha) = -\sin_\alpha(t^\alpha)$
7	$\frac{d^\alpha}{dt^\alpha} \sin_\alpha(t^\alpha) = \cos_\alpha(t^\alpha)$
8	$\frac{d^\alpha}{dt^\alpha} E_\alpha(t^\alpha) = E_\alpha(t^\alpha)$

2 Preliminaries and notations of local fractional calculus

Definition 1 Let $\wp : \Im \to \aleph$ be a function defined on a fractal set $\Im$ of fractal dimension α say $(0 < \alpha < 1)$. Then a real-valued function $\wp(t)$ on the fractal set $\Im$ is defined as [14, 15]

$$\wp(t) = t^\alpha, \tag{1}$$

where $t^\alpha \in \Im$ and $0 < \alpha < 1$.

Lemma 1 *Let F be a fractal and a subset of the real line. If $\nu : (F, d) \to (\Omega', d')$ is a bi-Lipschitz mapping, then there are constants $\rho, \tau > 0$, and $F \subset \mathbb{R}$,*

$$\rho^s H^s(F) \le H^s\big(\nu(F)\big) \le \tau^s H^s(F), \tag{2}$$

such that for all $t_1, t_2 \in F$,

$$\rho^\alpha |t_1 - t_2|^\alpha \le \big|\nu(t_1) - \nu(t_2)\big| \le \tau^\alpha |t_1 - t_2|^\alpha. \tag{3}$$

As a direct consequence of Lemma 2.1 [15, 19], we deduce

$$\big|\nu(t_1) - \nu(t_2)\big| \le \tau^\alpha |t_1 - t_2|^\alpha \tag{4}$$

such that

$$\big|\nu(t_1) - \nu(t_2)\big| < \varepsilon^\alpha, \tag{5}$$

where α denotes the fractal dimension of the set F. Besides, in fractal geometry the result is related to the fractal coarse-grained mass function $\gamma^\alpha[F, \beta_1, \beta_2]$ as

$$\gamma^\alpha[F, \beta_1, \beta_2] = \frac{H^\alpha(F \cap (\beta_1, \beta_2)}{\Gamma(\alpha + 1)}, \tag{6}$$

with

$$H^\alpha\big(F \cap (\beta_1, \beta_2)\big) = (\beta_2 - \beta_1), \tag{7}$$

where H^α denotes the α-dimensional Hausdorff measure.

Definition 2 Suppose that there exists [14, 15]

$$\left|\nu(t)-\nu(t_0)\right| < \varepsilon^{\alpha}, \tag{8}$$

with $|t-t_0| < \delta$, for $\delta, \varepsilon > 0$ and $\delta, \varepsilon \in \mathbb{R}$. Then the function $\nu(t)$ is called local fractional continuous at $t = t_0$ and is denoted by $\lim_{t\to t_0} \nu(t_0)$. Equivalently, the function $\nu(t)$ is called local fractional continuous function on the interval (β_1, β_2) and is denoted by

$$\nu(t) \in C_{\alpha}(\beta_1, \beta_2), \tag{9}$$

provided Eq. (8) is valid for $t \in (\beta_1, \beta_2)$.

Definition 3 The local fractional derivative of the function $\nu(t)$ of order α at $t = t_0$ is defined as follows [14, 15]:

$$\nu^{(\alpha)}(t) = \left.\frac{d^{\alpha}\nu}{dt^{\alpha}}\right|_{t=t_0} = \lim_{t\to t_0} \frac{\Delta^{\alpha}(\nu(t)-\nu(t_0))}{(t-t_0)^{\alpha}}, \tag{10}$$

where

$$\Delta^{\alpha}\big(\nu(t)-\nu(t_0)\big) \cong \Gamma(1+\alpha)\big[\nu(t)-\nu(t_0)\big]. \tag{11}$$

For any $t \in (\beta_1, \beta_2)$, there exists [14, 15]

$$\nu^{(\alpha)}(t) = D_t^{(\alpha)}\nu(t), \tag{12}$$

which is denoted by

$$\nu(t) \in D_t^{(\alpha)}(\beta_1, \beta_2). \tag{13}$$

Moreover, the local fractional derivatives of higher order are defined as follows [14, 15]:

$$D_t^{(n\alpha)}(t) = \nu^{(n\alpha)}(t) = \overbrace{D_t^{(\alpha)}\cdots D_t^{(\alpha)}}^{n\text{ times}}\nu(t), \tag{14}$$

and the local fractional partial derivative of higher order is defined as follows [14, 15]:

$$\frac{\partial^{n\alpha}\nu(t,x)}{\partial t^{n\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial t^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial t^{\alpha}}\nu(t,x)}^{n\text{ times}}. \tag{15}$$

Property 1 Suppose that $\nu^{(k+1)\alpha} \in C_{\alpha}(\beta_1, \beta_2)$ for $k = 0, 1, \ldots, n$ and $0 < \alpha \leq 1$, then

$$\nu(t) = \sum_{k=0}^{n} \frac{\nu^{(k\alpha)}(t_0)}{\Gamma(k\alpha+1)}(t-t_0) + \frac{\nu^{((k+1)\alpha)}(\xi)}{\Gamma((k+1)\alpha+1)}(t-t_0)^{(k+1)\alpha}, \tag{16}$$

with $\beta_1 < t_0 < \xi < t < \beta_2$, $\forall t \in (\beta_1, \beta_2)$, where $\nu^{k\alpha}(t) = \overbrace{\frac{d^{\alpha}}{dt^{\alpha}}\cdots\frac{d^{\alpha}}{dt^{\alpha}}\nu(t)}^{n+1\text{ times}}$.

Proof See [14, 15]. □

Remark 1 Property 1 does not hold if the function is only Holder continuous. See [9].

Definition 4 The local fractional integral of the function $\nu(t)$ of order α in the interval $[\gamma,\beta]$ is defined as follows [14, 15]:

$$ {}_{\beta}I_{\eta}^{(\alpha)} = \frac{1}{\Gamma(1+\alpha)}\int_{\eta}^{\beta}\nu(\tau)(d\tau)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\to 0}\sum_{i=0}^{N-1}\nu(\tau_i)(\Delta_{\tau i})^{\alpha}, \tag{17}$$

where $\Delta_{\tau i} = \tau_{i+1} - \tau_i$, $\Delta_{\tau} = \max \Delta_{\tau 0}, \Delta_{\tau 1}, \Delta_{\tau 2}, \ldots, [\tau_i, \tau_{i+1}]$, $\tau_0 = \eta$, $\tau_N = \beta$ is a partition of the interval $[\eta,\beta]$.

Based on the local fractional integral defined in Eq. (17), the following properties hold (see [14, 15]):

$$ {}_{\beta}I_{\eta}^{(\alpha)}\big[\nu(t) \pm w(t)\big] = {}_{\beta}I_{\eta}^{(\alpha)}\big[\nu(t)\big] \pm_{\eta} I_{\beta}^{(\alpha)}\big[w(t)\big]. \tag{18}$$

$$ {}_{\beta}I_{\eta}^{(\alpha)}\nu(t)w^{(\alpha)}(t) = \big[\nu(t)w(t)\big]_{\eta}^{\beta} - {}_{\beta}I_{\eta}^{(\alpha)}\nu^{(\alpha)}(t)w(t). \tag{19}$$

$$ {}_{\beta}I_{\eta}^{(\alpha)}\big[\nu(t)\big] = {}_{\beta}I_{\zeta}^{(\alpha)}\big[\nu(t)\big] +_{\zeta} I_{\eta}^{(\alpha)}\big[\nu(t)\big]. \tag{20}$$

$$ {}_{\beta}I_{\eta}^{(\alpha)}\big[\zeta\nu(t)\big] = \zeta_{\beta}I_{\eta}^{(\alpha)}\big[\nu(t)\big]. \tag{21}$$

$$ {}_{\beta}I_{\eta}^{(\alpha)}\big[\nu(t)\big] = \frac{(\beta-\eta)^{\alpha}}{\Gamma(\alpha+1)}, \tag{22}$$

where $\beta < \zeta < \eta$, $\nu(t), w(t) \in C_{\alpha}(\eta,\beta)$.

Definition 5 The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $f(t) \in C_{\tau}^{m}$ and $\tau \geq -1$ is defined as follows [39]:

$$ I^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\eta)^{\alpha-1}f(\eta)\,d\eta, & \alpha > 0, t > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{23}$$

Below we list some important properties of I^{α} (see [1–3]).

(i) If $f \in C_{\tau}$, $\tau \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$, then

$$ I^{\alpha}t^{x} = \frac{\Gamma(x+1)}{\Gamma(x+\alpha+1)}t^{\alpha+x}, \tag{24}$$

$$ I^{\alpha}I^{\beta}f(t) = I^{\alpha+\beta}, \qquad I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}f(t). \tag{25}$$

(ii) For $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_{\tau}^{m}$, $\tau \geq -1$, then

$$ D^{\alpha}I^{\alpha}f(t) = f(t), \qquad I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{i=0}^{m-1}f^{i}\big(0^{+}\big)\frac{t^{i}}{i!}, \quad t > 0. \tag{26}$$

Definition 6 The function $f(t)$ in the Caputo fractional derivative is defined as follows [39, 41]:

$$D^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\eta)^{m-\alpha-1} f^{(m)}(\eta)\,d\eta, \\ I^{m-\alpha} D^m f(t), \end{cases} \tag{27}$$

where $m-1 < \alpha < m$, $m \in \mathbb{N}$, $t > 0$.

3 Local fractional homotopy analysis method

In this section, we illustrate the basic idea of the local fractional homotopy analysis method. Consider the following nonlinear local fractional partial differential equation:

$$N\big[u(x,t)\big] = 0, \tag{28}$$

where N is the nonlinear operator, x and t denote the independent variables, and $u(x,t)$ denotes the local fractional unknown function. Using the fundamentals of the traditional homotopy analysis method proposed by Liao [63], we construct a convex non-differentiable homotopy called the zero order deformation equation

$$(1-p)\pounds_{\alpha}\big[\psi(x,t;p) - u_0(x,t)\big] = p\hbar H(x,t)N\big[\psi(x,t;p)\big], \tag{29}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is the nonzero convergence-control parameter, and $H(x,t) \neq 0$ is the local fractional nonzero auxiliary function, $\psi(x,t;p)$ is the local fractional unknown function, $u_0(x,t)$ is an initial guess of $u(x,t)$, and $\pounds_{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ is the linear local fractional operator with the property that

$$\pounds_{\alpha}\big[\psi(x,t)\big] = 0, \quad \text{when } \psi(x,t) = 0. \tag{30}$$

Based on the concept of homotopy analysis method, one has great freedom to choose the auxiliary linear operator and the initial guess. Obviously, when $p = 1$ and $p = 0$, it holds

$$\psi(x,t;0) = u_0(x,t) \quad \text{and} \quad \psi(x,t;1) = u(x,t), \tag{31}$$

respectively. Thus, as p increases from 0 to 1, the solution $\psi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\psi(x,t;p)$ using the local fractional Taylor series [14, 15] with respect to p, we deduce

$$\psi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)p^m, \tag{32}$$

where

$$u_m(x,t) = \left[\frac{1}{m!}\frac{\partial^m \psi(x,t;p)}{\partial p^m}\right]_{p=0}. \tag{33}$$

If the auxiliary linear operator, the initial guess, the auxiliary function, and the convergence-control parameter are chosen properly, then Eq. (32) converges at $p = 1$, and

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) \tag{34}$$

is the solution of the original problem Eq. (28). According to Eq. (32), the governing equation can be deduced from the zero deformation Eq. (29).

Define a local fractional vector

$$\mathbf{u}_m = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}. \tag{35}$$

Differentiating Eq. (29) m-times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing by $m!$, we obtain the so-called Mth-order deformation equation

$$\pounds_\alpha\big[u_m(x,t) - \chi_m u_{m-1}(x,t)\big] = pH(x,t)R_m(\mathbf{u}_{m-1}, x, t), \tag{36}$$

where

$$R_m(\mathbf{u}_{m-1}, x, t) = \left[\frac{1}{(m-1)!}\frac{\partial^{(m-1)!} N[\psi(x,t;p)]}{\partial p^{(m-1)}}\right]_{p=0} \tag{37}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{38}$$

Applying the local fractional integral operator on both sides of Eq. (36), we deduce

$$\begin{aligned} u_m(x,t) = {}& \chi_m u_{m-1}(x,t) - \chi_m \sum_{k=0}^{m-1} u_{m-1}^{(k)}\big(x, 0^+\big)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \\ & + \hbar H(x,t)\,{}_\beta I_\eta^{(\alpha)}\big[R_m(\mathbf{u}_{m-1}, x, t)\big]. \end{aligned} \tag{39}$$

Using computer algebra software such as Mathematica or Matlab, we can easily obtain the series solutions of $u_m(x,t)$ for $m \geq 1$ at Mth-order deformation equation as follows:

$$u(x,t) = \sum_{m=0}^{+\infty} u_m(x,t). \tag{40}$$

The cornerstone of the local fractional homotopy analysis method is the nonzero convergence-control parameter $\hbar$ which provides us with a convenient way to guarantee the convergence of the series solutions of Eq. (40).

In the next subsection, we prove the convergence analysis of Eq. (40).

3.1 Convergence analysis of the LFHAM

Lemma 2 *Suppose that the series solution of Eq.* (40) *is convergent, and let X be any set of local fractional continuous functions which satisfies Eq.* (8) *and $u_n \in X$. Then $B(X)$ is a Banach space.*

Proof Let $u_n \in B(X)$. Then we need to show that the local fractional continuous function u_n converges uniformly in $B(X)$. Let $x \in X$. Then, for all $n, m \in \mathbb{N}$, we have

$$\left|u_n(x) - u_m(x)\right| \le \|u_n - u_m\| < \varepsilon^{\alpha}, \tag{41}$$

which implies $(u_n(x))$ is a Cauchy sequence, hence converges. Let the limit of $(u_n(x))$ be $(u(x))$. Then we want show that $u_n(x) \to u(x)$ in $B(X)$.

For any given $\varepsilon^{\alpha} > 0$, there is a positive integer N such that, for every $n, m > N$,

$$\begin{aligned}\left|u_n(x) - u_m(x)\right| &= \left\|u_n(x) - u(x) + u(x) - u_m(x)\right\| \\ &\le \left\|u_n(x) - u(x)\right\| + \left\|u(x) - u_m(x)\right\| \\ &< \frac{1}{2\varepsilon^{\alpha}} + \frac{1}{2\varepsilon^{\alpha}} \\ &= \varepsilon^{\alpha}.\end{aligned}$$

Thus, for every x,

$$\left|u_n(x) - u(x)\right| \le \varepsilon^{\alpha}. \tag{42}$$

This implies that u_n uniformly converges to u. The proof is complete. □

Theorem 1 *If the series*

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t)p^m \tag{43}$$

converges to $\xi(x,t)$, where $v_m(x,t)$ is governed by Eq. (36) under the definition of Eq. (37) and Eq. (38), then $\xi(x,t)$ must be the exact solution of Eq. (28).

Proof Let

$$\lim_{M\to\infty} \sum_{m=1}^{M} \xi_m(x,t) = v_0(x,t) + \lim_{M\to\infty} \sum_{m=1}^{M} v_m(x,t) = \xi(x,t). \tag{44}$$

Then we deduce that $\lim_{M\to\infty} \sum_{m=1}^{M} v_m(x,t) = 0$. Besides, using Eq. (36), we get

$$\begin{aligned}&\lim_{M\to+\infty}\left[\hbar H(x,t)\sum_{m=1}^{M} R_m(\mathbf{v}_{m-1},x,t)\right] \\ &= \lim_{M\to+\infty}\left[\sum_{m=1}^{M} \pounds_{\alpha}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right]\right]\end{aligned}$$

$$= \pounds_\alpha\left[\lim_{M\to\infty}\sum_{m=1}^{M} v_m(x,t) - \lim_{M\to+\infty}\sum_{m=1}^{M}\chi_m v_{m-1}(x,t)\right]$$
$$= \pounds_\alpha\left[\lim_{M\to+\infty}\sum_{m=1}^{M} v_m(x,t)\right] = 0.$$

On the other hand, since $H(x,t) \neq 0$, $\hbar \neq 0$ and by the linearity property of Eq. (30), we obtain

$$\lim_{M\to\infty}\sum_{m=1}^{M} R_m(\mathbf{v}_{m-1},x,t) = 0. \tag{45}$$

Similarly, based on Eq. (37), we get

$$\lim_{M\to+\infty}\left[\sum_{m=1}^{M} R_m(\mathbf{v}_{m-1},x,t)\right]$$
$$= \lim_{M\to+\infty}\sum_{m=1}^{M}\left[\frac{1}{(m-1)!}\frac{\partial^{(m-1)}N[\psi(x,t;p)]}{\partial p^{(m-1)}}\right]_{p=0} = 0. \tag{46}$$

Generally, since $\psi(x,t;p) \neq N[u(x,t)]$ in Eq. (28), let the residual error $\epsilon(x,t;p) = N[u(x,t)]$. This implies

$$\epsilon(x,t;p) = 0, \tag{47}$$

which satisfies the solution of Eq. (28). Hence, the residual error of the local fractional Taylor series on the embedding parameter p yields

$$\lim_{M\to+\infty}\sum_{m=1}^{M}\left[\frac{1}{m!}\frac{\partial^m N[\epsilon(x,t;p)]}{\partial p^m}\right]_{p=0}$$
$$= \lim_{M\to+\infty}\sum_{m=1}^{M}\left[\frac{1}{m!}\frac{\partial^m N[\psi(x,t;p)]}{\partial p^m}\right]_{p=0}. \tag{48}$$

Then using Eq. (45) and the assumption that $p = 1$, we deduce

$$\epsilon(x,t;p) = \lim_{M\to+\infty}\left[\sum_{m=1}^{M} R_m(\mathbf{v}_{m-1},x,t)\right]$$
$$= \lim_{M\to+\infty}\sum_{m=1}^{M}\left[\frac{1}{m!}\frac{\partial^m N[\psi(x,t;p)]}{\partial p^m}\right]_{p=0} = 0. \tag{49}$$

Thus, Eq. (49) proved that $\xi(x,t)$ satisfies the exact solution of the original problem Eq. (28). □

4 Applications of the LFHAM

In this section, we demonstrate the applicability of the LFHAM to linear and nonlinear partial differential equations involving local fractional derivatives.

Example 1 Consider the following non-homogeneous local fractional heat conduction equation:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} v(x,t)}{\partial x^{2\alpha}} = E_{\alpha}\left(x^{\alpha}\right), \quad t > 0, x \in \mathbb{R} \tag{50}$$

subject to the initial condition

$$v(x,0) = E_{\alpha}\left(x^{\alpha}\right). \tag{51}$$

Based on Eq. (50) and Eq. (51) and the procedure of the LFHAM, it is natural to choose $v_0(x,t) = E_{\alpha}(x^{\alpha})$ to be the initial guess.

We choose the linear operator as

$$£_{\alpha}\left[\varphi(x,t;p)\right] = \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left[\varphi(x,t;p)\right], \tag{52}$$

with the property $£_{\alpha}[C] = 0$, where C is an integral constant.

We define the nonlinear operator as follows:

$$N\left[\varphi(x,t;p)\right] = \frac{\partial^{\alpha}\varphi(x,t;p)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha}\varphi(x,t;p)}{\partial x^{2\alpha}} - E_{\alpha}\left(x^{\alpha}\right). \tag{53}$$

We construct the zero-order deformation equation:

$$(1-p)£_{\alpha}\left[\varphi(x,t;p) - v_0(x,t)\right] = p\hbar H(x,t) N\left[\varphi(x,t;p)\right]. \tag{54}$$

Obviously, when $p = 0$ and $p = 1$,

$$\varphi(x,t;0) = v_0(x,t) \quad \text{and} \quad \varphi(x,t;1) = v(x,t). \tag{55}$$

Then the Mth-order deformation equation is defined as follows:

$$£_{\alpha}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] = \hbar H(x,t) R_m(\mathbf{v}_{m-1},x,t), \tag{56}$$

where

$$R_m(\mathbf{v}_{m-1},x,t) = \frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} - (1-\chi_m)E_{\alpha}\left(x^{\alpha}\right). \tag{57}$$

Setting $H(x,t) = 1$ and applying the local fractional integral on the Mth-order deformation Eq. (56), we get

$$\begin{aligned} v_m(x,t) &= (\chi_m + \hbar)v_{m-1}(x,t) - (\chi_m + \hbar)v_{m-1}(x,0) \\ &\quad - \hbar_{\beta} I_{\eta}^{(\alpha)}\left[\frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} + (1-\chi_m)E_{\alpha}\left(x^{\alpha}\right)\right]. \end{aligned} \tag{58}$$

Then Eq. (58) yields

$$\begin{cases} v_m(x,t) = -\hbar_{\beta} I_{\eta}^{(\alpha)}\left[\frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} + (1-\chi_m)E_{\alpha}(x^{\alpha})\right], & m = 1 \\ v_m(x,t) = (\chi_m + \hbar)v_{m-1}(x,t) - \hbar_{\beta} I_{\eta}^{(\alpha)}\left[\frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} + (1-\chi_m)E_{\alpha}(x^{\alpha})\right], & m > 1. \end{cases} \tag{59}$$

Hence

$$\begin{aligned}
v_0(x,t) &= E_\alpha\left(x^\alpha\right),\\
v_1(x,t) &= -2\hbar E_\alpha\left(x^\alpha\right)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
v_2(x,t) &= -2\hbar(\hbar+1)E_\alpha\left(x^\alpha\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^2 E_\alpha\left(x^\alpha\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\vdots
\end{aligned}$$

and so on.

Setting the convergence-control parameter $\hbar = -1$, the series solutions of Eq. (50) are given by

$$\begin{aligned}
v(x,t) &= v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t)\\
&= 2E_\alpha\left(x^\alpha\right)\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) + E_\alpha\left(x^\alpha\right)\\
&= E_\alpha\left(x^\alpha\right)\left(2E_\alpha\left(t^\alpha\right) - 1\right).
\end{aligned} \tag{60}$$

The results obtained in Eq. (60) were entirely in agreement with the local fractional homotopy analysis method [56].

Figures 1: The 3D surface solution of Eq. (50) for $\alpha = 1$ is presented in Fig. 1(a). The surface solution of Eq. (40) for ($\alpha = \frac{1}{2}$) is depicted in Fig. 1(b). The non-differentiable surface solution is depicted in Fig. 1(c). The surface solution behavior of $v(x,t)$ for different values of $\alpha = 1, \frac{1}{2}, \frac{\ln(2)}{\ln(3)}$ is given in Fig. 1(d). The absolute error analysis for $\alpha = 1$ of 10th and 20th-order approximations of the LFHAM is presented in Fig. 1(e) and Fig. 1(f), respectively. In Fig. 1(g) and Fig. 1(h), the absolute error analysis of 10th and 20th-order approximations of the non-differentiable problem for $\alpha = \frac{\ln(2)}{\ln(3)}$ is illustrated.

Example 2 Consider the following non-homogeneous local fractional heat conduction equation:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} v(x,t)}{\partial x^{2\alpha}} = -\cos_\alpha\left(x^\alpha\right), \quad t > 0, x \in \mathbb{R} \tag{61}$$

subject to the initial condition

$$v(x,0) = \sin_\alpha\left(x^\alpha\right). \tag{62}$$

According to the procedure of the LFHAM and based on Eq. (61) and Eq. (62), it is natural to choose $v_0(x,t) = \sin_\alpha(x^\alpha)$.

Let us choose the linear operator as follows:

$$\pounds_\alpha\left[\varphi(x,t;p)\right] = \frac{\partial^\alpha}{\partial x^\alpha}\left[\varphi(x,t;p)\right], \tag{63}$$

with the property $\pounds_\alpha[C] = 0$, where C is an integral constant.

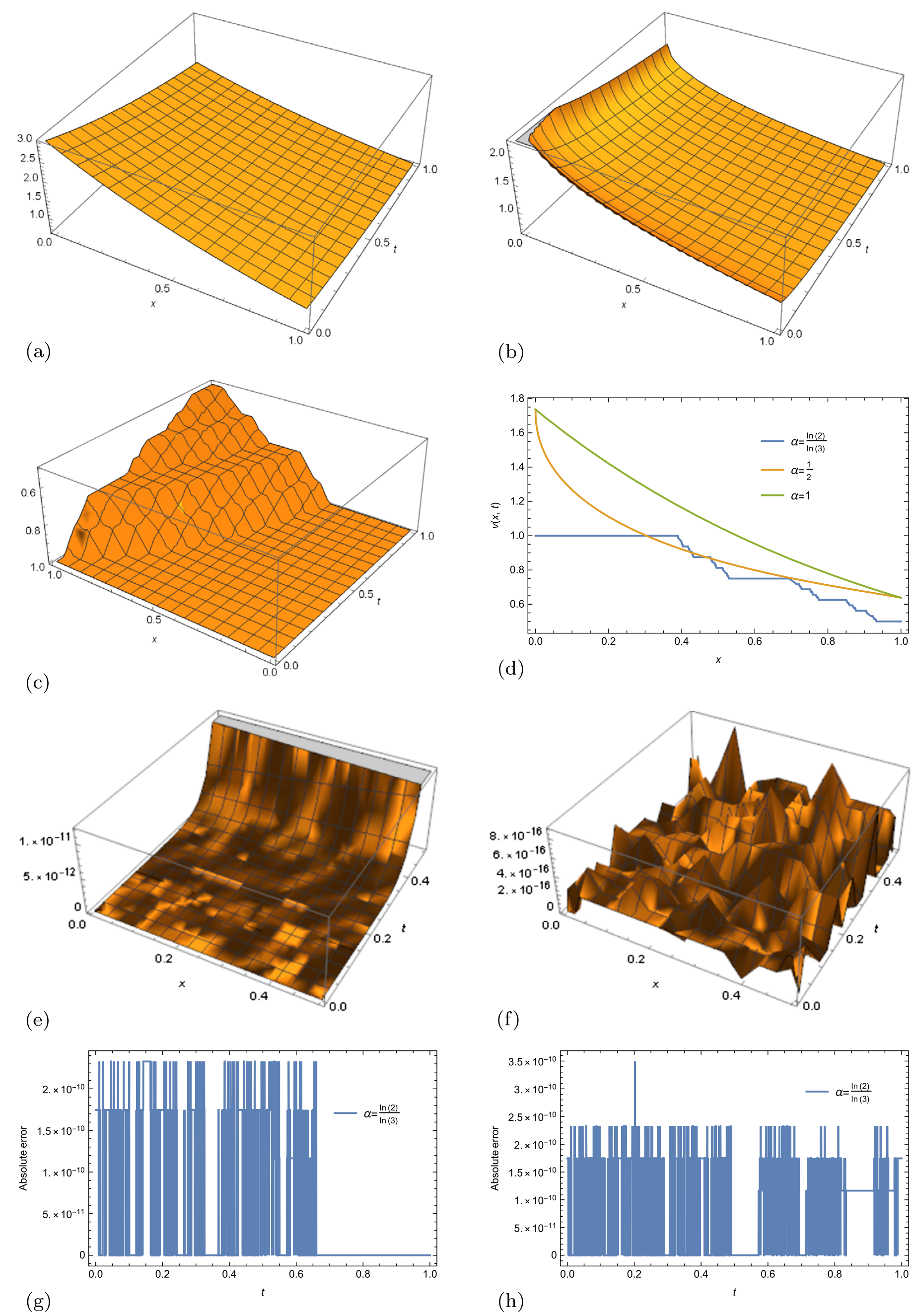

Figure 1 (**a**) Numerical simulation of Eq. (50) for $\alpha = 1$, (**b**) 3D surface solution for $\alpha = \frac{1}{2}$, (**c**) 3D non-differentiable surface solution behavior for $\alpha = \frac{\ln(2)}{\ln(3)}$, (**d**) 2D approximate solutions for $\alpha = 1, \frac{1}{2}$ and $\frac{\ln(2)}{\ln(3)}$, (**e**) Absolute error $E_{10}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$ for $\alpha = 1$, (**f**) Absolute error $E_{20}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$, $\alpha = 1$, (**g**) Absolute error of the LFHAM $E_{10}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$ when $\alpha = \frac{\ln(2)}{\ln(3)}$, (**h**) Absolute error of the LFHAM $E_{20}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$ when $\alpha = \frac{\ln(2)}{\ln(3)}$

We define the nonlinear operator:

$$N\big[\varphi(x,t;p)\big] = \frac{\partial^\alpha \varphi(x,t;p)}{\partial t^\alpha} - \frac{\partial^{2\alpha} \varphi(x,t;p)}{\partial x^{2\alpha}} + \cos_\alpha\big(x^\alpha\big). \tag{64}$$

We construct the zero-order deformation equation as follows:

$$(1-p)\pounds_{\alpha}\big[\varphi(x,t;p)-v_0(x,t)\big]=p\hbar H(x,t)N\big[\varphi(x,t;p)\big]. \tag{65}$$

Obviously, when $p=0$ and $p=1$,

$$\varphi(x,t;0)=v_0(x,t) \quad \text{and} \quad \varphi(x,t;1)=v(x,t). \tag{66}$$

Then the Mth-order deformation equation is defined as follows:

$$\pounds_{\alpha}\big[v_m(x,t)-\chi_m v_{m-1}(x,t)\big]=\hbar H(x,t)R_m(\mathbf{v}_{m-1},x,t), \tag{67}$$

where

$$R_m(\mathbf{v}_{m-1},x,t)=\frac{\partial^{\alpha}v_{m-1}(x,t)}{\partial t^{\alpha}}-\frac{\partial^{2\alpha}v_{m-1}(x,t)}{\partial x^{2\alpha}}+\cos_{\alpha}\big(x^{\alpha}\big). \tag{68}$$

Putting $H(x,t)=1$ and applying the local fractional integral on the Mth-order deformation Eq. (67), we obtain

$$\begin{aligned} v_m(x,t) &= (\chi_m+\hbar)v_{m-1}(x,t)-(\chi_m+\hbar)v_{m-1}(x,0)\\ &\quad +\hbar_{\beta}I_{\eta}^{(\alpha)}\left[\cos_{\alpha}\big(x^{\alpha}\big)-\frac{\partial^{2\alpha}v_{m-1}(x,t)}{\partial x^{2\alpha}}\right]. \end{aligned} \tag{69}$$

Then Eq. (69) yields

$$\begin{cases} v_m(x,t)=\hbar_{\beta}I_{\eta}^{(\alpha)}[\cos_{\alpha}(x^{\alpha})-\frac{\partial^{2\alpha}v_{m-1}(x,t)}{\partial x^{2\alpha}}], \quad m=1,\\ v_m(x,t)=(\chi_m+\hbar)v_{m-1}(x,t)-\hbar_{\beta}I_{\eta}^{(\alpha)}[(1-\chi_m)\cos_{\alpha}(x^{\alpha})-\frac{\partial^{2\alpha}v_{m-1}(x,t)}{\partial x^{2\alpha}}],\\ \quad m>1. \end{cases} \tag{70}$$

Hence

$$\begin{aligned} v_0(x,t) &= \sin_{\alpha}\big(x^{\alpha}\big),\\ v_1(x,t) &= \hbar\big(\sin_{\alpha}\big(x^{\alpha}\big)+\cos_{\alpha}\big(x^{\alpha}\big)\big)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\ v_2(x,t) &= \hbar(\hbar+1)\big(\sin_{\alpha}\big(x^{\alpha}\big)+\cos_{\alpha}\big(x^{\alpha}\big)\big)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\hbar^2\big(\sin_{\alpha}\big(x^{\alpha}\big)+\cos_{\alpha}\big(x^{\alpha}\big)\big)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\ &\vdots \end{aligned}$$

and so on.

Setting the convergence-control parameter $\hbar=-1$, the series solution of Eq. (61) is given by

$$\begin{aligned} v(x,t) &= v_0(x,t)+\sum_{m=1}^{+\infty}v_m(x,t)\\ &= \sin_{\alpha}\big(x^{\alpha}\big)+\big(\sin_{\alpha}\big(x^{\alpha}\big)+\cos_{\alpha}\big(x^{\alpha}\big)\big) \end{aligned}$$

$$\begin{aligned}
&\times\left(-\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right)\\
&=\sin_{\alpha}\left(x^{\alpha}\right)+\sum_{i=1}^{\infty}\frac{(-1)^{i}t^{i\alpha}}{\Gamma(i\alpha+1)}\left(\sin_{\alpha}\left(x^{\alpha}\right)+\cos_{\alpha}\left(x^{\alpha}\right)\right)\\
&=E_{\alpha}\left(-t^{\alpha}\right)\left(\sin_{\alpha}\left(x^{\alpha}\right)+\cos_{\alpha}\left(x^{\alpha}\right)\right)-\cos_{\alpha}\left(x^{\alpha}\right). \qquad (71)
\end{aligned}$$

The results obtained in Eq. (71) were in complete agreement with the local fractional homotopy perturbation method [56].

Figures 2: Surface solution of Eq. (61) for $\alpha = 1$ is given in Fig. 2(a). Surface solution behavior of Eq. (61) for ($\alpha = \frac{1}{2}$) is presented in Fig. 2(b). The non-differentiable surface solution behavior is depicted in Fig. 2(c). The 2D surface solution behavior for different values of $\alpha = 1, \frac{1}{2}, \frac{\ln(2)}{\ln(3)}$ is presented in Fig. 2(d). The absolute error analysis for 10th and 20th-order approximations of the LFHAM is given in Fig. 2(e) and Fig. 2(f), respectively. The 10th and 20th-order absolute error analysis of the non-differentiable problem for $\alpha = \frac{\ln(2)}{\ln(3)}$ is presented in Fig. 2(g) and Fig. 2(h), respectively.

Example 3 Consider the following nonlinear local fractional convection-diffusion equation:

$$\begin{aligned}
&\frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}}=\frac{\partial^{2\alpha}v(x,t)}{\partial x^{2\alpha}}-\frac{\partial^{\alpha}v(x,t)}{\partial x^{\alpha}}+v(x,t)\frac{\partial^{\alpha}v(x,t)}{\partial x^{\alpha}}-v^{2}(x,t)+v(x,t),\\
&\quad t>0, x\in\mathbb{R} \qquad (72)
\end{aligned}$$

subject to the initial condition

$$v(x,0)=E_{\alpha}\left(x^{\alpha}\right). \qquad (73)$$

Based on Eq. (72) and the initial condition Eq. (73), it is natural to choose $v_0(x,t) = E_\alpha(x^\alpha)$ to be the initial guess.

Let us choose the linear operator as follows:

$$\pounds_{\alpha}\left[\varphi(x,t;p)\right]=\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left[\varphi(x,t;p)\right], \qquad (74)$$

with the property $\pounds_\alpha[C] = 0$, where C is an integral constant.

We define the nonlinear operator:

$$\begin{aligned}
N\left[\varphi(x,t;p)\right]=&\frac{\partial^{\alpha}\varphi(x,t;p)}{\partial t^{\alpha}}-\frac{\partial^{2\alpha}\varphi(x,t;p)}{\partial x^{2\alpha}}+\frac{\partial^{\alpha}\varphi(x,t;p)}{\partial x^{\alpha}}\\
&-\varphi(x,t;p)\frac{\partial^{\alpha}\varphi(x,t;p)}{\partial x^{\alpha}}+\varphi^{2}(x,t;p)-\varphi(x,t;p). \qquad (75)
\end{aligned}$$

We construct the zero-order deformation equation as follows:

$$(1-p)\pounds_{\alpha}\left[\varphi(x,t;p)-v_{0}(x,t)\right]=p\hbar H(x,t)N\left[\varphi(x,t;p)\right]. \qquad (76)$$

Obviously, when $p = 0$ and $p = 1$,

$$\varphi(x,t;0)=v_{0}(x,t)\quad\text{and}\quad\varphi(x,t;1)=v(x,t). \qquad (77)$$

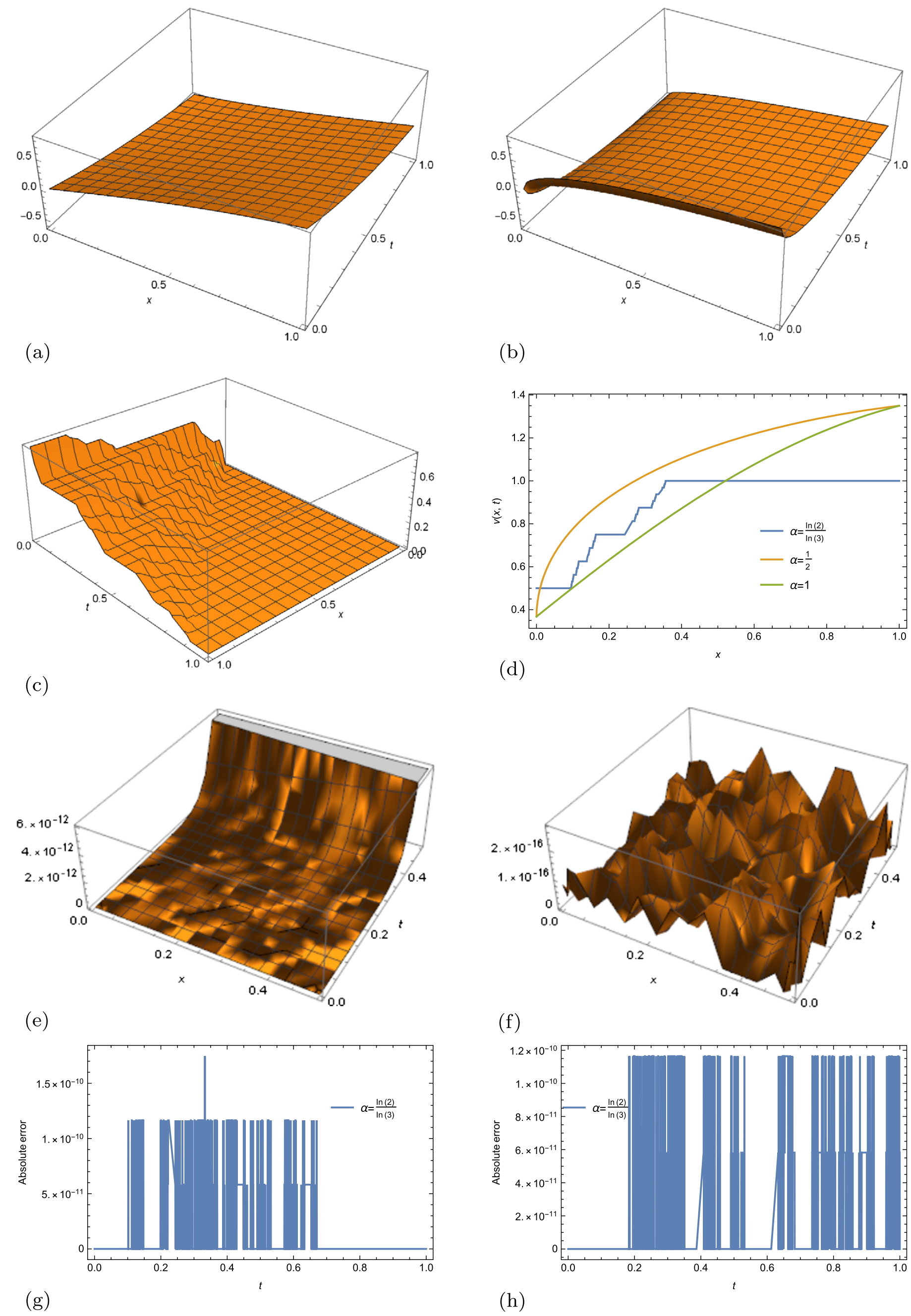

Figure 2 (**a**) Numerical simulation of Eq. (61) for $\alpha = 1$, (**b**) 3D surface solution for $\alpha = \frac{1}{2}$, (**c**) 3D non-differentiable surface solution behavior for $\alpha = \frac{\ln(2)}{\ln(3)}$, (**d**) 2D approximate solutions for $\alpha = 1, \frac{1}{2}$ and $\frac{\ln(2)}{\ln(3)}$, (**e**) Absolute error $E_{10}(v(x,t)) = |v_{ext.}(x,t) - v_{appr.}(x,t)|$, $\alpha = 1$, (**f**) Absolute error $E_{20}(v(x,t)) = |v_{ext.}(x,t) - v_{appr.}(x,t)|$ (**g**) Absolute error of the LFHAM $E_{10}(v(x,t)) = |v_{ext.}(x,t) - v_{appr.}(x,t)|$ for $\alpha = \frac{\ln(2)}{\ln(3)}$, (**h**) Absolute error of the LFHAM $E_{20}(v(x,t)) = |v_{ext.}(x,t) - v_{appr.}(x,t)|$ for $\alpha = \frac{\ln(2)}{\ln(3)}$

Then the Mth-order deformation equation is defined as follows:

$$\pounds_{\alpha}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] = \hbar H(x,t) R_m(\mathbf{v}_{m-1}, x, t), \tag{78}$$

where

$$R_m(\mathbf{v}_{m-1},x,t) = \frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial x^{\alpha}} - v_{m-1}(x,t)\frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial x^{\alpha}} + v_{m-1}^2(x,t) - v_{m-1}(x,t). \tag{79}$$

Putting $H(x,t) = 1$ and applying the local fractional integral on the Mth-order deformation Eq. (78), we deduce

$$v_m(x,t) = (\chi_m + \hbar)v_{m-1}(x,t) - (\chi_m + \hbar)v_{m-1}(x,0) + \hbar\, {}_{\beta}I_{\eta}^{(\alpha)}\Bigg[\frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial x^{\alpha}} - \frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} - \sum_{i=0}^{m-1} v_i(x,t)\frac{\partial^{2\alpha} v_{m-1-i}(x,t)}{\partial x^{2\alpha}} + \sum_{i=0}^{m} v_i(x,t)v_{m-i}(x,t) - v_{m-1}(x,t)\Bigg]. \tag{80}$$

For $m = 1$, Eq. (80) yields

$$v_m(x,t) = \hbar\, {}_{\beta}I_{\eta}^{(\alpha)}\Bigg[\frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial x^{\alpha}} - \frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} - \sum_{i=0}^{m-1} v_i(x,t)\frac{\partial^{2\alpha} v_{m-1-i}(x,t)}{\partial x^{2\alpha}} + \sum_{i=0}^{m} v_i(x,t)v_{m-i}(x,t) - v_{m-1}(x,t)\Bigg]. \tag{81}$$

And for $m \geq 2$, Eq. (80) yields

$$v_m(x,t) = (\chi_m + \hbar)v_{m-1}(x,t) + \hbar\, {}_{\beta}I_{\eta}^{(\alpha)}\Bigg[\frac{\partial^{\alpha} v_{m-1}(x,t)}{\partial x^{\alpha}} - \frac{\partial^{2\alpha} v_{m-1}(x,t)}{\partial x^{2\alpha}} - \sum_{i=0}^{m-1} v_i(x,t)\frac{\partial^{2\alpha} v_{m-1-i}(x,t)}{\partial x^{2\alpha}} + \sum_{i=0}^{m} v_i(x,t)v_{m-i}(x,t) - v_{m-1}(x,t)\Bigg]. \tag{82}$$

Thus

$$\begin{aligned}
v_0(x,t) &= E_{\alpha}(x^{\alpha}),\\
v_1(x,t) &= \hbar\, {}_{\beta}I_{\eta}^{(\alpha)}\Bigg[\frac{\partial^{\alpha} v_0(x,t)}{\partial x^{\alpha}} - \frac{\partial^{2\alpha} v_0(x,t)}{\partial x^{2\alpha}} - v_0(x,t)\frac{\partial^{2\alpha} v_0(x,t)}{\partial x^{2\alpha}} + v_0^2(x,t) - v_0(x,t)\Bigg]\\
&= -\hbar E_{\alpha}(x^{\alpha})\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
v_2(x,t) &= (\hbar+1)v_1(x,t)\\
&\quad - \hbar\, {}_{\beta}I_{\eta}^{(\alpha)}\Bigg[\frac{\partial^{\alpha} v_1(x,t)}{\partial x^{\alpha}} - \frac{\partial^{2\alpha} v_1(x,t)}{\partial x^{2\alpha}} - v_0(x,t)\frac{\partial^{2\alpha} v_1(x,t)}{\partial x^{2\alpha}} - v_1(x,t)\frac{\partial^{2\alpha} v_0(x,t)}{\partial x^{2\alpha}}\\
&\quad + 2v_0(x,t)v_1(x,t) - v_1(x,t)\Bigg]\\
&= \hbar^2 E_{\alpha}(x^{\alpha})\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\vdots
\end{aligned}$$

and so on.

Then, choosing the convergence-control parameter $\hbar = -1$, the series solutions of Eq. (72) are given by

$$\begin{aligned} v(x,t) &= v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t) \\ &= E_\alpha\left(x^\alpha\right)\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) \\ &= E_\alpha\left(x^\alpha\right)\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \\ &= E_\alpha\left(x^\alpha\right)E_\alpha\left(t^\alpha\right). \end{aligned} \tag{83}$$

The result obtained in Eq. (83) is the same as that of the local fractional homotopy analysis method [56].

Figures 3: The surface solution behavior of Eq. (72) for $\alpha = 1$ is presented in Fig. 3(a). Surface solution behavior of Eq. (72) for ($\alpha = \frac{1}{2}$) is illustrated in Fig. 3(b). The non-differentiable surface solution behavior for $\alpha = \frac{\ln(2)}{\ln(3)}$ is depicted in Fig. 3(c). 2D surface solutions for different values of $\alpha = 1, \frac{1}{2}, \frac{\ln(2)}{\ln(3)}$ are presented in Fig. 3(d). The absolute error analysis for 10th and 20th-order approximations of the LFHAM is given in Fig. 3(e) and Fig. 3(f), respectively. The 10th and 20th-order absolute error analysis of the non-differentiable problem for $\alpha = \frac{\ln(2)}{\ln(3)}$ is depicted in Fig. 3(g) and Fig. 3(h), respectively.

5 Conclusion

In this paper, we introduced a modified version of the well-known homotopy analysis method (HAM) called the local fractional homotopy analysis method (LFHAM) for solving non-differential models arising on Cantor sets. The suggested method was successfully applied to some non-differentiable problems, and the results obtained were entirely in agreement with the results of the existing methods. It is further shown that when the nonzero convergence-control parameter $\hbar = -1$, the results of the local fractional homotopy perturbation method (LFHPM) are recovered as a particular case of the proposed technique. The most significant advantage of this technique over the existing methods is not only the highest degree of freedom to adjust and control the convergence of the series solutions, but also the great privilege to choose the initial approximation, the deformation-functions, and the auxiliary linear operator. Thus, we conclude that the LFHAM is a powerful semi-analytical technique for solving non-differentiable partial differential equations and can be regarded as a modification of the homotopy analysis method.

Appendix

Here, we present some important results.

Definition 7 Let φ be a function of the homotopy-parameter p, then

$$\Theta_m(\varphi) = \frac{1}{m!}\frac{d^m\varphi}{dp^m}\bigg|_{p=0} \tag{84}$$

is called the Mth-order homotopy-derivative of φ, where $m \geq 0$ is an integer, and Θ_m is called the operator of the mth-order homotopy-derivative [67].

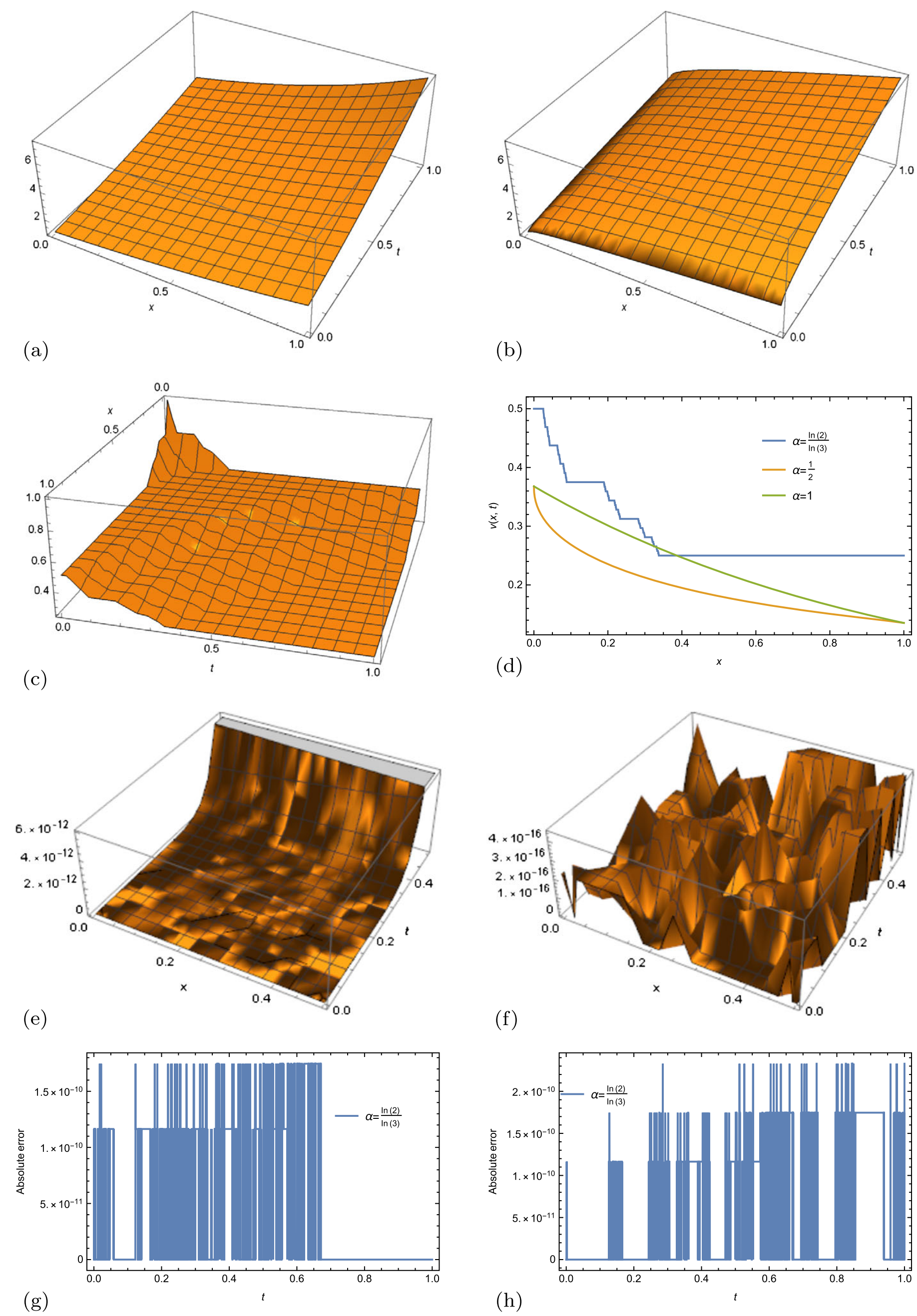

Figure 3 (**a**) Numerical solution of Eq. (72) for $\alpha = 1$, (**b**) 3D surface solution for $\alpha = \frac{1}{2}$, (**c**) Non-differentiable surface solution behavior for $\alpha = \frac{\ln(2)}{\ln(3)}$, (**d**) Approximate solutions for $\alpha = 1, \frac{1}{2}$ and $\frac{\ln(2)}{\ln(3)}$, (**e**) Absolute error $E_{10}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$, $\alpha = 1$, (**f**) Absolute error $E_{20}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$, $\alpha = 1$, (**g**) Absolute error of the LFHAM $E_{10}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$ for $\alpha = \frac{\ln(2)}{\ln(3)}$, (**h**) Absolute error of the LFHAM $E_{20}(v(x,t)) = |v_{\text{ext.}}(x,t) - v_{\text{appr.}}(x,t)|$ for $\alpha = \frac{\ln(2)}{\ln(3)}$

Theorem 2 *For three arbitrary Maclaurin series*

$$\varphi = \sum_{i=0}^{+\infty} v_i p^i, \qquad \varphi^2 = \sum_{i=0}^{+\infty} v_i^2 p^i, \qquad \frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}} = \sum_{i=0}^{+\infty} \frac{\partial^{2\alpha} v_i}{\partial x^{2\alpha}} p^i, \tag{85}$$

it holds

$$\Theta_m(\varphi) = \nu_m, \tag{86}$$

$$\Theta_m\left(\varphi^2\right) = \sum_{i=0}^{m} \nu_i \nu_{m-i}, \tag{87}$$

$$\Theta_m\left(\varphi \frac{\partial^{2\alpha}\varphi}{\partial x^{2\alpha}}\right) = \sum_{i=0}^{m} \nu_i \frac{\partial^{2\alpha}\nu_{m-i}}{\partial x^{2\alpha}}. \tag{88}$$

Proof For the proof of Eq. (86) and Eq. (87), the reader should refer to Liao [67]. According to Leibnitz's rule for derivative of product, it holds

$$\begin{aligned} \frac{\partial^m(\nu\frac{\partial^{2\alpha}\nu}{\partial x^{2\alpha}})}{\partial p^m} &= \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} \frac{\partial^n(\nu)}{\partial p^n} \frac{\partial^{m-n}(\frac{\partial^{2\alpha}\nu}{\partial x^{2\alpha}})}{\partial p^{m-n}} \\ &= \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} \frac{\partial^n(\frac{\partial^{2\alpha}(\nu)}{\partial x^{2\alpha}})}{\partial p^n} \frac{\partial^{m-n}(\nu)}{\partial p^{m-n}}. \end{aligned}$$

Then, according to Eq. (84) and Eq. (88), we deduce

$$\begin{aligned} \Theta_m\left(\nu\frac{\partial^{2\alpha}\nu}{\partial x^{2\alpha}}\right) &= \frac{1}{m!}\left[\frac{\partial^m}{\partial p^m}\left[\nu\frac{\partial^{2\alpha}\nu}{\partial x^{2\alpha}}\right]\bigg|_{p=0}\right] \\ &= \frac{1}{m!}\left[\binom{n}{x}\frac{\partial^n(\nu)}{\partial p^n}\frac{\partial^{m-n}(\frac{\partial^{2\alpha}(\nu)}{\partial x^{2\alpha}})}{\partial p^{m-n}}\right]\bigg|_{p=0} \\ &= \left\{\left[\sum_{n=0}^{n}\frac{1}{m!}\frac{\partial^n(\nu)}{\partial p^n}\right]\left[\frac{1}{(m-n)!}\frac{\partial^{m-n}(\frac{\partial^{2\alpha}(\nu)}{\partial x^{2\alpha}})}{\partial p^{m-n}}\right]\bigg|_{p=0}\right\} \\ &= \left\{\left[\sum_{n=0}^{n}\frac{1}{m!}\frac{\partial^n(\nu)}{\partial p^n}\right]\bigg|_{p=0}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[\frac{1}{(m-n)!}\frac{\partial^{m-n}(\nu)}{\partial p^{m-n}}\right]\bigg|_{p=0}\right\} \\ &= \sum_{n=0}^{m}\nu_n\frac{\partial^{2\alpha}\nu_{m-n}}{\partial x^{2\alpha}}. \end{aligned}$$

□

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Competing interests
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Authors' contributions
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