Khan *Advances in Difference Equations* (2019) 2019:56
https://doi.org/10.1186/s13662-019-1995-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Bifurcations of a two-dimensional discrete-time predator–prey model

Abdul Qadeer Khan[1*]

*Correspondence:
abdulqadeerkhan1@gmail.com
[1]Department of Mathematics, University of Azad Jammu & Kashmir, Muzaffarabad, Pakistan

Abstract

We study the local dynamics and bifurcations of a two-dimensional discrete-time predator–prey model in the closed first quadrant $\mathbb{R}_+^2$. It is proved that the model has two boundary equilibria: $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1},0)$ and a unique positive equilibrium $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ under some restriction to the parameter. We study the local dynamics along their topological types by imposing the method of linearization. It is proved that a fold bifurcation occurs about the boundary equilibria: $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1},0)$ and a period-doubling bifurcation in a small neighborhood of the unique positive equilibrium $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. It is also proved that the model undergoes a Neimark–Sacker bifurcation in a small neighborhood of the unique positive equilibrium $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ and meanwhile a stable invariant closed curve appears. From the viewpoint of biology, the stable closed curve corresponds to the periodic or quasi-periodic oscillations between predator and prey populations. Numerical simulations are presented to verify not only the theoretical results but also to exhibit the complex dynamical behavior such as the period-2, -4, -11, -13, -15 and -22 orbits. Further, we compute the maximum Lyapunov exponents and the fractal dimension numerically to justify the chaotic behaviors of the discrete-time model. Finally, the feedback control method is applied to stabilize chaos existing in the discrete-time model.

MSC: 39A10; 40A05; 92D25; 70K50; 35B35

Keywords: Discrete-time predator–prey model; Stability and bifurcations; Center manifold theorem; Fractal dimension; Chaos control; Numerical simulation

1 Introduction

Different models have been invoked to understand the mechanism of competition between populations of two-species. In 1931, Volterra proposed a famous prey–predator model which is represented by the following system of ordinary differential equations [1]:

$$\left.\begin{aligned}\dot{X} &= aX - bXY,\\ \dot{Y} &= -cY + dXY,\end{aligned}\right\} \tag{1}$$

where X denotes the number of prey and Y denotes the number of predator. Moreover, a, b, c, d are positive parameters. It has been shown that the number of prey grows exponentially in the absence of predators, while the number of predators decreases exponentially in the absence of a prey population. The terms bXY and dXY explain the prey–predators encounters which are conducive to predators and lethal to prey. It is noted that model (1) then takes the following form, if one consider some harvesting effect [2]:

$$\left.\begin{aligned} \dot{X} &= aX - bXY - \gamma X, \\ \dot{Y} &= -cY + dXY - \gamma Y, \end{aligned}\right\} \tag{2}$$

and as a result the reasonable harvesting effect is favorable to prey population. There are also some other prey–predator models which are more fascinating and effective for a number of interacting species greater than two or which assume a parasitic infection of the populations [3, 4].

It is a well-known fact that discrete-time models described by difference equations are more beneficial and reliable than continuous-time models whenever there are non-overlapping generations in the populations. Moreover, these models also provide efficient computational results for numerical simulations and provide a rich dynamics as compared to the continuous ones [5–10]. In the last few years, many interesting papers have appeared in the literature that discuss the stability, bifurcation and chaos phenomena in discrete-time models (see [11–20] and the references cited therein).

This paper deals with the study of stability, bifurcations and chaos control of the following discrete-time predator–prey model [21]:

$$\left.\begin{aligned} X_{n+1} &= rX_n(1 - X_n) - bX_nY_n, \\ Y_{n+1} &= dX_nY_n. \end{aligned}\right\} \tag{3}$$

It is noted that after using the following re-scaling transformations:

$$X_n = x_n, \qquad Y_n = \frac{y_n}{b},$$

the discrete-time model (3) then takes the form

$$\left.\begin{aligned} x_{n+1} &= \alpha_1 x_n(1 - x_n) - x_ny_n, \\ y_{n+1} &= \alpha_2 x_ny_n, \end{aligned}\right\} \tag{4}$$

where $\alpha_1 = r > 0$ and $\alpha_2 = d > 0$.

The rest of the paper is organized as follows: Sect. 2 deals with the study of the existence of equilibria and local stability along their different topological types of the discrete-time model (4). In Sect. 3, we study the existence of bifurcations about equilibria of the model (4). Section 4 deals with a bifurcation analysis about the unique positive equilibrium of the model (4). In Sect. 5, numerical simulations are presented to verify the theoretical results. This also includes the study of fractal dimensions which characterize the strange attractors of the model (4). In Sect. 6, we study the chaos control by the feedback control method to stabilize chaos at unstable trajectories. A conclusion is given in Sect. 7.

2 Existence of equilibria and local stability of the discrete-time model (4)

Lemma 1 *System* (4) *has at least two boundary equilibria and one unique positive equilibrium in* $\mathbb{R}_+^2$. *More precisely,*

(i) *for all parametric values* α_1 *and* α_2, *system* (4) *has the boundary equilibrium* $O(0,0)$;

(ii) *if* $\alpha_1 > 1$ *then system* (4) *has the boundary equilibrium* $A(\frac{\alpha_1-1}{\alpha_1},0)$;

(iii) *system* (4) *has a unique positive equilibrium* $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *if* $\alpha_1 > \frac{\alpha_2}{\alpha_2-1}$ *and* $\alpha_2 > 1$.

Now will study the local dynamics of (4) about $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1},0)$ and $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. The Jacobian matrix $J_{(\widehat{x},\widehat{y})}$ of the discrete-time model (4) about equilibrium $(\widehat{x},\widehat{y})$ is given by

$$J_{(\widehat{x},\widehat{y})} = \begin{pmatrix} \alpha_1(1-2\widehat{x})-\widehat{y} & -\widehat{x} \\ \alpha_2\widehat{y} & \alpha_2\widehat{x} \end{pmatrix}.$$

Its characteristic equation is

$$\kappa^2 - p(\widehat{x},\widehat{y})\kappa + q(\widehat{x},\widehat{y}) = 0, \tag{5}$$

where

$$p(\widehat{x},\widehat{y}) = \alpha_1(1-2\widehat{x}) - \widehat{y} + \alpha_2\widehat{x},$$
$$q(\widehat{x},\widehat{y}) = \bigl(\alpha_1(1-2\widehat{x}) - \widehat{y}\bigr)\alpha_2\widehat{x} + \alpha_2\widehat{x}\widehat{y}.$$

Lemma 2 *For equilibrium* O, *the following statements hold*:

(i) O *is a sink if* $0 < \alpha_1 < 1$;

(ii) O *is never a source*;

(iii) O *is a saddle if* $\alpha_1 > 1$;

(iv) O *is non-hyperbolic if* $\alpha_1 = 1$.

Lemma 3 *For* $A(\frac{\alpha_1-1}{\alpha_1},0)$, *the following statements hold*:

(i) $A(\frac{\alpha_1-1}{\alpha_1},0)$ *is a sink if* $\alpha_1 \in (1,3)$ *and* $0 < \alpha_2 < \frac{\alpha_1}{\alpha_1-1}$;

(ii) $A(\frac{\alpha_1-1}{\alpha_1},0)$ *is a source if* $\alpha_1 > 3$ *and* $\alpha_2 > \frac{\alpha_1}{\alpha_1-1}$;

(iii) $A(\frac{\alpha_1-1}{\alpha_1},0)$ *is a saddle if* $\alpha_1 > 3$ *and* $0 < \alpha_2 < \frac{\alpha_1}{\alpha_1-1}$;

(iv) $A(\frac{\alpha_1-1}{\alpha_1},0)$ *is non-hyperbolic if* $\alpha_2 = \frac{\alpha_1}{\alpha_1-1}$.

Hereafter we will investigate the local dynamics of the discrete-time model (4) about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ by using Lemma 2.2 of [22]. The Jacobian matrix $J_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$ of the linearized system of the discrete-time model (4) about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ is

$$\kappa^2 - p\kappa + q = 0, \tag{6}$$

where

$$p = \frac{2\alpha_2-\alpha_1}{\alpha_2},$$
$$q = \frac{\alpha_1(\alpha_2-2)}{\alpha_2}.$$

Moreover, the eigenvalues of $J_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ are given by

$$\kappa_{1,2}=\frac{p\pm\sqrt{\Delta}}{2}$$

where

$$\begin{aligned}\Delta&=p^2-4q\\&=\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}.\end{aligned}$$

Hereafter if $\Delta \gtreqless 0$, we will study the topological classification about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ of the discrete-time model (4) as follows:

Lemma 4 *For* $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$, *the following statements hold*:

(i) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is a locally asymptotically stable focus if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0 \quad \textit{and} \quad 0<\alpha_1<\frac{\alpha_2}{\alpha_2-2};$$

(ii) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is an unstable focus if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0 \quad \textit{and} \quad \alpha_1>\frac{\alpha_2}{\alpha_2-2};$$

(iii) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is non-hyperbolic if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0 \quad \textit{and} \quad \alpha_1=\frac{\alpha_2}{\alpha_2-2}.$$

Lemma 5 *For* $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$, *the following statements hold*:

(i) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is locally asymptotically node if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0 \quad \textit{and} \quad 0<\alpha_1<\frac{3\alpha_2}{3-\alpha_2};$$

(ii) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is unstable node if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0 \quad \textit{and} \quad \alpha_1>\frac{3\alpha_2}{3-\alpha_2};$$

(iii) $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ *is non-hyperbolic if*

$$\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0 \quad \textit{and} \quad \alpha_1=\frac{3\alpha_2}{3-\alpha_2}.$$

3 Existence of bifurcations about equilibria of the discrete-time model (4)

In this section based on theoretical studies in Sect. 2, we will study the existence of bifurcations about equilibria. The existence of corresponding bifurcations about the equilibria $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1},0)$ and $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ can be summarized as follows:

(i) From Lemma 2, we can see that when $\alpha_1=1$, one of the eigenvalues about the equilibrium $O(0,0)$ is 1. So a fold bifurcation may occur when the parameter varies in the small neighborhood of $\alpha_1=1$.

(ii) From Lemma 3, we can easily see that if $\alpha_2=\frac{\alpha_1}{\alpha_1-1}$ holds then one of the eigenvalues about $A(\frac{\alpha_1-1}{\alpha_1},0)$ is 1. So a fold bifurcation occurs when the parameter varies in a small neighborhood of $\alpha_2=\frac{\alpha_1}{\alpha_1-1}$. And we denote the parameters satisfying $\alpha_2=\frac{\alpha_1}{\alpha_1-1}$ as

$$F_{A(\frac{\alpha_1-1}{\alpha_1},0)}=\left\{(\alpha_1,\alpha_2):\alpha_2=\frac{\alpha_1}{\alpha_1-1},\alpha_1,\alpha_2>0\right\}.$$

(iii) From Lemma 4, we see that if $\alpha_1=\frac{\alpha_2}{\alpha_2-2}$ holds then the eigenvalues of $J_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ are a pair of complex conjugate with modulus 1. So a Neimark–Sacker bifurcation exists by the variation of parameter in a small neighborhood of $\alpha_1=\frac{\alpha_2}{\alpha_2-2}$. Precisely we represent the parameters satisfying $\alpha_1=\frac{\alpha_2}{\alpha_2-2}$ as

$$\begin{aligned}&N_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}\\&\quad=\left\{(\alpha_1,\alpha_2):\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0\text{ and }\alpha_1=\frac{\alpha_2}{\alpha_2-2}\right\}.\end{aligned}$$

(iv) From Lemma 5, we see that if $\alpha_1=\frac{3\alpha_2}{3-\alpha_2}$ holds we see that one of the eigenvalues of $J_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ is -1 and other is neither 1 nor -1. So a period-doubling bifurcation exists by the variation of parameter in a small neighborhood of $\alpha_1=\frac{3\alpha_2}{3-\alpha_2}$. More precisely we can also represent the parameters satisfying $\alpha_1=\frac{3\alpha_2}{3-\alpha_2}$ as

$$\begin{aligned}&P_{B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}\\&\quad=\left\{(\alpha_1,\alpha_2):\left(\frac{2\alpha_2-\alpha_1}{\alpha_2}\right)^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0\text{ and }\alpha_1=\frac{3\alpha_2}{3-\alpha_2}\right\}.\end{aligned}$$

4 Bifurcations analysis

This section deals with the study of Neimark–Sacker bifurcation and period-doubling bifurcation, respectively, about the unique positive equilibrium $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$.

4.1 Neimark–Sacker bifurcation about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$

Here we study the Neimark–Sacker bifurcation of the discrete-time model (4) about $B(\frac{1}{\alpha_2},\frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. Consider the parameter α_1 in a neighborhood of α_1^*, i.e., $\alpha_1=\alpha_1^*+\epsilon$, where $\epsilon\ll 1$, then the discrete-time model (4) becomes

$$\left.\begin{aligned}x_{n+1}&=(\alpha_1^*+\epsilon)x_n(1-x_n)-x_ny_n,\\y_{n+1}&=\alpha_2x_ny_n.\end{aligned}\right\}\tag{7}$$

The characteristic equation of $J_{B(\frac{1}{\alpha_2},\frac{(\alpha_1^*+\epsilon)\alpha_2-\alpha_1^*-\epsilon-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2},\frac{(\alpha_1^*+\epsilon)\alpha_2-\alpha_1^*-\epsilon-\alpha_2}{\alpha_2})$ of the discrete-time model (7) is

$$\kappa^2 - p(\epsilon)\kappa + q(\epsilon) = 0,$$

where

$$p(\epsilon) = \frac{2\alpha_2 - \alpha_1^* - \epsilon}{\alpha_2}, \qquad q(\epsilon) = \frac{(\alpha_1^* + \epsilon)(\alpha_2 - 2)}{\alpha_2}.$$

The roots of characteristic equation of $J_{B(\frac{1}{\alpha_2},\frac{(\alpha_1^*+\epsilon)\alpha_2-\alpha_1^*-\epsilon-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2},\frac{(\alpha_1^*+\epsilon)\alpha_2-\alpha_1^*-\epsilon-\alpha_2}{\alpha_2})$ are

$$\begin{aligned}\kappa_{1,2} &= \frac{p(\epsilon) \pm \iota\sqrt{4q(\epsilon) - p^2(\epsilon)}}{2}, \\ &= \frac{2\alpha_2 - \alpha_1^* - \epsilon}{2\alpha_2} \pm \frac{\iota}{2}\sqrt{\frac{4(\alpha_1^* + \epsilon)(\alpha_2 - 2)}{\alpha_2} - \left(\frac{2\alpha_2 - \alpha_1^* - \epsilon}{\alpha_2}\right)^2}\end{aligned}$$

and

$$|\kappa_{1,2}| = \sqrt{q(\epsilon)}, \qquad \left.\frac{d|\kappa_{1,2}|}{d\epsilon}\right|_{\epsilon=0} = \frac{\alpha_2 - 2}{2\alpha_2} > 0.$$

Additionally, we required that when $\epsilon = 0$, $\kappa_{1,2}^m \neq 1$, $m = 1, 2, 3, 4$, which corresponds to $p(0) \neq -2, 0, 1, 2$, which is true by computation.

Let $u_n = x_n - x^*$, $v_n = y_n - y^*$ then the equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ of the discrete-time model (4) transforms into $O(0, 0)$. By manipulation, one gets

$$\left.\begin{aligned} u_{n+1} &= (\alpha_1^* + \epsilon)(u_n + x^*)(1 - u_n - x^*) - (u_n + x^*)(v_n + y^*) - x^*, \\ v_{n+1} &= \alpha_2(u_n + x^*)(v_n + y^*) - y^*, \end{aligned}\right\} \tag{8}$$

where $x^* = \frac{1}{\alpha_2}$, $y^* = \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2}$. Hereafter when $\epsilon = 0$, the normal form of system (8) is studied. Expanding (8) about $(u_n, v_n) = (0, 0)$ by Taylor series, we get

$$\left.\begin{aligned} u_{n+1} &= \Gamma_{11}u_n + \Gamma_{12}v_n + \Gamma_{13}u_n^2 + \Gamma_{14}u_nv_n, \\ v_{n+1} &= \Gamma_{21}u_n + \Gamma_{22}v_n + \Gamma_{23}u_nv_n, \end{aligned}\right\} \tag{9}$$

where

$$\begin{aligned} &\Gamma_{11} = \alpha_1^*(1 - 2x^*) - y^*, \qquad \Gamma_{12} = -x^*, \qquad \Gamma_{13} = -\alpha_1^*, \qquad \Gamma_{14} = -1, \\ &\Gamma_{21} = \alpha_2 y^*, \qquad \Gamma_{22} = \alpha_2 x^*, \qquad \Gamma_{23} = \alpha_2. \end{aligned}$$

Now, let

$$\begin{aligned} \eta &= \frac{2\alpha_2 - \alpha_1^*}{2\alpha_2}, \\ \zeta &= \frac{1}{2}\sqrt{\frac{4\alpha_1^*(\alpha_2 - 2)}{\alpha_2} - \left(\frac{2\alpha_2 - \alpha_1^*}{\alpha_2}\right)^2}, \end{aligned}$$

and the invertible matrix T defined by

$$T = \begin{pmatrix} \Gamma_{12} & 0 \\ \eta - \Gamma_{11} & -\zeta \end{pmatrix}.$$

Using the following translation:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} \Gamma_{12} & 0 \\ \eta - \Gamma_{11} & -\zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

(9) gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \eta & -\zeta \\ \zeta & \eta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \Phi(X_n, Y_n) \\ \Psi(X_n, Y_n) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} \Phi(X_n, Y_n) &= \Pi_{11} {X_n}^2 + \Pi_{12} X_n Y_n, \\ \Psi(X_n, Y_n) &= \Pi_{21} {X_n}^2 + \Pi_{22} X_n Y_n, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \Pi_{11} &= \Gamma_{12}\Gamma_{13} + \Gamma_{14}(\eta - \Gamma_{11}), \\ \Pi_{12} &= -\zeta \Gamma_{14}, \\ \Pi_{21} &= \frac{\eta - \Gamma_{11}}{\zeta}\big[\Gamma_{12}(\Gamma_{13} - \Gamma_{23}) + \Gamma_{14}(\eta - \Gamma_{11})\big], \\ \Pi_{22} &= \Gamma_{12}\Gamma_{23} - \Gamma_{14}(\eta - \Gamma_{11})). \end{aligned}$$

In addition,

$$\begin{aligned} &\Phi_{X_n X_n}|_{(0,0)} = 2\Pi_{11}, \qquad \Phi_{X_n Y_n}|_{(0,0)} = \Pi_{12}, \qquad \Phi_{Y_n Y_n}|_{(0,0)} = 0, \\ &\Phi_{X_n X_n X_n}|_{(0,0)} = \Phi_{X_n X_n Y_n}|_{(0,0)} = \Phi_{X_n Y_n Y_n}|_{(0,0)} = \Phi_{Y_n Y_n Y_n}|_{(0,0)} = 0, \end{aligned}$$

and

$$\begin{aligned} &\Psi_{X_n X_n}|_{(0,0)} = 2\Pi_{21}, \qquad \Psi_{X_n Y_n}|_{(0,0)} = \Pi_{22}, \qquad \Psi_{Y_n Y_n}|_{(0,0)} = 0, \\ &\Psi_{X_n X_n X_n}|_{(0,0)} = \Psi_{X_n X_n Y_n}|_{(0,0)} = \Psi_{X_n Y_n Y_n}|_{(0,0)} = \Psi_{Y_n Y_n Y_n}|_{(0,0)} = 0. \end{aligned}$$

In order for (10) to undergo a Neimark–Sacker bifurcation, it is mandatory that the following discriminatory quantity, i.e., $\chi \neq 0$ (see [22–32]),

$$\chi = -\mathrm{Re}\left[\frac{(1 - 2\bar{\kappa})\bar{\kappa}^2}{1 - \kappa}\tau_{11}\tau_{20}\right] - \frac{1}{2}\|\tau_{11}\|^2 - \|\tau_{02}\|^2 + \mathrm{Re}(\bar{\kappa}\tau_{21}), \tag{12}$$

where

$$\begin{aligned}
\tau_{02} &= \frac{1}{8}\big[\Phi_{X_nX_n} - \Phi_{Y_nY_n} + 2\Psi_{X_nY_n} + \iota(\Psi_{X_nX_n} - \Psi_{Y_nY_n} + 2\Phi_{X_nY_n})\big]\big|_{(0,0)},\\
\tau_{11} &= \frac{1}{4}\big[\Phi_{X_nX_n} + \Phi_{Y_nY_n} + \iota(\Psi_{X_nX_n} + \Psi_{Y_nY_n})\big]\big|_{(0,0)},\\
\tau_{20} &= \frac{1}{8}\big[\Phi_{X_nX_n} - \Phi_{Y_nY_n} + 2\Psi_{X_nY_n} + \iota(\Psi_{X_nX_n} - \Psi_{Y_nY_n} - 2\Phi_{X_nY_n})\big]\big|_{(0,0)},\\
\tau_{21} &= \frac{1}{16}\big[\Phi_{X_nX_nX_n} + \Phi_{X_nY_nY_n} + \Psi_{X_nX_nY_n} + \Psi_{Y_nY_nY_n}\\
&\quad + \iota(\Psi_{X_nX_nX_n} + \Psi_{X_nY_nY_n} - \Phi_{X_nX_nY_n} - \Phi_{Y_nY_nY_n})\big]\big|_{(0,0)}.
\end{aligned} \tag{13}$$

After calculating, we get

$$\begin{aligned}
\tau_{02} &= \frac{1}{4}\big[\Pi_{11} + \Pi_{22} + \iota(\Pi_{21} + \Pi_{12})\big],\\
\tau_{11} &= \frac{1}{2}[\Pi_{11} + \iota\Pi_{21}],\\
\tau_{20} &= \frac{1}{4}\big[\Pi_{11} + \Pi_{22} + \iota(\Pi_{21} - \Pi_{12})\big],\\
\tau_{21} &= 0.
\end{aligned} \tag{14}$$

Based on this analysis and the Neimark–Sacker bifurcation theorem discussed in [30, 31], we arrive at the following theorem.

Theorem 1 *If $\chi \neq 0$ then the discrete-time model (4) undergoes a Neimark–Sacker bifurcation about $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ as (α_1, α_2) go through $N_{B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$. Additionally, an attracting (resp. repelling) closed curve bifurcates from $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ if $\chi < 0$ (resp. $\chi > 0$).*

Remark According to bifurcation theory discussed in [30, 31], the bifurcation is called a supercritical Neimark–Sacker bifurcation if the discriminatory quantity $\chi < 0$. In the following section, numerical simulations guarantee that a supercritical Neimark–Sacker bifurcation occurs for the discrete-time model (4).

4.2 Period-doubling bifurcation about $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$

This section deals with the study of period-doubling bifurcation of the discrete-time model (4) about the unique positive equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. Consider α_1^* as a bifurcation parameter, then the discrete-time model (4) becomes

$$\left.\begin{aligned}
x_{n+1} &= \big(\alpha_1 + \alpha_1^*\big)x_n(1 - x_n) - x_ny_n,\\
y_{n+1} &= \alpha_2x_ny_n,
\end{aligned}\right\} \tag{15}$$

where $\alpha_1^* \ll 1$. Let $u_n = x_n - x^*$, $v_n = y_n - y^*$. Then we transformed $B(x^*, y^*)$, where $x^* = \frac{1}{\alpha_2}$, $y^* = \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2}$ of (15) into origin. By calculating we get

$$\left.\begin{aligned}
u_{n+1} &= \widehat{\Gamma_{11}}u_n + \widehat{\Gamma_{12}}v_n + \widehat{\Gamma_{13}}u_n^2 + \widehat{\Gamma_{14}}u_nv_n + \Upsilon_{01}u_n\alpha_1^* + \Upsilon_{02}u_n^2\alpha_1^*,\\
v_{n+1} &= \widehat{\Gamma_{21}}u_n + \widehat{\Gamma_{22}}v_n + \widehat{\Gamma_{23}}u_nv_n,
\end{aligned}\right\} \tag{16}$$

where

$$\widehat{\Gamma_{11}} = \alpha_1\left(1-2x^*\right) - y^*, \qquad \widehat{\Gamma_{12}} = -x^*, \qquad \widehat{\Gamma_{13}} = -\alpha_1^*, \qquad \Gamma_{14} = -1,$$
$$\Upsilon_{01} = 1-2x^*, \qquad \Upsilon_{02} = -1,$$
$$\widehat{\Gamma_{21}} = \alpha_2 y^*, \qquad \widehat{\Gamma_{22}} = \alpha_2 x^*, \qquad \widehat{\Gamma_{23}} = \alpha_2.$$

Now, construct an invertible matrix T

$$T = \begin{pmatrix} \widehat{\Gamma_{12}} & \widehat{\Gamma_{12}}, \\ 1-\widehat{\Gamma_{11}} & \kappa_2-\widehat{\Gamma_{11}} \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} \widehat{\Gamma_{12}} & \widehat{\Gamma_{12}} \\ 1-\widehat{\Gamma_{11}} & \kappa_2-\widehat{\Gamma_{11}} \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

(16) gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}(u_n, v_n, \alpha_1^*) \\ \widehat{\Psi}(u_n, v_n, \alpha_1^*) \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned}
\widehat{\Phi}\left(u_n, v_n, \alpha_1^*\right) &= \frac{\widehat{\Gamma_{13}}(\kappa_2-\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n^2 + \frac{\widehat{\Gamma_{14}}(\kappa_2-\widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n v_n \\
&\quad + \frac{\Upsilon_{01}(\kappa_2-\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n\alpha_1^* + \frac{\Upsilon_{02}(\kappa_2-\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n^2\alpha_1^*, \\
\widehat{\Psi}\left(u_n, v_n, \alpha_1^*\right) &= \frac{\widehat{\Gamma_{13}}(1+\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n^2 + \frac{\widehat{\Gamma_{14}}(1+\widehat{\Gamma_{11}}) + \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n v_n \\
&\quad + \frac{\Upsilon_{01}(1+\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n\alpha_1^* + \frac{\Upsilon_{02}(1+\widehat{\Gamma_{11}})}{\widehat{\Gamma_{12}}(1+\kappa_2)}u_n^2\alpha_1^*, \qquad (18) \\
u_n^2 &= \widehat{\Gamma_{12}}^2\left(X_n^2 + 2X_nY_n + Y_n^2\right), \\
u_n v_n &= -\widehat{\Gamma_{12}}(1+\widehat{\Gamma_{11}})X_n^2 + \left(\widehat{\Gamma_{12}}(\kappa_2-\widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}(1+\widehat{\Gamma_{11}})\right)X_nY_n \\
&\quad + \widehat{\Gamma_{12}}(\kappa_2-\widehat{\Gamma_{11}})Y_n^2, \\
u_n\alpha_1^* &= \widehat{\Gamma_{12}}X_n\alpha_1^* + \widehat{\Gamma_{12}}Y_n\alpha_1^*, \\
u_n^2\alpha_1^* &= \widehat{\Gamma_{12}}^2\left(X_n^2\alpha_1^* + 2X_nY_n\alpha_1^* + Y_n^2\alpha_1^*\right).
\end{aligned}$$

Hereafter we determine the center manifold $W^c(0,0)$ of (17) about $(0,0)$ in a small neighborhood of α_1^*. By center manifold theorem, there exists a center manifold $W^c(0,0)$ that can be represented as follows:

$$W^c(0,0) = \left\{(X_n, Y_n) : Y_n = c_0\alpha_1^* + c_1X_n^2 + c_2X_n\alpha_1^* + c_2\alpha_1^{*3} + O\left(\left(|X_n| + \left|\alpha_1^*\right|\right)^3\right)\right\},$$

where $O((|X_n| + |\alpha_1^*|)^3)$ is a function with order at least three in their variables (X_n, α_1^*), and

$$\begin{aligned} c_0 &= 0, \\ c_1 &= \frac{(1+\widehat{\Gamma_{11}})[\widehat{\Gamma_{12}}\widehat{\Gamma_{13}} - \widehat{\Gamma_{14}}(1+\widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}]}{1-\kappa_2^2}, \\ c_2 &= \frac{\Upsilon_{01}(1+\widehat{\Gamma_{11}})}{1-\kappa_2^2}, \\ c_3 &= 0. \end{aligned} \tag{19}$$

Therefore, we consider the map (17) restricted to $W^c(0,0)$ as follows:

$$f(x_n) = -x_n + h_1 x_n^2 + h_2 x_n \alpha_1^* + h_3 x_n^2 \alpha_1^* + h_4 x_n \alpha_1^{*2} + h_5 x_n^3 + O\big(\big(|X_n| + |\alpha_1^*|\big)^4\big), \tag{20}$$

where

$$\begin{aligned} h_1 &= \frac{1}{1+\kappa_2}\big[\widehat{\Gamma_{12}}\widehat{\Gamma_{13}}(\kappa_2 - \widehat{\Gamma_{11}}) - (1+\widehat{\Gamma_{11}})\big(\widehat{\Gamma_{14}}(\kappa_2 - \widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}\big)\big], \\ h_2 &= \frac{1}{1+\kappa_2}\big[\Upsilon_{01}(\kappa_2 - \widehat{\Gamma_{11}})\big], \\ h_3 &= \frac{1}{1+\kappa_2}\big[(2c_2\widehat{\Gamma_{12}}\widehat{\Gamma_{13}} + \Upsilon_{01}c_1 + \Upsilon_{02}\widehat{\Gamma_{12}})(\kappa_2 - \widehat{\Gamma_{11}}) \\ &\quad + c_2\big(\widehat{\Gamma_{14}}(\kappa_2 - \widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}\big)(\kappa_2 - 2\widehat{\Gamma_{11}} - 1)\big], \\ h_4 &= \frac{1}{1+\kappa_2}\big[\Upsilon_{01}c_2(\kappa_2 - \widehat{\Gamma_{11}})\big], \\ h_5 &= \frac{c_1}{1+\kappa_2}\big[2\widehat{\Gamma_{12}}\widehat{\Gamma_{13}}(\kappa_2 - \widehat{\Gamma_{11}}) + \big(\widehat{\Gamma_{14}}(\kappa_2 - \widehat{\Gamma_{11}}) - \widehat{\Gamma_{12}}\widehat{\Gamma_{23}}\big)(\kappa_2 - 2\widehat{\Gamma_{11}} - 1)\big]. \end{aligned} \tag{21}$$

In order for the map (20) to undergo a period-doubling bifurcation, we require that the following discriminatory quantities are non-zero:

$$\begin{aligned} \Lambda_1 &= \left(\frac{\partial^2 f}{\partial x_n \partial \alpha_1^*} + \frac{1}{2}\frac{\partial f}{\partial \alpha_1^*}\frac{\partial^2 f}{\partial x_n^2}\right)\bigg|_{(0,0)}, \\ \Lambda_2 &= \left(\frac{1}{6}\frac{\partial^3 f}{\partial x_n^3} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_n^2}\right)^2\right)\bigg|_{(0,0)}. \end{aligned}$$

After calculating we get

$$\Lambda_1 = \frac{(\alpha_2 - 2)(15 - 7\alpha_2)}{\alpha_2(9 - 4\alpha_2)} \neq 0$$

and

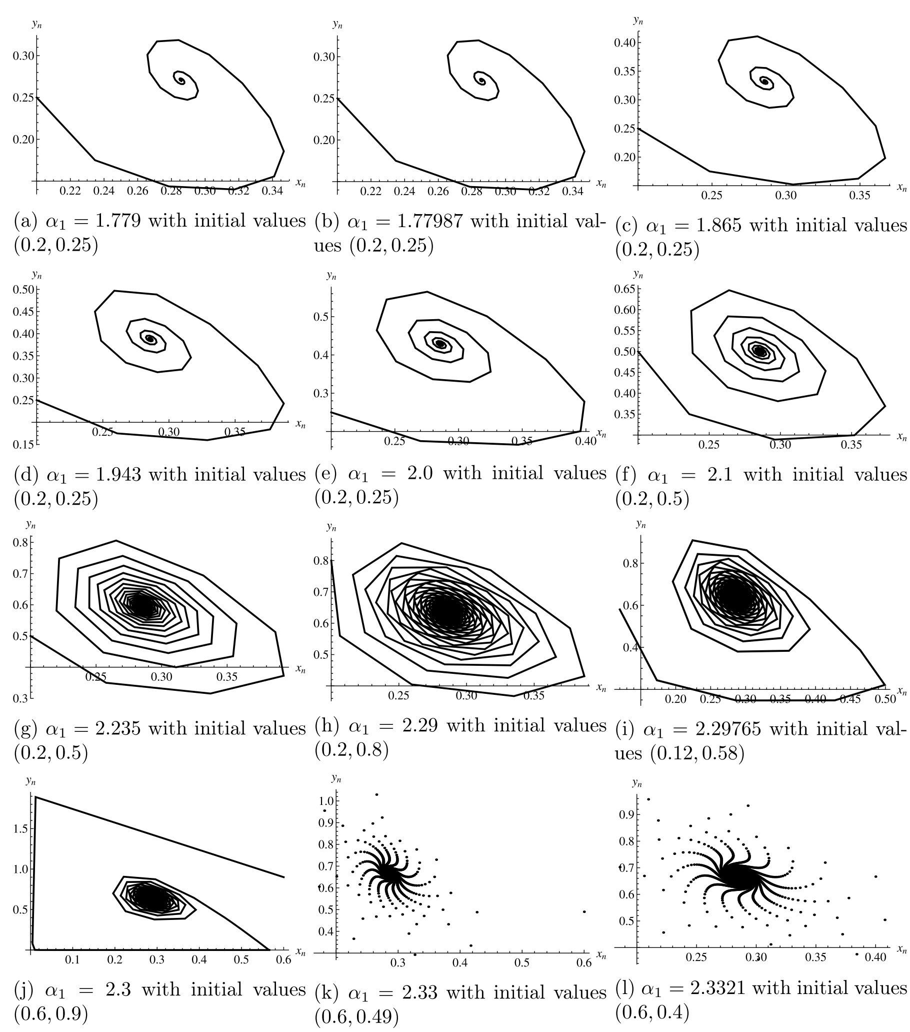
$$\Lambda_2 = \frac{4\alpha_2^2 - 18\alpha_2 + 27}{(9 - 4\alpha_2)^2}.$$

From the above analysis and Theorem in [30, 31], we have the following theorem.

Theorem 2 *If $\varLambda_2 \neq 0$, the map* (15) *undergoes a period-doubling bifurcation about the unique positive equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ when α_1^* varies in a small neighborhood of $O(0,0)$. Moreover, if $\varLambda_2 > 0$ (resp. $\varLambda_2 < 0$), then the period-2 points that bifurcate from $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ are stable (resp. unstable).*

5 Numerical simulations

In this section, some simulations are given to verify the obtained results. If $\alpha_2 = 3.5$ then from non-hyperbolic condition $\alpha_1 = \frac{\alpha_2}{\alpha_2-2}$ of Lemma 4 one gets $\alpha_1 = 2.33333$. From a theoretical point of view the equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ is a locally asymptotically stable focus if $\alpha_1 < 2.33333$. To see this if $\alpha_1 = 1.779 < 2.33333$, then it is clear from Fig. 1(a) that the equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ is a locally asymptotically stable focus. That means that all the orbits attract towards the unique positive equilibrium. Similarly for the other values of the parameter α_1, it is also observed that the equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ of the discrete-time model (4) is a locally asymptotical focus (see Figs. 1(b)–1(l)). But when α_1 goes



(a) $\alpha_1 = 1.779$ with initial values $(0.2, 0.25)$
(b) $\alpha_1 = 1.77987$ with initial values $(0.2, 0.25)$
(c) $\alpha_1 = 1.865$ with initial values $(0.2, 0.25)$
(d) $\alpha_1 = 1.943$ with initial values $(0.2, 0.25)$
(e) $\alpha_1 = 2.0$ with initial values $(0.2, 0.25)$
(f) $\alpha_1 = 2.1$ with initial values $(0.2, 0.5)$
(g) $\alpha_1 = 2.235$ with initial values $(0.2, 0.5)$
(h) $\alpha_1 = 2.29$ with initial values $(0.2, 0.8)$
(i) $\alpha_1 = 2.29765$ with initial values $(0.12, 0.58)$
(j) $\alpha_1 = 2.3$ with initial values $(0.6, 0.9)$
(k) $\alpha_1 = 2.33$ with initial values $(0.6, 0.49)$
(l) $\alpha_1 = 2.3321$ with initial values $(0.6, 0.4)$

Figure 1 Phase portraits of the discrete-time model (4) through 2.33333, equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ of the discrete-time model (4) is unstable focus. Meanwhile an attracting closed invariant curve bifurcates from $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ of the discrete-time model (4). In particular the existence of an attracting closed invariant curve implies that the discrete-time model (4) undergoes a supercritical Neimark–Sacker bifurcation about $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. To see this if $\alpha_1 = 2.34 > 2.33333$ then the eigenvalues of $J_{B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})}$ about $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ are

$$\kappa_{1,2} = 0.6657142857142857 \pm 0.7481187289816109\iota, \tag{22}$$

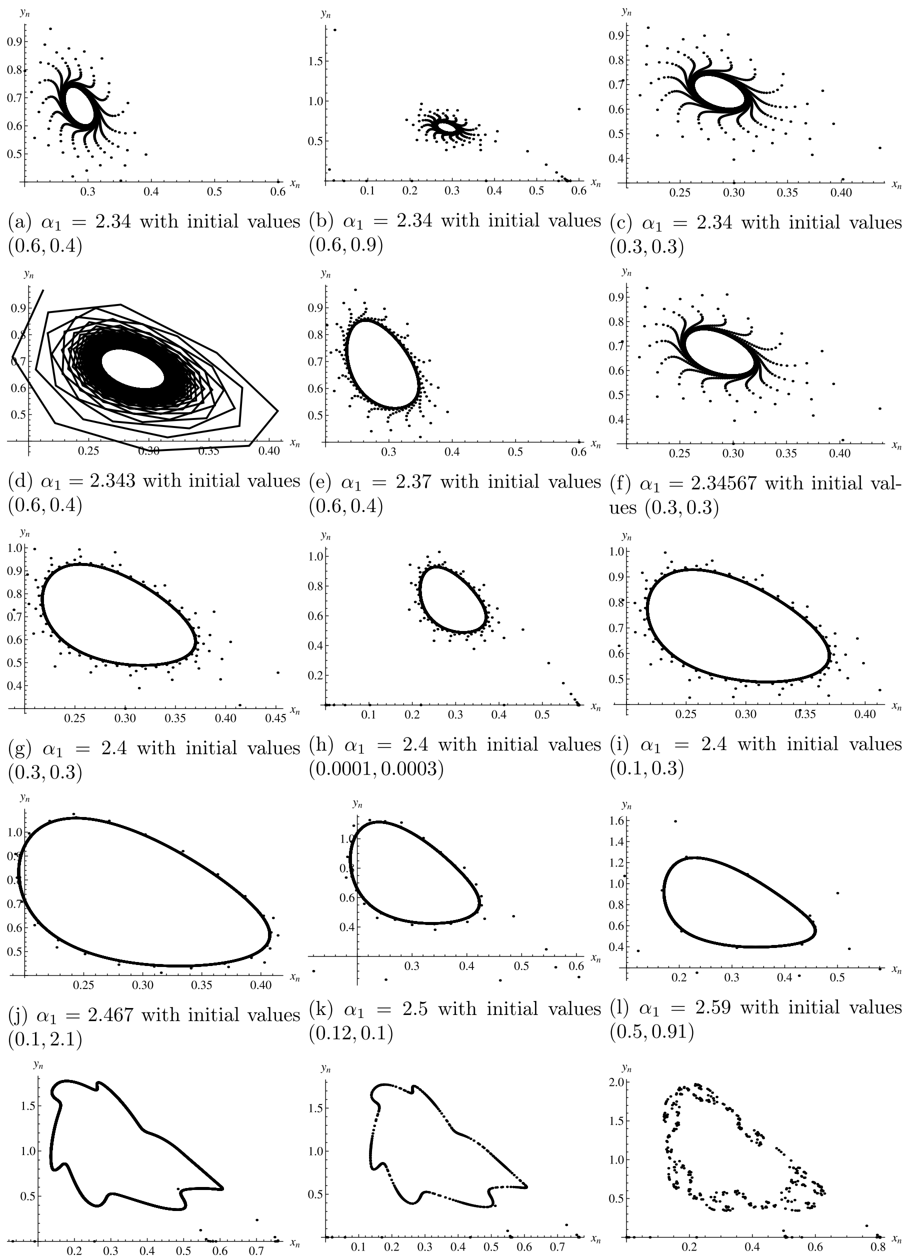
and a non-degenerate condition for the existence of Neimark–Sacker bifurcation of the discrete-time model (4), i.e., $\frac{\alpha_2-2}{2\alpha_2} = 0.5341880341880343 > 0$ hold. Moreover, after some manipulations from Mathematica one gets

$$\begin{aligned} \tau_{02} &= -0.08285714285714288 + 0.33608110810276415\iota, \\ \tau_{11} &= 0.1671428571428572 + 0.29810285171472284\iota, \\ \tau_{20} &= -0.08285714285714288 - 0.03797825638804131\iota, \\ \tau_{21} &= 0. \end{aligned} \tag{23}$$

In view of (22) and (23) the value of the discriminatory quantity from (12) is $\chi = -0.11221718590370466 < 0$. Therefore if $\alpha_1 = 2.34 > 2.33333$ then the discrete-time model (4) undergoes a supercritical Neimark–Sacker bifurcation and in fact a stable invariant close curve appears, which is presented in Fig. 2(a). Similarly for other choices of bifurcation parameter the value of discriminatory quantity is less than 0 (see Table 1) and their corresponding attracting close invariant curves are depicted in Figs. 2(b)–2(o). In the context of biology, attracting closed invariant curve bifurcations from the supercritical Neimark–Sacker bifurcation imply that host and parasitoid populations will coexist under periodic or quasi-periodic oscillations with long time.

Hereafter we will provide the numerical simulation in order to verify the theoretical results obtained in Sect. 4.2 by fixing $\alpha_2 = 1.53$ and varying $1.5 \le \alpha_1 \le 18.5$. Fixing $\alpha_2 = 1.53$, then from the non-hyperbolic condition (iii) of Lemma 5 one gets $\alpha_1 = 3.1224489795918364$. From a theoretical point of view the unique positive equilibrium point $(0.6535947712418301, 0.08163300653594788)$ of (4) is stable if $\alpha_1 < 3.1224489795918364$; bifurcation occurs if $\alpha_1 = 3.1224489795918364$, and there is a period-doubling bifurcation if $\alpha_1 > 3.1224489795918364$.

From Figs. 3(a)–3(b), we see that the equilibrium point is stable if $\alpha_1 < 3.1224489795918364$, and loses its stability at the period-doubling bifurcation parameter value $\alpha_1 = 3.1224489795918364$. The maximum Lyapunov exponents corresponding to Figs. 3(a)–3(b) are plotted in Fig. 3(c). Moreover, $3D$ bifurcation diagrams are also plotted in Figs. 4(a)–4(c). The phase portraits which are associated with Figs. 3(a)–3(b) are depicted in Figs. 5(a)–5(f), which indicates that the discrete-time model (4) exhibits a complex dynamics such as period-2, -4, -11, -13, -15 and -22 orbits.



(a) $\alpha_1 = 2.34$ with initial values $(0.6, 0.4)$ (b) $\alpha_1 = 2.34$ with initial values $(0.6, 0.9)$ (c) $\alpha_1 = 2.34$ with initial values $(0.3, 0.3)$

(d) $\alpha_1 = 2.343$ with initial values $(0.6, 0.4)$ (e) $\alpha_1 = 2.37$ with initial values $(0.6, 0.4)$ (f) $\alpha_1 = 2.34567$ with initial values $(0.3, 0.3)$

(g) $\alpha_1 = 2.4$ with initial values $(0.3, 0.3)$ (h) $\alpha_1 = 2.4$ with initial values $(0.0001, 0.0003)$ (i) $\alpha_1 = 2.4$ with initial values $(0.1, 0.3)$

(j) $\alpha_1 = 2.467$ with initial values $(0.1, 2.1)$ (k) $\alpha_1 = 2.5$ with initial values $(0.12, 0.1)$ (l) $\alpha_1 = 2.59$ with initial values $(0.5, 0.91)$

(m) $\alpha_1 = 3.1$ with initial values $(0.03, 0.001)$ (n) $\alpha_1 = 3.1$ with initial values $(0.002, 0.001)$ (o) $\alpha_1 = 3.22$ with initial values $(0.0000002, 0.000001)$

Figure 2 Phase portraits of the discrete-time model (4)

5.1 Fractal dimension

The fractal dimension which characterized the strange attractors of the discrete-time system is defined by (see [33, 34])

$$d_L = j + \frac{\sum_{i=1}^{j} \kappa_j}{|\kappa_j|}, \tag{24}$$

Table 1 Numerical values of χ for $\alpha_1 > 2.33333$

Value of bifurcation parameter when $\alpha_1 > 2.33333$	Numerical value of χ
2.34	−0.11221718590370466 < 0
2.343	−0.17385157491789316 < 0
2.37	−0.17587912984797172 < 0
2.34567	−0.23250816667811489 < 0
2.4	−0.23591771770576037 < 0
2.467	−0.24392011825143836 < 0
2.5	−0.24740313673037914 < 0
2.59	−0.25764617193199524 < 0
3.1	−0.3335331833515125 < 0
3.22	−0.35545528698159623 < 0

where $\kappa_1, \kappa_2, \ldots, \kappa_n$ are Lyapunov exponents and j is the largest integer such that $\sum_{i=1}^{j} \kappa_j \geq 0$ and $\sum_{i=1}^{j+1} \kappa_j < 0$. For our under consideration discrete-time model (4), the fractal dimension takes the following form:

$$d_L = 1 + \frac{\kappa_1}{|\kappa_2|}, \qquad \kappa_1 > 0 > \kappa_2. \tag{25}$$

If $\alpha_2 = 1.53$ then after some manipulation two Lyapunov exponents are $\kappa_1 = 1.3153221370247266$ (resp. $\kappa_1 = 1.2928270741698256$) and $\kappa_2 = -0.3806816141489094$ (resp. $\kappa_2 = -0.40393818528093656$) for $\alpha_1 = 1.63$ (resp. $\alpha_1 = 1.7$). So the fractal dimension for the discrete-time model (4) is

$$\begin{aligned} d_L &= 1 + \frac{1.3153221370247266}{0.3806816141489094} \\ &= 4.4551764207614 66 \quad \text{for } \alpha_1 = 1.63, \\ d_L &= 1 + \frac{1.2928270741698256}{0.40393818528093656} \\ &= 4.200556721991193 \quad \text{for } \alpha_1 = 1.7. \end{aligned} \tag{26}$$

The strange attractors for the above fixed parametric values are also plotted and presented in Figs. 6(a)–6(b), which illustrate that the discrete model (4) has a complex dynamical behavior as the parameter α_1 increases.

6 Chaos control

In this section, we will study the chaos control by applying the state feedback control method [35, 36]. By adding a feedback control law as the control force u_n to the discrete-time model (4), the controlled model (4) takes the following form:

$$\left.\begin{aligned} x_{n+1} &= \alpha_1 x_n(1 - x_n) - x_n y_n + u_n, \\ y_{n+1} &= \alpha_2 x_n y_n, \\ u_n &= -k_1\bigl(x_n - x^*\bigr) - k_2\bigl(y_n - y^*\bigr), \end{aligned}\right\} \tag{27}$$

where the feedback gains are denoted by k_1 and k_2 and (x^*, y^*) is the unique positive equilibrium point, i.e., $(x^*, y^*) = B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2 - \alpha_1 - \alpha_2}{\alpha_2})$ of the discrete-time model (4). The Jacobian

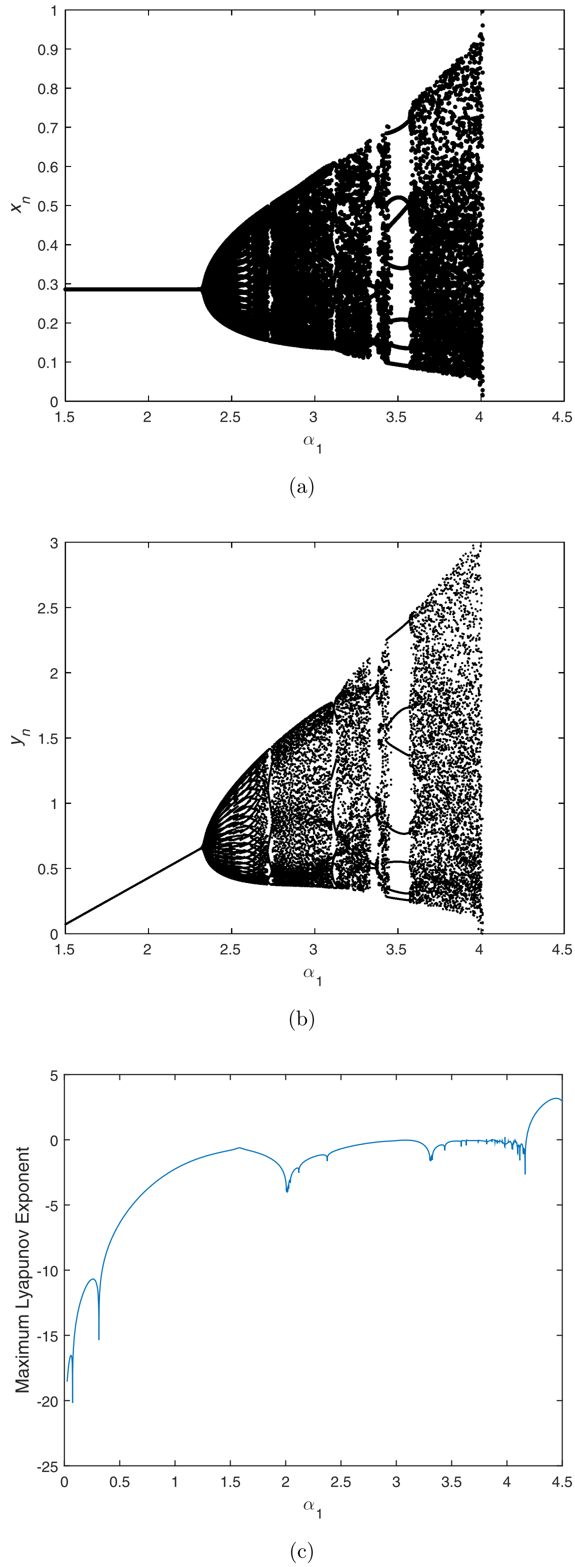


Figure 3 (**a**), (**b**) Bifurcation diagram of the discrete-time model (4) with $\alpha_1 \in [1.5, 18.5]$, $\alpha_2 = 1.53$ and initial value is $(0.2, 0.15)$. (**b**) Maximum Lyapunov exponent corresponding to (**a**) and (**b**)

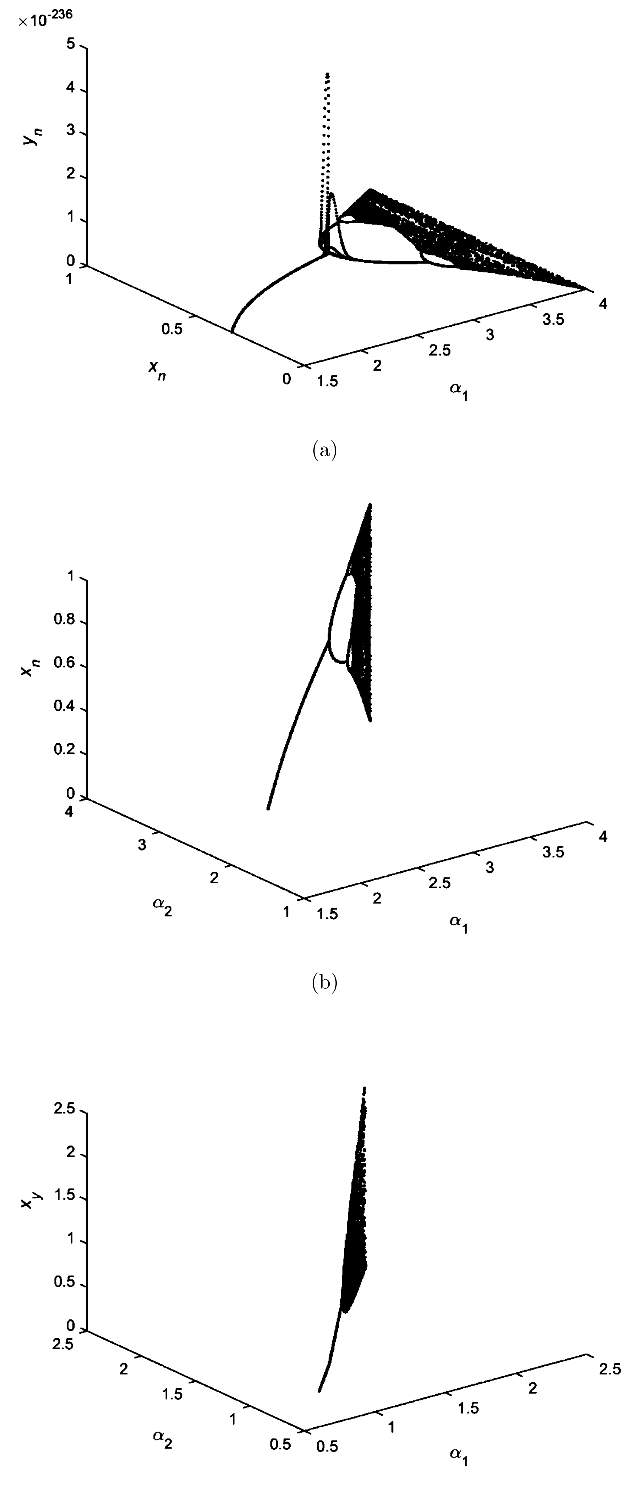


Figure 4 Bifurcation diagram in $3D$ for $\alpha_1 \in [1.5, 18.5]$, $\alpha_2 = 1.53$ and initial value is $(0.2, 0.15)$

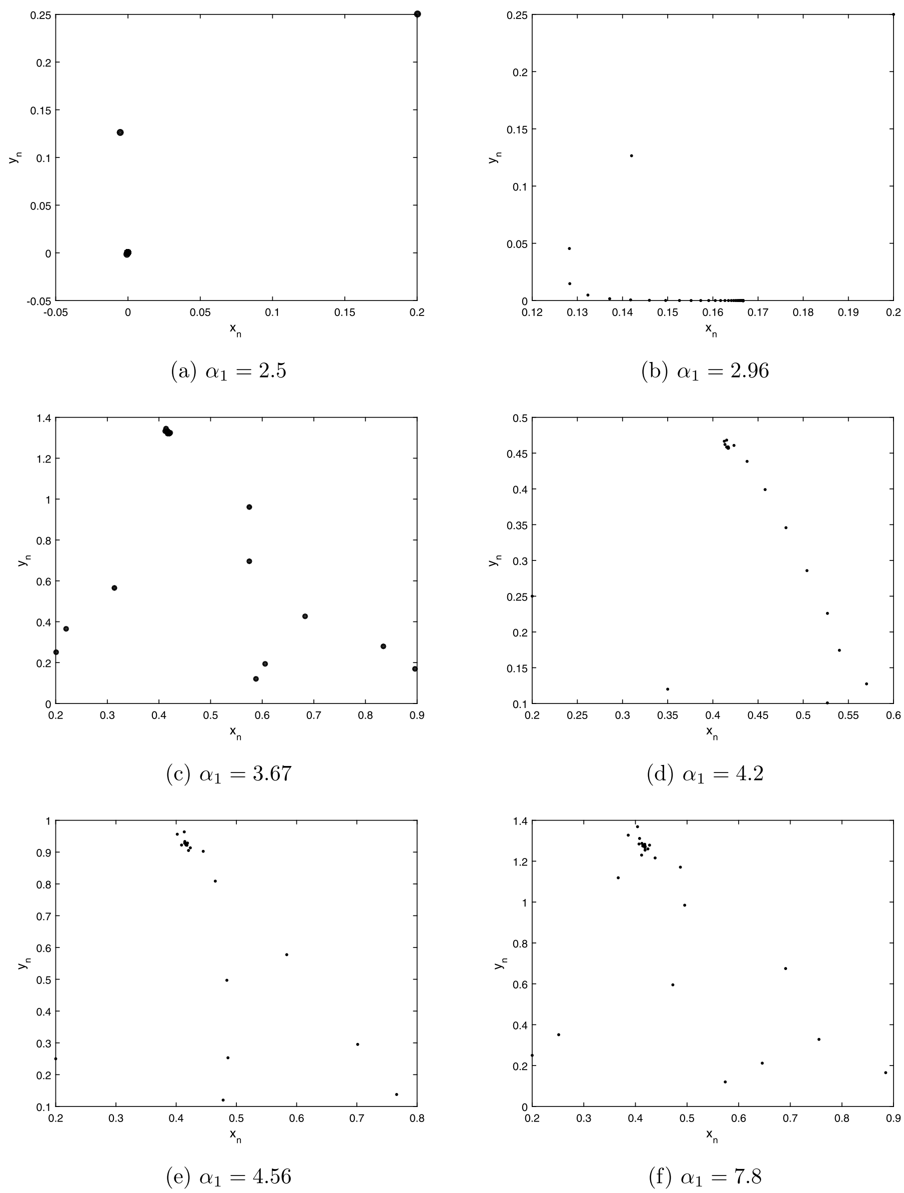


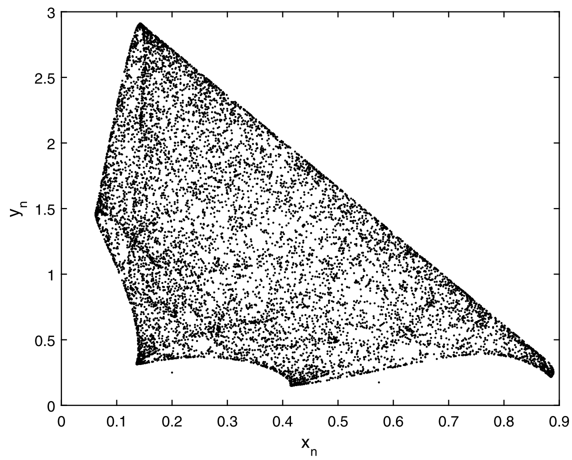
Figure 5 Phase portraits for values of α_1 corresponding to Figs. 3(a)–3(b)

matrix J_c of the controlled system (27) is

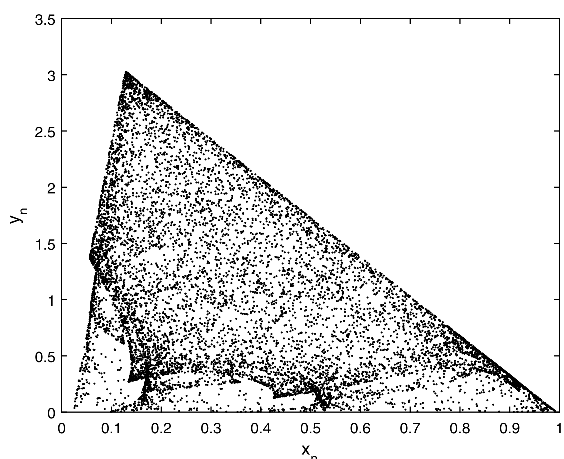
$$J_c(x^*, y^*) = \begin{pmatrix} a_{11} - k_1 & a_{12} - k_2 \\ a_{21} & a_{22} \end{pmatrix}, \tag{28}$$

where $a_{11} = \frac{\alpha_2 - \alpha_1}{\alpha_2}$, $a_{12} = -\frac{1}{\alpha_2}$, $a_{21} = \alpha_1\alpha_2 - \alpha_1 - \alpha_2$, $a_{22} = 1$. The characteristic equation of $J_c(x^*, y^*)$ about (x^*, y^*) is

$$\kappa^2 - \operatorname{tr}\big(J_c(x^*, y^*)\big)\kappa + \det\big(J_c(x^*, y^*)\big) = 0, \tag{29}$$



(a)



(b)

Figure 6 Strange attractor of the discrete-time model (4) for $\alpha_1 = 1.63$ (resp. $\alpha_1 = 1.7$) with (0.2, 0.25)

where

$$\left.\begin{aligned} \operatorname{tr}\big(J_c(x^*, y^*)\big) &= a_{11} + a_{22} - k_1, \\ \det\big(J_c(x^*, y^*)\big) &= a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2). \end{aligned}\right\} \tag{30}$$

Let κ_1 and κ_2 be the roots of (29) then

$$\kappa_1 + \kappa_2 = a_{11} + a_{22} - k_1, \tag{31}$$

$$\kappa_1\kappa_2 = a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2). \tag{32}$$

The solutions of the equations $\kappa_1 = \pm 1$ and $\kappa_1\kappa_2 = 1$ determine the lines of marginal stability. These conditions confirm that $|\kappa_{1,2}| < 1$. Suppose that $\kappa_1\kappa_2 = 1$, then from (32) one gets

$$l_1 : \frac{\alpha_2 - \alpha_1}{\alpha_2} - k_1 + (\alpha_1\alpha_2 - \alpha_1 - \alpha_2)\left(\frac{1}{\alpha_2} + 1\right) - 1 = 0. \tag{33}$$

Now assuming that $\kappa_1 = 1$ then from (31) and (32) one gets

$$l_2 : (\alpha_1\alpha_2 - \alpha_1 - \alpha_2)\left(k_2 + \frac{1}{\alpha_2}\right) = 0. \tag{34}$$

Finally, assume that $\kappa_1 = -1$, then again from (31) and (32) one gets

$$l_3 : 2k_1 - (\alpha_1\alpha_2 - \alpha_1 - \alpha_2)\left(k_2 + \frac{1}{\alpha_2}\right) - 2\left(1 + \frac{\alpha_2 - \alpha_1}{\alpha_2}\right) = 0. \tag{35}$$

Then the lines l_1, l_2 and l_3 in the (k_1, k_2) plane determine a triangular region which gives $|\kappa_{1,2}| < 1$ (see Fig. 7(a)).

In order to check how the implementation of feedback control method works and how to control chaos at an unstable state, we have performed numerical simulations. Figures 7(b)–7(c) show that about the unique positive equilibrium the chaotic trajectories are stabilized.

7 Conclusion

This work deals with the study of local dynamics, bifurcations and chaos control of a discrete-time predator–prey model (4) in $\mathbb{R}^2_+$. It is proved that the model has the boundary equilibria $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1}, 0)$ and a unique positive equilibrium $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$. We have investigated local stability along their topological types about $O(0,0)$, $A(\frac{\alpha_1-1}{\alpha_1}, 0)$, $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ by imposing the method of linearization and conclusions are presented in Table 2. We proved that about $A(\frac{\alpha_1-1}{\alpha_1}, 0)$ there exists a fold bifurcation when the parameters of the discrete model (4) are located in the following set:

$$F_{A(\frac{\alpha_1-1}{\alpha_1},0)} = \left\{(\alpha_1, \alpha_2) : \alpha_2 = \frac{\alpha_1}{\alpha_1 - 1}, \alpha_1, \alpha_2 > 0\right\}.$$

We have also shown that about $B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})$ the discrete-time model (4) undergoes both a Neimark–Sacker bifurcation and a period-doubling bifurcation when the parameters, respectively, are located in the following sets:

$$\begin{aligned} &N_{B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})} \\ &\quad = \left\{(\alpha_1, \alpha_2) : \left(\frac{2\alpha_2 - \alpha_1}{\alpha_2}\right)^2 - \frac{4\alpha_1(\alpha_2 - 2)}{\alpha_2} < 0 \text{ and } \alpha_1 = \frac{\alpha_2}{\alpha_2 - 2}\right\}, \\ &P_{B(\frac{1}{\alpha_2}, \frac{\alpha_1\alpha_2-\alpha_1-\alpha_2}{\alpha_2})} \\ &\quad = \left\{(\alpha_1, \alpha_2) : \left(\frac{2\alpha_2 - \alpha_1}{\alpha_2}\right)^2 - \frac{4\alpha_1(\alpha_2 - 2)}{\alpha_2} \geq 0 \text{ and } \alpha_1 = \frac{3\alpha_2}{3 - \alpha_2}\right\}. \end{aligned}$$

Numerical simulations have verified not only the theoretical results but also exhibited a complex dynamical behavior such as the period-2, -4, -11, -13, -15 and -22 orbits. Further, we have computed maximum Lyapunov exponents numerically. Finally, the feedback control method is applied to stabilize chaos existing in the discrete-time model (4).

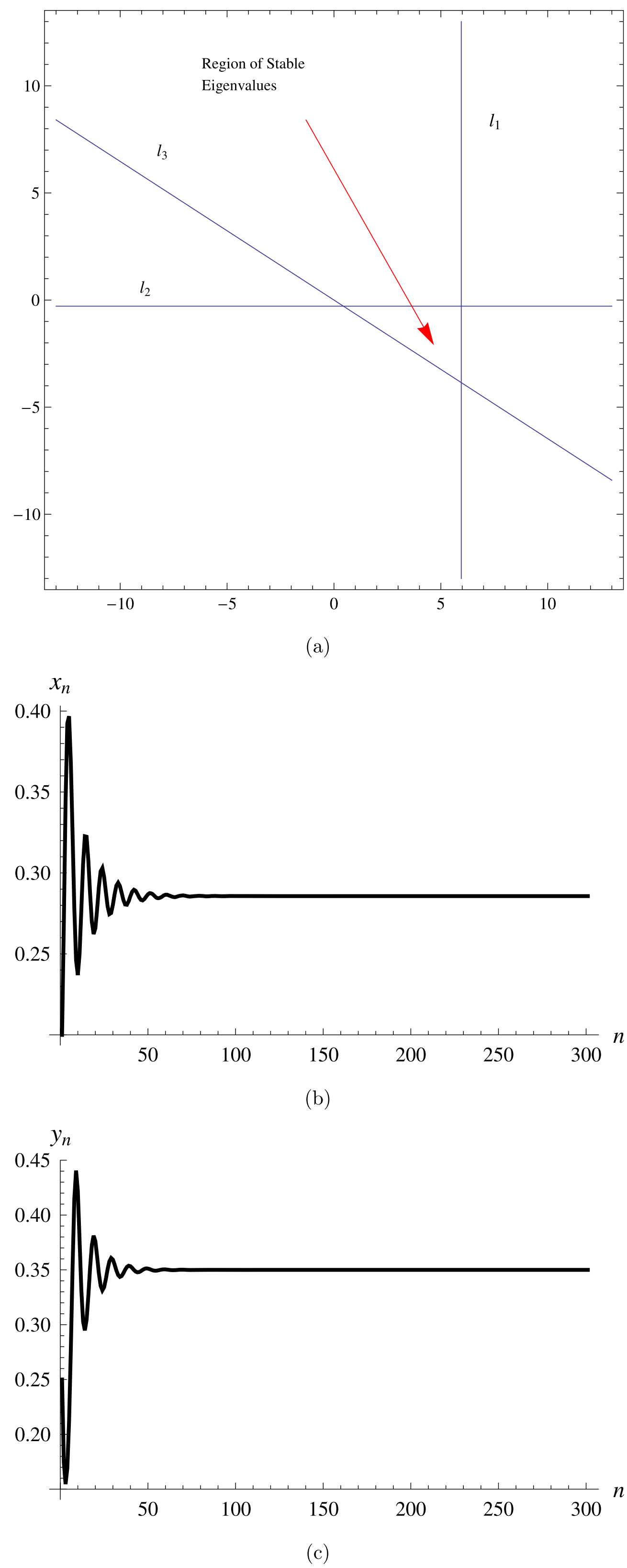


Figure 7 Control of chaotic trajectories of the controlled discrete-time model (27) for $\alpha_1 = 1.53$, $\alpha_2 = 3.75$ with initial values $(0.2, 0.25)$ (**a**) stability region in (k_1, k_2)-plan. (**b**)–(**c**) Time series for states x_n and y_n, respectively

Table 2 Number of equilibria along their qualitative behavior of the discrete-time model (4)

E.P	Corresponding behavior
O	sink if $0<\alpha_1<1$; never source; saddle if $\alpha_1>1$; non-hyperbolic if $\alpha_1=1$.
A	sink if $\alpha_1\in(1,3)$ and $0<\alpha_2<\frac{\alpha_1}{\alpha_1-1}$; source if $\alpha_1>3$ and $\alpha_2>\frac{\alpha_1}{\alpha_1-1}$; saddle if $\alpha_1>3$ and $0<\alpha_2<\frac{\alpha_1}{\alpha_1-1}$; non-hyperbolic if $\alpha_2=\frac{\alpha_1}{\alpha_1-1}$.
B	locally asymptotically stable focus if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0$ and $0<\alpha_1<\frac{\alpha_2}{\alpha_2-2}$; unstable focus if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0$ and $\alpha_1>\frac{\alpha_2}{\alpha_2-2}$; non-hyperbolic (under which $\kappa_{1,2}$ are a pair of complex conjugate with modulus 1) if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}<0$ and $\alpha_1=\frac{\alpha_2}{\alpha_2-2}$; locally asymptotically stable node if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0$ and $0<\alpha_1<\frac{3\alpha_2}{3-\alpha_2}$; unstable node if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0$ and $\alpha_1>\frac{3\alpha_2}{3-\alpha_2}$; non-hyperbolic (under which the real eigenvalues with modulus 1) if $(\frac{2\alpha_2-\alpha_1}{\alpha_2})^2-\frac{4\alpha_1(\alpha_2-2)}{\alpha_2}\geq 0$ and $\alpha_1=\frac{3\alpha_2}{3-\alpha_2}$.

Acknowledgements
The author thanks the main editor and referees for their valuable comments and suggestions leading to improvement of this paper. This work was supported by the Higher Education Commission (HEC) of Pakistan.

Funding
The author declares that he got no funding on any part of this research.

Competing interests
The author declares that he has no conflict of interests regarding the publication of this paper.

Authors' contributions
The author carried out the proof of the main results and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 September 2018 Accepted: 28 January 2019 Published online: 08 February 2019

References
1. Volterra, V.: Leçons sur la Théorie Mathématique de la Lutte pour la Vie. Gauthier-Villars, Paris (1931)
2. Martelli, M.: Discrete Dynamical Systems and Chaos. Longman, New York (1992)
3. Landa, P.S.: Self oscillatory models of some natural and technical processes. In: Kreuzer, E., Schmidt, G. (eds.) Mathematical Research, vol. 72, p. 23
4. Freedman, H.I.: A model of predator–prey dynamics as modified by the action of a parasite. Math. Biosci. **99**, 143–155 (1990)
5. May, R.M.: Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton (2001)
6. Volterra, V.: Fluctuations in the abundance of a species considered mathematically. Nature **118**, 558–560 (1926)
7. Alebraheem, J., Hasan, Y.A.: Dynamics of a two predator-one prey system. Comput. Appl. Math. **33**, 67–780 (2014)
8. Sinha, S., Misra, O., Dhar, J.: Modelling a predator–prey system with infected prey in polluted environment. Appl. Math. Model. **34**(7), 1861–1872 (2010)
9. Chen, Y., Changming, S.: Stability and Hopf bifurcation analysis in a prey–predator system with stage-structure for prey and time delay. Chaos Solitons Fractals **38**(4), 1104–1114 (2008)
10. Gakkhar, S., Singh, A.: Complex dynamics in a prey–predator system with multiple delays. Commun. Nonlinear Sci. Numer. Simul. **17**(2), 914–929 (2012)
11. Yan, J., Li, C., Chen, X., Ren, L.: Dynamic complexities in 2-dimensional discrete-time predator–prey systems with Allee effect in the prey. Discrete Dyn. Nat. Soc. **2016**, Article ID 4275372 (2016)
12. Zhao, J., Yan, Y.: Stability and bifurcation analysis of a discrete predator–prey system with modified Holling–Tanner functional response. Adv. Differ. Equ. **2018**, 402 (2018)
13. Fang, Q., Li, X.: Complex dynamics of a discrete predator–prey system with a strong Allee effect on the prey and a ratio-dependent functional response. Adv. Differ. Equ. **2018**, 320 (2018)
14. Kangalgi, F., Kartal, S.: Stability and bifurcation analysis in a host–parasitoid model with Hassell growth function. Adv. Differ. Equ. **2018**, 240 (2018)
15. Li, L., Shen, J.: Bifurcations and dynamics of a predator–prey model with double Allee effects and time delays. Int. J. Bifurc. Chaos **28**(11), 1–14 (2018)
16. Zhao, M., Li, C., Wang, J.: Complex dynamic behaviors of a discrete-time predator–prey system. J. Appl. Anal. Comput. **7**(2), 478–500 (2017)
17. Cheng, L., Cao, H.: Bifurcation analysis of a discrete-time ratio-dependent predator–prey model with Allee effect. Commun. Nonlinear Sci. Numer. Simul. **38**, 288–302 (2016)
18. Liu, W., Cai, D., Shi, J.: Dynamic behaviors of a discrete-time predator–prey bioeconomic system. Adv. Differ. Equ. **2018**, 133 (2018)
19. Liu, X., Chu, Y., Liu, Y.: Bifurcation and chaos in a host–parasitoid model with a lower bound for the host. Adv. Differ. Equ. **2018**, 31 (2018)
20. Sohel Rana, S.M.: Chaotic dynamics and control of discrete ratio-dependent predator–prey system. Discrete Dyn. Nat. Soc. **2017**, Article ID 4537450 (2017)
21. Zhao, M., Du, Y.: Stability of a discrete-time predator–prey system with Allee effect. Nonlinear Analy. Diff. Equ. **4**(5), 225–233 (2016)
22. Liu, X., Xiao, D.: Complex dynamic behaviors of a discrete-time predator–prey system. Chaos Solitons Fractals **32**(1), 80–94 (2007)
23. Khan, A.Q., Ma, J., Xiao, D.: Bifurcations of a two-dimensional discrete time plant-herbivore system. Commun. Nonlinear Sci. Numer. Simul. **39**, 185–198 (2016)
24. Khan, A.Q., Ma, J., Xiao, D.: Global dynamics and bifurcation analysis of a host–parasitoid model with strong Allee effect. J. Biol. Dyn. **11**(1), 121–146 (2017)
25. Khan, A.Q.: Stability and Neimark–Sacker bifurcation of a ratio-dependence predator–prey model. Math. Methods Appl. Sci. **40**(11), 3833–4232 (2017)
26. Hu, Z., Teng, Z., Zhang, L.: Stability and bifurcation analysis of a discrete predator–prey model with non-monotonic functional response. Nonlinear Anal., Real World Appl. **12**(4), 2356–2377 (2011)
27. Jing, Z., Yang, J.: Bifurcation and chaos in discrete-time predator–prey system. Chaos Solitons Fractals **27**(1), 259–277 (2006)
28. Zhang, C.H., Yan, X.P., Cui, G.H.: Hopf bifurcations in a predator–prey system with a discrete delay and a distributed delay. Nonlinear Anal., Real World Appl. **11**(5), 4141–4153 (2010)

29. Sen, M., Banerjee, M., Morozov, A.: Bifurcation analysis of a ratio-dependent prey–predator model with the Allee effect. Ecol. Complex. **11**, 12–27 (2012)
30. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields. Springer, New York (1983)
31. Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory, 3rd edn. Springer, New York (2004)
32. Khan, A.Q.: Supercritical Neimark–Sacker bifurcation of a discrete-time Nicholson–Bailey model. Math. Methods Appl. Sci. **41**(12), 4841–4852 (2018)
33. Cartwright, J.H.E.: Nonlinear stiffness Lyapunov exponents and attractor dimension. Phys. Lett. A **264**, 298–304 (1999)
34. Kaplan, J.L., Yorke, J.A.: Preturbulence: a regime observed in a fluid flow model of Lorenz. Commun. Math. Phys. **67**(2), 93–108 (1979)
35. Elaydi, S.N.: An Introduction to Difference Equations. Springer, New York (1996)
36. Lynch, S.: Dynamical Systems with Applications Using Mathematica. Birkhäuser, Boston (2007)