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Complex dynamics in an SIS epidemic model with nonlinear incidence

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Abstract

We study an epidemic model with nonlinear incidence rate, describing the saturated mass action and the psychological effect of certain serious diseases on the community. Firstly, the existence and local stability of disease-free and endemic equilibria are investigated. Then we prove the occurrence of backward bifurcations, saddle-node bifurcations, Hopf bifurcations and cusp type Bogdanov–Takens bifurcations of codimension 3. Finally, numerical simulations, including one limit cycle, two limit cycles, an unstable homoclinic loop and many other phase portraits are presented. These results show that the psychological effect of diseases and the behavior change of the susceptible individuals may affect the final spread level of an epidemic.

Keywords: Epidemic; Nonlinear incidence rate; Saturated mass action; Psychological effect; Bifurcation

1 Introduction

In the well-known SIS epidemic model, the population is always separated into two compartments, susceptible and infective individuals. In most SIS epidemic models (see Anderson and May [1]), the incidence takes the mass-action form with bilinear interactions. However, in a practical application, to describe the transmission process more realistically, it is necessary to introduce the nonlinear contact rates [2].

Actually, various forms of nonlinear incidence rates have been proposed recently [3–10]. For example, in order to incorporate the effect of behavioral changes, Liu, Levin, and Iwasa [6] used a nonlinear incidence rate of the form

$$g(I)S = \frac{\kappa I^{\iota} S}{1 + \alpha I^{h}},$$

where κI^{ι} represents the infection force of the disease, $1/(1 + \alpha I^{h})$ is a description of the suppression effect from the behavioral change of susceptible individuals when the infective population increases. ι, h and κ are all positive constants, and α is a nonnegative constant. See also Hethcote and van den Driessche [7], Moghadas [8] and Alexander and Moghadas [9, 10], etc.

To describe the effects of psychology effect caused by protection measures and intervention policies, etc., when a serious disease arouses widespread horror, in [11], Ruan

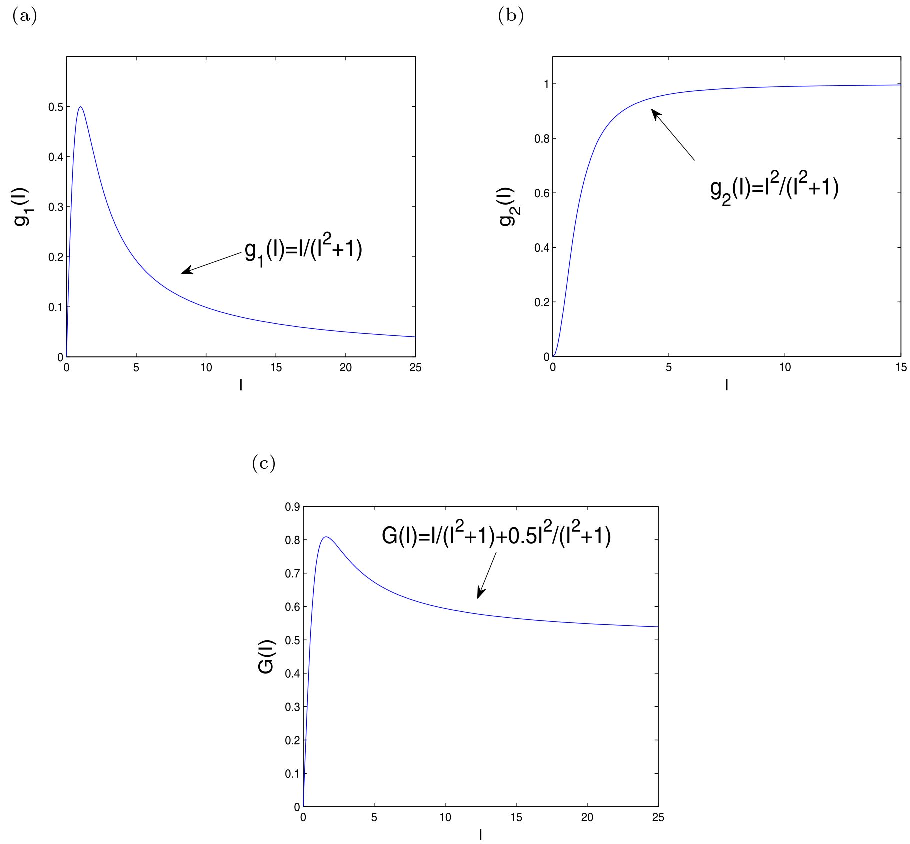

Figure 1 Graphs of different incidence rate functions. (**a**) Non-monotone incidence $g_1(I) = \frac{\kappa I}{1+\alpha I^2}$. (**b**) Nonlinear incidence $g_2(I) = \frac{\kappa I^2}{1+\alpha I^2}$. (**c**) Nonlinear and non-monotone incidence rate $G(I) = \frac{aI^2}{I^2+c} + \frac{bI}{I^2+c}$

discussed a specific infection force

$$g_1(I) = \frac{\kappa I}{1 + \alpha I^2};$$

see Fig. 1(a). Obviously, $g_1(I)$ is increasing with small I and decreasing with large I, that is, $g_1(I)$ is non-monotone. It can be used to interpret the "psychological" effect: for a very large number of infective individuals, the infection force may decrease as the number of infective individuals increases, since a large number of infectives may lead to the reducing of the number of contacts per unit time. For example, in 2003, the epidemic outbreak of severe acute respiratory syndrome (SARS) had such psychological effects on the general public (see [12]), and aggressive measures and policies had been taken, such as border screening, mask wearing, quarantine, isolation, etc. One showed that either the number of infective individuals tends to zero as time evolves or the disease persists.

Furthermore, Li, Zhao and Zhu (see [13]) studied the following SIS model, which describes behavior change effect of susceptible individual when infectious population increases:

$$\begin{cases} \frac{dS}{dt} = \Lambda - d_0 S - g_2(I)S + \delta I, \\ \frac{dI}{dt} = g_2(I)S - (d_0 + d' + \delta)I, \end{cases} \tag{1.1}$$

where

$$g_2(I) = \frac{\alpha I^2}{1 + \beta_0 I^2};$$

see Fig. 1(b). By the qualitative and bifurcation analyses, they showed that the maximal multiplicity of weak focus is 2, and proved that the model can undergo a Bogdanov–Takens bifurcation of codimension 2. These results illustrate that the behavior change of the susceptible individuals may affect the final spread level of an epidemic.

Actually, both the effect of psychology and the behavior change of susceptible individuals have influence on the transmission of the disease. Thus, motivated by the above research, we consider a nonlinear incidence rate of a SIS model as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda - SG(I) - dS + \sigma I, \\ \frac{dI}{dt} = SG(I) - (d + \mu + \sigma)I, \end{cases} \tag{1.2}$$

where

$$G(I) = \frac{aI^2}{c + I^2} + \frac{bI}{c + I^2},$$

see Fig. 1(c), which can describe the effect of psychology and behavior change of susceptible individuals. S and I represent the number of susceptible individuals and infected individuals, respectively. Λ is the recruitment rate of population, d is the natural death, μ is the disease-induced death rate, and σ represents the recovered rate, a, b and c are all positive constants.

The organization of this paper is as follows. In Sect. 2, we analyze the existence of the equilibria and local stability of the equilibria. In Sect. 3, we study the existence of Hopf bifurcation around the positive equilibrium at the critical value under the conditions of $R_0 < 1$ and $R_0 > 1$. We also show that these positive equilibria can be weak focus for some parameter values and a cusp type of Bogdanov–Takens bifurcation of codimension 3. In Sect. 4, we give some brief discussions.

2 Types and stability of the equilibria

Firstly, we make scalings: $(\Lambda', a', d', \sigma', c') = (\frac{\Lambda}{b}, \frac{a}{d+\mu+\sigma}, \frac{d}{d+\mu+\sigma}, \frac{\sigma}{d+\mu+\sigma}, \frac{c(d+\mu+\sigma)^2}{b^2})$, and $(x, y, \tau) = (\frac{d+\mu+\sigma}{b}S, \frac{d+\mu+\sigma}{b}I, (d + \mu + \sigma)t)$. To avoid the abuse of mathematical notation, we still denote $(\Lambda', a', d', \sigma', c', \tau)$ by $(\Lambda, a, d, \sigma, c, t)$. Then model (1.2) becomes

$$\begin{cases} \frac{dx}{dt} = \Lambda - \frac{xy(ay+1)}{y^2+c} - dx + \sigma y, \\ \frac{dy}{dt} = \frac{xy(ay+1)}{y^2+c} - y, \end{cases} \tag{2.1}$$

where $d + \sigma < 1$.

Lemma 2.1 *The set* $D = \{(x, y) | x \geq 0, y \geq 0, x + y \leq \frac{\Lambda}{d}\}$ *is an invariant manifold of system* (2.1), *which is attracting in the first octant of* $\mathbb{R}^2$.

Proof Summing up the two equations in (2.1), we can get

$$\frac{d(x+y)}{dt} = \Lambda - d(x+y) - (1-d-\sigma)y \le \Lambda - d(x+y).$$

Thus, $\limsup_{t\to\infty}(x+y) \le \frac{\Lambda}{d}$, which implies the conclusion. □

Obviously, system (2.1) always has a unique disease-free equilibrium $E_0 = (\frac{\Lambda}{d}, 0)$. The positive equilibria of (2.1) can be obtained by solving the following algebraic equations:

$$\Lambda - \frac{xy(ay+1)}{y^2+c} - dx + \sigma y = 0, \qquad \frac{xy(ay+1)}{y^2+c} - y = 0,$$

which yields

$$\bigl(d + a(1-\sigma)\bigr)y^2 + (1-\sigma - a\Lambda)y + dc - \Lambda = 0. \tag{2.2}$$

Denote the basic reproduction number as follows:

$$R_0 = \frac{\Lambda}{dc}.$$

And, for convenience, we define the following quantity:

$$R^* = 1 - \frac{[(1-\sigma) - a\Lambda]^2}{4dc(d+a(1-\sigma))}.$$

Then, computing the discriminant of (2.2), we get

$$\begin{aligned}\Delta &= (1-\sigma - a\Lambda)^2 - 4\bigl[d + a(1-\sigma)\bigr](dc - \Lambda)\\ &= (1-\sigma - a\Lambda)^2 - 4dc\bigl[d + a(1-\sigma)\bigr](1 - R_0)\\ &= 4dc\bigl[d + a(1-\sigma)\bigr]\bigl(R_0 - R^*\bigr),\end{aligned}$$

which implies that $\Delta > 0$ if and only if $R_0 > R^*$, $\Delta = 0$ if and only if $R_0 = R^*$, and that $\Delta < 0$ if and only if $R_0 < R^*$. It is clear that $R^* < 1$ and we can obtain the following theorem.

Theorem 2.2 *Model (2.1) always has a disease-free equilibrium E_0 and the following conclusions hold.*

(i) *When $R_0 < 1$, we have*
   (a) *if $R_0 < R^*$, then system (2.1) has no positive equilibrium*;
   (b) *if $R_0 = R^*$ and $\Lambda > (1-\sigma)/a$, then system (2.1) has a unique positive equilibrium $E_1(x_1, y_1)$, where $x_1 = \frac{y_1^2+c}{ay_1+1}$ and $y_1 = \frac{a\Lambda-(1-\sigma)}{d+a(1-\sigma)}$*;
   (c) *if $R_0 > R^*$ and $\Lambda > (1-\sigma)/a$, then system (2.1) has two positive equilibria $E_2(x_2, y_2)$ and $E_3(x_3, y_3)$, where $x_k = \frac{y_k^2+c}{ay_k+1}$ $(k = 2, 3)$ and $y_2 = \frac{a\Lambda-(1-\sigma)-\sqrt{\Delta}}{2(d+a(1-\sigma))}$, $y_3 = \frac{a\Lambda-(1-\sigma)+\sqrt{\Delta}}{2(d+a(1-\sigma))}$*;

(ii) *When $R_0 = 1$ and $\Lambda > (1-\sigma)/a$, then system (2.1) has a unique positive equilibrium $E_4(x_4, y_4)$, where $x_4 = \frac{y_4^2+c}{ay_4+1}$ and $y_4 = \frac{a\Lambda-(1-\sigma)}{d+a(1-\sigma)}$*;

(iii) *When* $R_0 > 1$, *then system* (2.1) *has a unique positive equilibrium* $E_5(x_5, y_5)$, *where* $x_5 = \frac{y_5^2+c}{ay_5+1}$ *and* $y_5 = \frac{a\Lambda-(1-\sigma)+\sqrt{\Delta}}{2(d+a(1-\sigma))}$.

In the following, we discuss the local stability of $E_k(x_k, y_k)$ ($k = 0, 1, 2, 3, 4, 5$) and present the corresponding phase portrait. By directly calculating, the Jacobian matrix at equilibrium E_k is

$$J_k = \begin{pmatrix} -d - \frac{y_k(ay_k+1)}{y_k^2+c} & \sigma - \frac{-y_k^2+2acy_k+c}{(y_k^2+c)^2}x_k \\ \frac{y_k(ay_k+1)}{y_k^2+c} & -1 + \frac{-y_k^2+2acy_k+c}{(y_k^2+c)^2}x_k \end{pmatrix}.$$

We have

$$\operatorname{tr} J_k = \frac{\psi_k}{(y_k^2+c)(ay_k+1)}, \qquad \det J_k = \frac{y_k\phi_k}{(y_k^2+c)(ay_k+1)},$$

where

$$\psi_k = -a(d+1+a)y_k^3 - (d+2+2a)y_k^2 - (acd+1-ac)y_k - cd,$$
$$\phi_k = a\bigl(d+a(1-\sigma)\bigr)y_k^2 + 2\bigl(d+a(1-\sigma)\bigr)y_k + (1-\sigma) - dac.$$

In addition, after some complicated computations, we get

$$\phi_2 = \frac{\sqrt{\Delta}(a(\sqrt{\Delta}-(a\Lambda+1-\sigma))-2d)}{2(d+a(1-\sigma))}$$

and

$$\phi_k = \frac{\sqrt{\Delta}(a(\sqrt{\Delta}+(a\Lambda+1-\sigma))+2d)}{2(d+a(1-\sigma))} > 0 \quad (k = 3, 4, 5).$$

Theorem 2.3 *The disease-free equilibrium* E_0 *of system* (2.1) *is*

(i) *an attracting node if* $R_0 < 1$;
(ii) *a hyperbolic saddle if* $R_0 > 1$;
(iii) *a saddle-node of codimension* 1 *if* $R_0 = 1$ *and* $2a - \frac{1-\sigma}{dc} \neq 0$; *a repelling node if* $R_0 = 1$ *and* $2a - \frac{1-\sigma}{dc} = 0$.

Proof Obviously, at equilibrium E_0, we have $\det(J_0) = d(1-R_0)$ and $\operatorname{Tr}(J_0) = -d-(1-R_0)$. Therefore, E_0 is a stable node if $R_0 < 1$ and a hyperbolic saddle if $R_0 > 1$, and degenerate if $R_0 = 1$.

When $R_0 = 1$, we let $u = x - \frac{\Lambda}{d}$, $v = y$, then system (2.1) becomes

$$\begin{cases} \frac{du}{dt} = -du - (1-\sigma)v - \frac{1}{c}uv - 2av^2 + \mathcal{O}(|(u,v)|^3), \\ \frac{dv}{dt} = 2av^2 + \frac{1}{c}uv + \mathcal{O}(|(u,v)|^3). \end{cases} \tag{2.3}$$

Indeed, if $R_0 = 1$, the Jacobian J_0 is diagonalizable with eigenvalues $\lambda_1 = -d$ and $\lambda_2 = 0$ and respective eigenvectors $v_1 = (1, 0)$ and $v_2 = (-\frac{1-\sigma}{d}, 1)$. By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-\sigma}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

system (2.3) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = -dx - (\frac{1}{c} + \frac{1-\sigma}{d})xy - (\frac{1-\sigma}{cd} - 2a + \frac{2a(1-\sigma)}{d})y^2 + \mathcal{O}(|(x,y)|^3), \\ \frac{dy}{dt} = (2a - \frac{1-\sigma}{dc})y^2 + \frac{1}{c}xy + \mathcal{O}(|(x,y)|^3). \end{cases}$$

If $2a - \frac{1-\sigma}{dc} \neq 0$, according to the calculation of center manifold [14], we know that the center manifold $x = h(y)$ of (2.3) begins with quadratic term of y. In addition, from the second equation of (2.3), we can easily see that the equation restricted to the center manifold is as follows:

$$\frac{dy}{dt} = \left(2a - \frac{1-\sigma}{dc}\right)y^2 + \mathcal{O}(y^3).$$

By applying Theorem 7.1 in Zhang et al. [14], E_0 is a saddle-node.

If $2a - \frac{1-\sigma}{dc} = 0$, then the center manifold turns into

$$\frac{dy}{dt} = \frac{1}{c}y^3 + \mathcal{O}(y^4).$$

Since $\frac{1}{c} > 0$ and the first nonzero item is uneven. Thus, the equilibrium E_0 is a repelling node, according to Theorem 7.1 in Zhang et al. [14]. □

From the expression of ψ_1 and ϕ_1, we can see that one of the eigenvalues of the characteristic matrix of E_1 is zero and the other is nonzero if $\psi_1 \neq 0$. The type of E_1 can be directed proved by checking the conditions in Zhang et al. ([14], Theorems 7.1–7.3). So, we have the following results.

Theorem 2.4 *If $R_0 = R^*$, then system (2.1) has a unique positive equilibrium E_1. More precisely,*

(a) *if $\psi_1 \neq 0$, then E_1 is a saddle-node;*
(b) *if $\psi_1 = 0$, then E_1 is a cusp.*

Theorem 2.5 *Suppose that $R^* < R_0 < 1$ and $a\Lambda > 1 - \sigma$, then system (2.1) has two positive equilibria E_2 and E_3, and equilibrium E_2 is a hyperbolic saddle for all permissible choices of the parameters, equilibrium E_3 is not degenerate. Moreover,*

(i) *E_3 is a stable focus or node if $\psi_3 < 0$;*
(ii) *E_3 is a weak focus or center if $\psi_3 = 0$;*
(iii) *E_3 is an unstable focus or node if $\psi_3 > 0$.*

Proof Note that ϕ_2 is less than zero since $\Delta < (a\Lambda + 1 - \sigma)^2$, then E_2 is a hyperbolic saddle for any choices of the parameters. And at E_3, we have $\phi_3 > 0$. Thus, the stability of the equilibrium E_3 depends on the sign of ψ_3. □

Theorem 2.6 *When $R_0 = 1$ and $c > \frac{1-\sigma}{a}$, then system (2.1) has a unique positive equilibrium $E_4(x_4, y_4)$, and the equilibrium E_4 is stable if $\psi_4 < 0$.*

Proof In fact, when $R_0 = 1$ and $c > \frac{1-\sigma}{a}$, then $\phi_4 > 0$. Thus, the stability of E_4 is determined by the sign of ψ_4. □

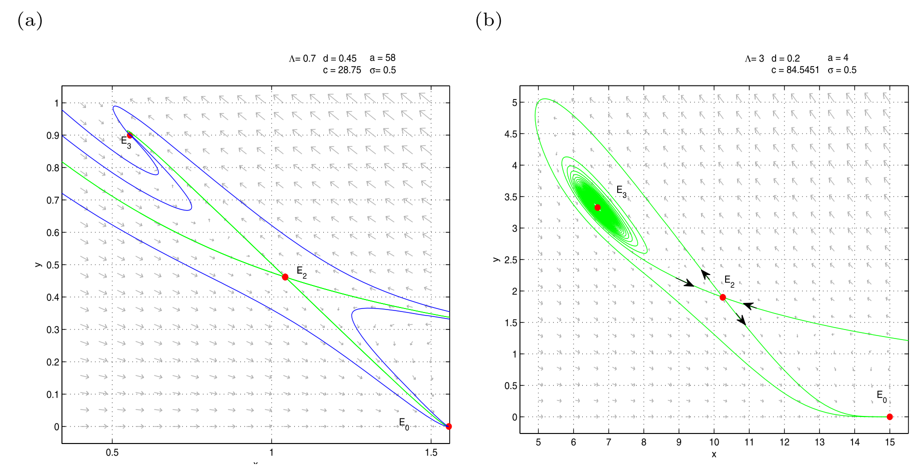

Figure 2 When $R^* < R_0 < 1$, $\Lambda > (1-\sigma)/a$. (**a**) $\psi_3 < 0$, equilibria E_0 and E_3 are locally stable and E_2 is unstable. (**b**) $\psi_3 > 0$, equilibrium E_0 is locally stable and E_2 and E_3 are unstable

Theorem 2.7 *Assume $R_0 > 1$, then system (2.1) has a unique positive equilibrium E_5. Moreover,*

(i) *E_5 is stable if $\psi_5 < 0$;*
(ii) *E_5 is a weak focus or center if $\psi_5 = 0$;*
(iii) *E_5 is unstable if $\psi_5 > 0$.*

Proof Obviously, when $R_0 > 1$, then $\phi_5 > 0$, and then E_5 is stable if $\psi_5 < 0$. □

Lemma 2.8 *From the expression of ψ_k $(k = 1,3,5)$, we can see that E_k $(k = 1,3,5)$ is always stable if $d \geq 1 - \frac{1}{ac}$.*

When $\psi_i \neq 0$ $(i = 3,5)$, the dynamics of system (2.1) can easily be seen in Fig. 2, Fig. 3 and Fig. 4, respectively. The dynamical behaviors of system (2.1) when $\psi_i = 0$ $(i = 3,5)$ will be discussed in detail in the next section.

Remark 2.9 In fact, Fig. 2(a) shows the occurrence of bi-stability, in which solution may converge to one of the two equilibria, depending on the initial conditions. And in practical cases, this interesting phenomenon implies that initial states determine whether the disease dies out or not.

Remark 2.10 From Fig. 2(a), we can see that there exist two separatrices. All solutions tend to the disease-free equilibrium E_0 except the two green lines tend to equilibrium E_2.

Theorem 2.11 *Suppose that $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, then system (2.1) has a unique positive equilibrium E_4. If $\psi_4 > 0$, then there exists at least one stable limit cycle in the interior of the first quadrant.*

Proof Indeed, the Jacobian J_0 is diagonalizable with eigenvalues $\lambda_1 = -d$ and $\lambda_2 = 0$ and respective eigenvectors $v_1 = (1,0)$ and $v_2 = (-\frac{1-\sigma}{d}, 1)$. By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-\sigma}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

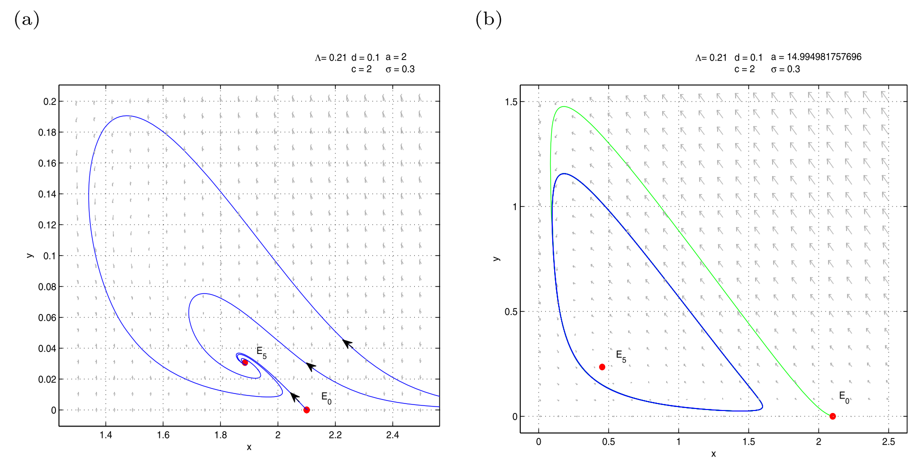

Figure 3 When $R_0 > 1$. (**a**) $\psi_5 < 0$, the disease-free equilibrium E_0 is unstable and the unique positive equilibrium E_5 is globally stable. (**b**) $\psi_5 > 0$, E_0 and E_5 are unstable and there exists a stable limit cycle

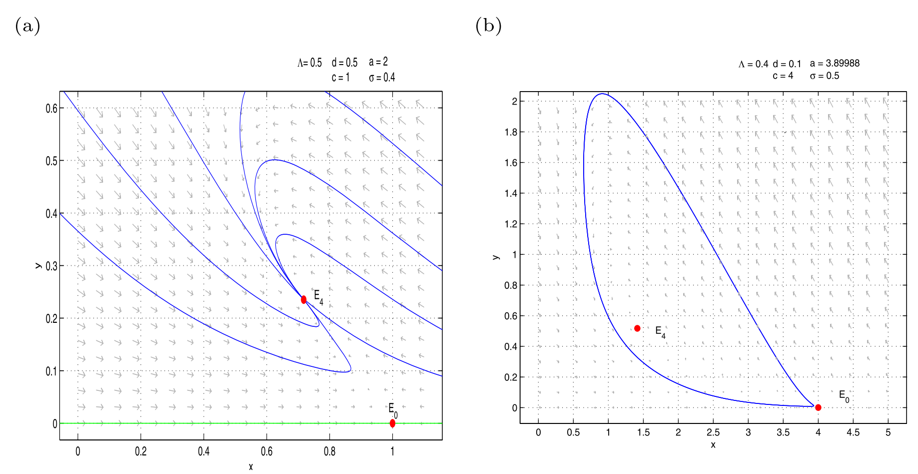

Figure 4 When $R_0 = 1$, and $\Lambda > (1 - \sigma)/a$. (**a**) $\psi_4 < 0$, the disease-free equilibrium E_0 is unstable and the unique positive equilibrium E_4 is globally stable. (**b**) $\psi_4 > 0$, E_0 and E_4 are unstable and there exists a stable limit cycle

system (2.3) becomes

$$\begin{cases} \frac{dx}{dt} = -dx - (\frac{1}{c} + \frac{1-\sigma}{d})xy - (\frac{1-\sigma}{cd} - 2a + \frac{2a(1-\sigma)}{d})y^2 + \mathcal{O}(|(x,y)|^3), \\ \frac{dy}{dt} = (2a - \frac{1-\sigma}{dc})y^2 + \frac{1}{c}xy + \mathcal{O}(|(x,y)|^3). \end{cases} \tag{2.4}$$

The theorem of Chochitaichvili [15] shows directly that system (2.4) is topologically equivalent to the system

$$\begin{cases} \frac{dx}{dt} = -dx, \\ \frac{dy}{dt} = (2a - \frac{1-\sigma}{dc})y^2 + \mathcal{O}(|y|^3). \end{cases}$$

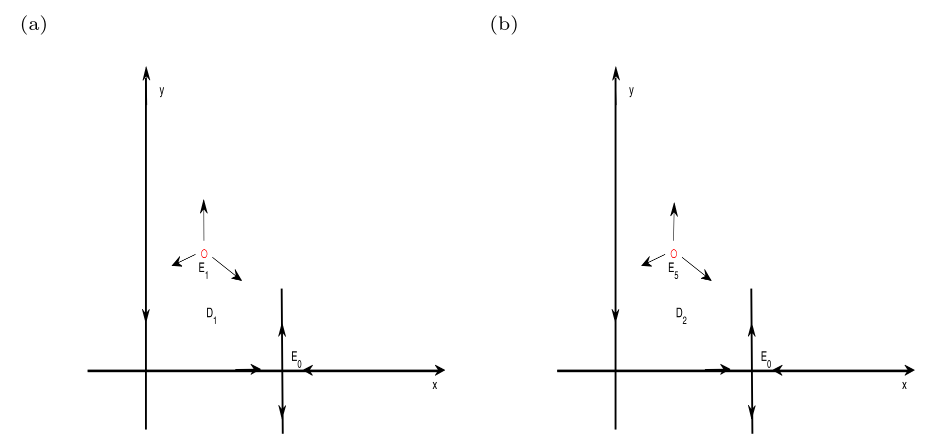

Figure 5 When $R_0 = 1$, and $\Lambda > (1-\sigma)/a$. (**a**) When $R_0 = 1$, $\Lambda > \frac{1-\sigma}{a}$ and $\psi_4 > 0$, there exists a stable limit cycle. (**b**) When $R_0 > 1$ and $\psi_5 > 0$, there exists a stable limit cycle

It is easy to see that $2a - \frac{1-\sigma}{dc} > 0$ according to the condition of $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, and then we find that there exists a unique repelling equilibrium E_4 in the region D_1 shown in (a) of Fig. 5. Consequently, by the Poincaré–Bendixson theorem, at least one stable limit cycle appears in the interior of the first quadrant. □

Similarly, when $\psi_5 > 0$, we have the following result.

Theorem 2.12 *Suppose that $R_0 > 1$. If $\psi_5 > 0$, then there exists at least one stable limit cycle in the interior of the first quadrant.*

Proof Indeed, the Jacobian J_5 has eigenvalues $\lambda_1 = -d$ and $\lambda_2 = R_0 - 1 > 0$, when $R_0 > 1$. Thus, we find that there exists a E_5, which is the unique repelling equilibrium in the region D_2 shown in (b) of Fig. 5. Consequently, by the Poincaré–Bendixson theorem, at least one stable limit cycle appears in the interior of the first quadrant. □

Remark 2.13 From (b) of Fig. 3, we can see clearly that there exists a stable limit cycle enclosing the equilibrium E_5 even though E_0 is a saddle node. Similarly, from (b) of Fig. 4, we also find that there exists a stable limit cycle if E_4 is unstable.

3 Bifurcations

3.1 Backward bifurcation

Theorem 3.1 *When $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, model (2.1) exhibits a backward bifurcation at equilibrium E_0.*

Remark 3.2 When $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, system (2.1) exhibits a unique positive equilibrium E_4, which means that once R_0 crosses 1, the disease can invade to a relatively high level. And this is one of the main characters of backward bifurcation [16].

Remark 3.3 Actually, backward bifurcation did not emerge with $a = 0$, which is considered in [13]. This indicates that introducing the non-motonic incidence into model (1.1) makes the epidemic model more complex and exhibits richer dynamical behaviors.

3.2 Hopf bifurcation

In this subsection, we will study the Hopf bifurcation of system (2.1) for (i) $R^* < R_0 < 1$ and $\Lambda > (1-\sigma)/a$; (ii) $R_0 > 1$. From the discussion in Sect. 2, it can be seen that Hopf bifurcation may occur at E_3, E_5. The expressions of the equilibria E_3 and E_5 are the same, not considering the values of every parameters. Based on Theorem 2.5, Theorem 2.6 and Theorem 2.7, we know that the stability of E_3 and that of E_5 are similar and when $\psi_k = 0$, E_k $(k = 3, 5)$ is a weak focus or center. Thus, we show the existence of a Hopf bifurcation around E_k $(k = 3, 5)$.

Theorem 3.4 *Suppose E_k $(k = 3, 5)$ exist, then model (1.2) undergoes a Hopf bifurcation at equilibrium E_k if $\psi_k = 0$. Moreover,*

(a) *if $\eta < 0$, there is a family of stable periodic orbits of model (2.1) as ψ_k decreases from 0;*
(b) *if $\eta = 0$, there are at least two limit cycles in (2.1), where η will be defined below;*
(c) *if $\eta > 0$, there is a family of unstable periodic orbits of (2.1) as ψ_k increases from 0.*

Proof From the above discussions, we can see that $\operatorname{tr} J_k = 0$ if and only if $\psi_k = 0$, and $\det J_k > 0$ when equilibrium E_k exists. Therefore, the eigenvalues of J_k are a pair of pure imaginary roots if $\psi_k = 0$. From direct calculations we have

$$\left.\frac{d(\operatorname{tr} J_k)}{d\psi_k}\right|_{\psi_k=0} = \frac{1}{(y_k^2 + c)(y_k + 1)} \neq 0.$$

By Theorem 3.4.2 in [17], $\Psi_k = 0$ is the Hopf bifurcation point for (2.1).

Next, similar to [13], we introduce a new time variable τ by $dt = (y^2 + c)\,d\tau$. By rewriting τ as t, we obtain the following equivalent system of (2.1):

$$\begin{cases} \frac{dx}{dt} = \Lambda(y^2 + c) - xy(ay + 1) - dx(y^2 + c) + \sigma y(y^2 + c), \\ \frac{dy}{dt} = xy(ay + 1) - y(y^2 + c). \end{cases} \tag{3.1}$$

Let $X = x - x_k$ and $Y = y - y_k$, still use (x, y) to express (X, Y), then system (3.1) becomes

$$\begin{cases} \frac{dx}{dt} = b_{11}x + b_{12}y + c_1xy + c_2y^2 + c_3xy^2 + c_4y^3, \\ \frac{dy}{dt} = b_{21}x + b_{22}y + c_5xy + c_6y^2 + c_7xy^2 + c_8y^3, \end{cases}$$

where

$$\begin{gathered} b_{11} = -d\bigl((y_k)^2 + c\bigr) - y_k(ay_k + 1), \qquad b_{21} = y_k(ay_k + 1), \qquad b_{22} = (ax_k - 2y_k)y_k, \\ b_{12} = 2\Lambda y_k - 2(a + d)x_ky_k - x_k + \sigma\bigl(3(y_k)^2 + c\bigr), \qquad c_1 = -1 - 2(a + d)y_k, \\ c_2 = \Lambda + 3\sigma y_k - (a + d)x_k, \qquad c_3 = -(a + d), \qquad c_4 = \sigma, \qquad c_5 = 2ay_k + 1, \\ c_6 = ax_k - 3y_k, \qquad c_7 = a, \qquad c_8 = -1. \end{gathered}$$

Let $\hat{E}$ denote the origin of $x - y$ plane. Since E_k satisfies Eq. (2.1), we obtain

$$\det J(\hat{E}) = b_{11}b_{22} - b_{12}b_{21} = \frac{y_k\psi_k}{y_k + b} > 0,$$

and it is easy to verify that $b_{11} + b_{22} = 0$ if and only if $\psi_k = 0$. Let $\omega = (\det J(\hat{E}))^{\frac{1}{2}}$, $u = -x$ and $v = \frac{b_{11}}{\omega}x + \frac{b_{12}}{\omega}y$, then the normal form of system (3.1) reads

$$\begin{cases} \frac{du}{dt} = -\omega v + f(u,v), \\ \frac{dv}{dt} = \omega u + g(u,v), \end{cases} \tag{3.2}$$

where

$$\begin{aligned} f(u,v) = {} & \left(\frac{b_{11}c_1}{b_{12}} - \frac{b_{11}^2 c_2}{b_{12}^2}\right)u^2 + \omega\left(\frac{c_1}{b_{12}} - \frac{2b_{11}c_2}{b_{12}^2}\right)uv - \frac{c_2\omega^2}{b_{12}^2}v^2 + b_{11}^2\left(\frac{c_3}{b_{12}^2} - \frac{b_{11}c_4}{b_{12}^3}\right)u^3 \\ & + b_{11}\omega\left(\frac{2c_3}{b_{12}^2} - \frac{3b_{11}c_4}{b_{12}^3}\right)u^2 v + \omega^2\left(\frac{c_3}{b_{12}^2} - \frac{3b_{11}c_4}{b_{12}^3}\right)uv^2 - \frac{c_4\omega^3}{b_{12}^3}v^3, \\ g(u,v) = {} & \frac{1}{\omega}\left(\frac{b_{11}^3 c_2}{b_{12}^2} - \frac{b_{11}^2 c_1}{b_{12}} - c_5 b_{11} + \frac{b_{11}^2 c_6}{b_{12}}\right)u^2 + \left(\frac{2b_{11}^2 c_2}{b_{12}^2} - \frac{b_{11}c_1}{b_{12}} + \frac{2b_{11}c_6}{b_{12}} - c_5\right)uv \\ & + \omega\left(\frac{b_{11}c_2}{b_{12}^2} + \frac{c_6}{b_{12}}\right)v^2 + \frac{b_{11}^2}{b_{12}\omega}\left(\frac{b_{11}^2 c_4}{b_{12}^2} - \frac{b_{11}c_3}{b_{12}} - c_7 + \frac{b_{11}c_8}{b_{12}}\right)u^3 \\ & + \left(\frac{3b_{11}^3 c_4}{b_{12}^3} - \frac{2b_{11}^2 c_3}{b_{12}^2} - \frac{b_{11}c_7}{b_{12}} + \frac{3b_{11}^2 c_8}{b_{12}^2}\right)u^2 v \\ & + \omega\left(\frac{3b_{11}^2 c_4}{b_{12}^3} - \frac{b_{11}c_3}{b_{12}^2} - \frac{c_7}{b_{12}} + \frac{3b_{11}c_8}{b_{12}^2}\right)uv^2 + \omega^2\left(\frac{b_{11}c_4}{b_{12}^3} + \frac{c_8}{b_{12}^2}\right)v^3. \end{aligned}$$

Set

$$\begin{aligned} \Gamma = {} & \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}] \\ & + \frac{1}{16\omega}\left[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\right], \end{aligned}$$

where f_{uv} denotes $\frac{\partial^2 f}{\partial u \partial v}(0,0)$, etc. Then by computing we obtain

$$\Gamma = \frac{\eta}{8b_{12}^2\omega^2},$$

where

$$\begin{aligned} \eta = {} & \left[c_3 + 3c_8 - \frac{c_2(c_1 + 2c_6)}{b_{12}}\right]\omega^4 + \left[\left(c_3 + 3c_8 - \frac{2c_2(c_1 + 2c_6)}{b_{12}}\right)b_{11}^2 \right. \\ & \left. + b_{11}\left(c_1^2 - 2b_{12}c_7 + c_2c_5 + c_1c_6 - 2c_6^2\right) + b_{12}c_5c_6\right]\omega^2 \\ & + \left[c_5 - \frac{b_{11}(c_1 + 2c_6)}{b_{12}}\right]\left[b_{11}^2(b_{12}c_6 - c_1) + b_{11}\left(b_{11}^2 c_2 - b_{12}^2 c_5\right)\right]. \end{aligned}$$

By Theorem 3.4.2 and Theorem 3.4.11 in [17], the rest of the claims in Theorem 3.4 are proven. □

Remark 3.5 What we need to note here is that the expression b_{12} in Theorem 3.4 is nonzero. Otherwise, we have $\det J_k = b_{11}b_{22} < 0$, since $b_{11} + b_{22} = 0$, which is a contradiction.

Next, we present examples to show that equilibrium E_k can be a stable weak focus of multiplicity two, and under a small perturbation, system (2.1) undergoes a degenerate Hopf bifurcation and produces two limit cycles.

Firstly, fix $y_k = 1/2$ and solve for $a = -2 + \frac{6}{(2-2\Lambda-\sigma)}$. Also, fix $x_k = 1$, based on $a = -2 + \frac{6}{(2-2\Lambda-\sigma)}$, we can get $d = \frac{1}{2(2\Lambda+1-\sigma)}$. Then $\Psi_k = 0$ if and only if $c = \frac{-10-2l-\sigma}{4(2\Lambda+\sigma-2)}$. Secondly, substituting these expressions into η and through complicated computation, we obtain

$$\begin{aligned}\eta &\triangleq L_1 \\ &= \frac{81(1+2\Lambda+\bar{\sigma})^2}{64(1-2\Lambda+\bar{\sigma})^6}\big(32\Lambda^5 - 80\Lambda^4(-1+\bar{\sigma}) + 8\Lambda^3\big(-39+2\bar{\sigma}(-13+5\bar{\sigma})\big) \\ &\quad + 4\Lambda^2\big(-103+\bar{\sigma}\big(99+2(24-5\bar{\sigma})\bar{\sigma}\big)\big) + \bar{\sigma}\big(140-\bar{\sigma}\big(139+\bar{\sigma}\big(-21 \\ &\quad + (-11+\bar{\sigma})\bar{\sigma}\big)\big)\big) + 2\Lambda\big(-80+\bar{\sigma}\big(242+\bar{\sigma}\big(-81+\bar{\sigma}(-38+5\bar{\sigma})\big)\big)\big)\big),\end{aligned}$$

with $\bar{\sigma} = 1-\sigma > 0$, which is the first Liapunov number of the equilibrium $(0,0)$ of (3.2). Then we fix $\bar{\sigma} = 4/5$ and solve the equation $\eta = 0$, then we get only one suitable value 0.5859 for Λ. That is to say, if $(\sigma,\Lambda,d,a,c) = (1/5, 5859, 0.2, 8, 4.741)$, then $L_1 = 0$. Furthermore, it can be seen that $E_k = E_3$ under this group of parameters.

In the following, we further compute the second Liapunov number of the equilibrium $(0,0)$ of system (3.2) by the successor function method. It is convenient to introduce polar coordinates (r,θ) and rewrite system (3.2) in polar coordinates by $x = r\cos\theta$, $y = r\sin\theta$. It is clear that in a small neighborhood of the origin, the successor function $D(c_0)$ of system (3.2) can be expressed by

$$D(c_0) = r(2\pi, c_0) - r(0, c_0),$$

where $r(\theta, c_0)$ is the solution of the following Cauchy problem:

$$\begin{aligned}&\frac{dr}{d\theta} = R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + R_5(\theta)r^5 + \cdots, \\ &r(0) = c_0, \qquad 0 < |c_0| \ll 1,\end{aligned}$$

where $R_i(\theta)$ $(i = 1,2,3,\ldots)$ is a polynomial of $(\sin\theta, \cos\theta)$, whose coefficients can be expressed by the coefficients of system (3.2). We omit them here, since the expressions are too long.

With the aid of Mathematica, we get

$$L_2 \doteq -0.334275$$

and

$$\begin{vmatrix} \frac{\partial\,\mathrm{tr} J_3}{\partial\Lambda} & \frac{\partial\,\mathrm{tr} J_3}{\partial c} \\ \frac{\partial L_1}{\partial\Lambda} & \frac{\partial L_1}{\partial c}\end{vmatrix} = -42{,}982.27165,$$

when $(\sigma,\Lambda,d,a,c) = (1/5, 5859, 0.2, 8, 4.741)$. Therefore, the interior equilibrium E_3 is a stable weak focus of multiplicity two if $(\sigma,\Lambda,d,a,c) = (1/5, 5859, 0.2, 8, 4.741)$. The phase portraits of system (2.1) under this group of parameters are shown in Fig. 6.

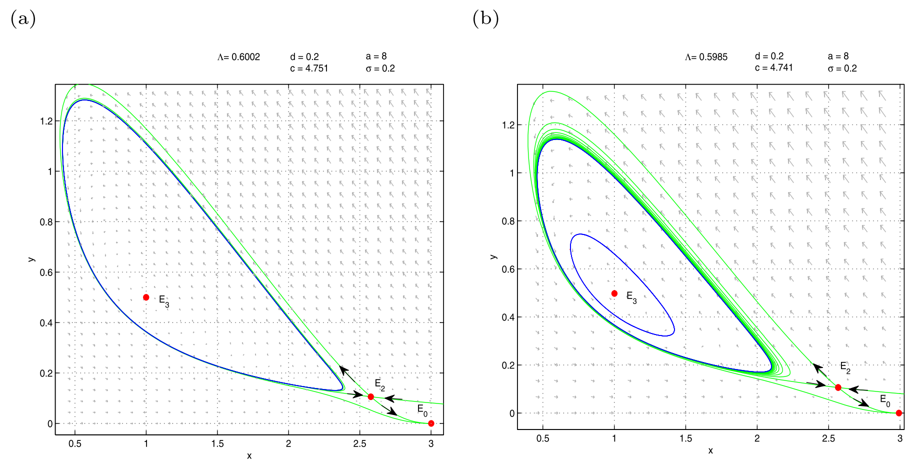

Figure 6 (**a**) One limit cycle enclosing an unstable hyperbolic focus E_3. (**b**) Two limit cycles enclosing an unstable hyperbolic focus E_3, and the small one is stable, the large one is unstable

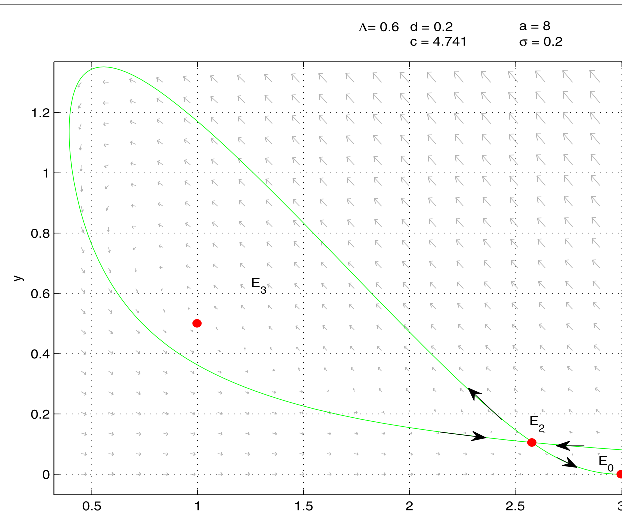

Figure 7 Equilibrium E_3 is stable and there exists an unstable homoclinic loop

Besides, we give the numerical simulation graphs for one limit cycle and two limit cycles under small perturbations of some parameters. From Fig. 6(a), we can see that there exists only one limit cycle around the endemic E_3, and Fig. 6(b) shows us that a new limit cycle emerges with small perturbations of the parameters Λ and c. It is worth emphasizing that if we change the values of the parameters Λ and c, an unstable homoclinic loop arise, which is shown in Fig. 7.

Around equilibrium E_5, we obtain the same result from Fig. 8. Under the condition that parameter a, d, σ take value 5.6923, 0.11, 0.2 and change value of Λ and c from 0.45102, 3.6062 to 0.51, 3.5972, respectively, which is a minor change, then the number of limit cycles will add one. Thus, the interior equilibrium E_5 is a stable weak focus of multiplicity two if $(\sigma, \Lambda, d, a, c) = (0.2, 0.51, 0.11, 5.6923, 3.5962)$.

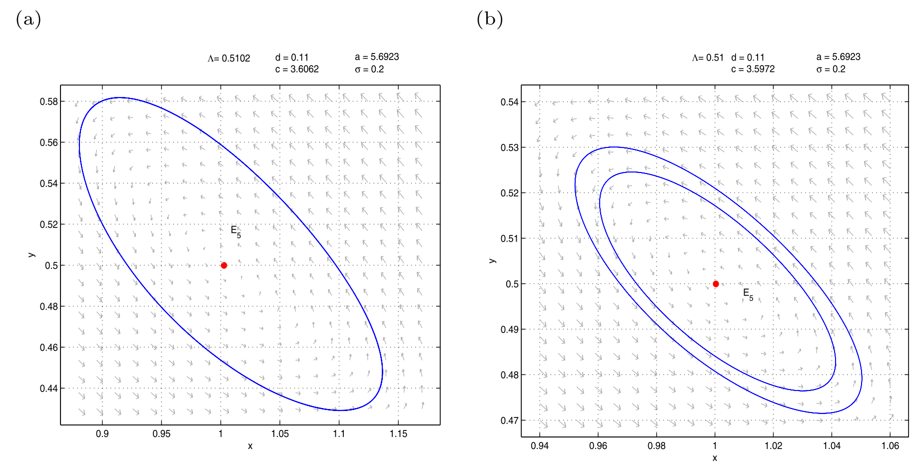

Figure 8 (**a**) One limit cycle enclosing an unstable hyperbolic focus E_5. (**b**) Two limit cycles enclosing an unstable hyperbolic focus E_5, and the small one is stable, the large one is unstable

Remark 3.6 As a matter of fact, the reproduction number is equal to zero in [13, 18], which simplifies the condition that a Hopf bifurcation occur. In our model, we also comprehensively discuss the existence of a Hopf bifurcation when $R_0 < 1$, $R_0 = 1$ and $R_0 > 1$. Besides, the authors in [13] did not show the appearance of a homoclinic loop, which is an interesting bifurcation phenomenon given in Fig. 7.

3.3 Bogdanov–Takens bifurcation

In this subsection, we investigate the Bogdanov–Takens bifurcation in system (2.1). Lemma 3.7 is from Perko [19], and Lemma 3.8 is Proposition 5.3 in Lamontage et al. [20].

Lemma 3.7 *The system*

$$\begin{cases} \frac{dx}{dt} = y + Ax^2 + Bxy + Cy^2 + \mathcal{O}(|(X,Y)|^3), \\ \frac{dy}{dt} = Dx^2 + Exy + Fy^2 + \mathcal{O}(|(X,Y)|^3), \end{cases}$$

is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = Dx^2 + (E + 2A)xy + \mathcal{O}(|(X,Y)|^3), \end{cases}$$

in some small neighborhood of $(0,0)$ *after changes of coordinates.*

Lemma 3.8 *The system*

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + a_{30}x^3 + a_{40}x^4 + y(a_{21}x^2 + a_{31}x^3) + y^2(a_{12}x + a_{22}x^2) + \mathcal{O}(|(x,y)|^4), \end{cases}$$

is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + (a_{31} - a_{30}a_{21})xy + \mathcal{O}(|(X,Y)|^3), \end{cases}$$

in some small neighborhood of $(0,0)$ *after changes of coordinates.*

From Theorem 2.2, we can see that there exists a unique positive equilibrium $E_1(x_1, y_1)$ when $R_0 = R^*$, where

$$x_1 = \frac{y_1^2 + c}{ay_1 + 1}, \qquad y_1 = \frac{a\Lambda - (1-\sigma)}{2(d + a(1-\sigma))}.$$

From the proving process of Theorem 2.5, it is easily to see that $\det J_1 = 0$. And Theorem 2.4 suggests that the characteristic matrix J_1 possesses a zero eigenvalue with multiplicity 2 when $\psi_1 = 0$, which shows that system (2.1) may admit a Bogdanov–Takens bifurcation. Thus, we can prove the following theorem.

Define two functions:

$$\begin{aligned} f(y) &= (1 + a + d)y^4 + \bigl(1 - ac(3 + 2a + 2d)\bigr)y^3 - 3c(1 + a)y^2 \\ &\quad - c\bigl(1 + ac(-1 + 2d)\bigr)y - c^2 d, \\ g(y) &= c^2 d + 2c(1 + 6a - 2c + cd)y + 6c(2 + a)y^2 \\ &\quad + 2\bigl(2a^2 c + acd - 1\bigr)y^3 - (4 + 2a + d)y^4. \end{aligned}$$

Theorem 3.9 *Suppose that* $R_0 = R^*$ *and* $\psi_1 = 0$, *then the only interior equilibrium* E_1 *of system* (2.1) *is a cusp. Moreover,*

(a) *if* $f(y_1)g(y_1) \neq 0$, *then* E_1 *is a cusp of codimension* 2;
(b) *if* $f(y_1)g(y_1) = 0$, *then* E_1 *is a cusp of codimension greater than or equal to* 3.

Proof Changing the variables as $X = x - x_1$, $Y = y - y_1$, system (2.1) becomes

$$\begin{cases} \frac{dX}{dt} = b_{11}X + b_{12}Y + c_1 XY + c_3 Y^2 + \mathcal{O}(|(X,Y)|^3), \\ \frac{dY}{dt} = b_{21}X + b_{22}Y - c_1 XY + c_2 Y^2 + \mathcal{O}(|(X,Y)|^3), \end{cases} \tag{3.3}$$

where

$$\begin{aligned} &b_{11} = -d - \frac{y_1(ay_1 + 1)}{y_4^2 + c}, \qquad b_{12} = \sigma - \frac{-y_1^2 + 2acy_1 + c}{(y_1^2 + c)^2} x_1, \qquad b_{21} = \frac{y_1(ay_1 + 1)}{y_4^2 + c}, \\ &b_{22} = -1 + \frac{-y_1^2 + 2acy_1 + c}{(y_1^2 + c)^2} x_1, \qquad c_1 = -\frac{-y_1^2 + 2acy_1 + c}{(y_1^2 + c)^2}, \\ &c_2 = \frac{2(y_1^3 - 3acy_1^2 - 3cy_1 + ac^2)x_1}{(y_1^2 + c)^3}, \qquad c_3 = -\frac{x_1(ac^2 - 3cy_1 - 3acy_1^2 + y_1^3)}{(c + y_1^2)^3}. \end{aligned}$$

Make the non-singular linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{b_{11}}{b_{21}} & \frac{1}{b_{21}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then system (3.3) is transformed into

$$\begin{cases} \frac{dx}{dt} = y + b_1x^2 + b_2xy, \\ \frac{dy}{dt} = b_3x^2 + b_4xy + Q_2(x,y), \end{cases} \tag{3.4}$$

where $Q_2(x,y)$ is a smooth function in (x,y) at least of the third order and

$$b_1 = c_2 - \frac{c_1b_{11}}{b_{21}}, \qquad b_2 = -\frac{c_1}{b_{21}}, \qquad b_3 = b_{21}c_3 + b_{11}(c_1 - c_2) + \frac{b_{11}^2c_1}{b_{21}}, \qquad b_4 = \frac{-dc_1}{b_{21}}.$$

By Lemma 3.7, we obtain a topologically equivalent system of system (3.4)

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = b_3u^2 + (b_4 + 2b_1)uv + Q_3(u,v), \end{cases}$$

where

$$b_3 = \frac{df(y_1)}{(c+y_1^2)^3}, \qquad b_4 + 2b_1 = -\frac{g(y_1)}{y_1(ay_1+1)(y_1^2+c)^2}.$$

Therefore, E_1 is a cusp of codimension 2 if $f(y_1)g(y_1) \neq 0$, by the results in Perko [19], or else, E_1 is a cusp of codimension at least 3. □

Remark 3.10 In fact, Theorem 3.9(b) includes the following three cases:

(1) If $f(y_1) \neq 0$ and $g(y_1) = 0$, E_1 is a cusp point;

(2) If $f(y_1) = 0$ and $g(y_1) \neq 0$, E_1 is nilpotent focus/elliptic point;

(3) If $f(y_1) = g(y_1) = 0$, E_1 is a nilpotent focus.

Unfortunately, due to the complexity of $f(y_1)$ and $g(y_1)$, we cannot determine which of these three situations occurs theoretically. But we will show for some parameter values that $f(y_1) \neq 0$ and $g(y_1) = 0$, i.e. E_1 is a cusp point.

In the following, we will give an example to show that Theorem 3.9(b) occurs.

In the first place, fix $y_1 = 1/5$, then we can solve for $\sigma = 1 - \frac{-2d+5a\Lambda}{5+2a}$. By assumptions $\psi_1 = 0$ and $R_0 = R^*$, solve for parameters d and Λ,

$$d = \frac{(5+a)^2\Lambda}{-1+25c+10ac}, \qquad \Lambda = \frac{(35+a(11+a-25c))(1-5(5+2a)c)}{(5+a)^3(1+25c)}.$$

Then

$$f(y_1) = -\frac{(1+25c)(1+a^2c)}{125(5+a)} < 0$$

for any $a, c > 0$. We solve $g(y_1) = 0$ for the parameter c and denote the corresponding solution with respect to c by $c_{\pm}^g$, where

$$c_{\pm}^g = \frac{(375 + a(1630 + a(323 + 2a)) \pm \sqrt{(5+a)(28{,}125 + a(236{,}775 + a(531{,}415 + a(104{,}449 + 4a(318 + a)))))}}{50a(15+2a)}.$$

Actually, if $c = c^g_-$, we have $d = d(a) < 0$ for any $a > 0$. Thus, $g(y_1) = 0$ if and only if $c = c^g_+ \triangleq c^g(a)$. In addition, with the help of Mathematica, for any $0 < a < 1.86433$ or $a > 123.449$, one can have all other parameters positive and satisfying $d + \sigma < 1$.

For example, take $a = 1$, then we can get

$$\begin{aligned}(a, d, c, \Lambda, \sigma) &\doteq (1, 0.108676, 5.478144, 0.575787, 0.619774)\\ &\triangleq (a_0, d_0, c_0, \Lambda_0, \sigma_0),\end{aligned}$$

which satisfies $c = c^g(a)$, i.e. Theorem 3.9(b) is in order.

Theorem 3.11 *When $(a, d, c, \Lambda, \sigma) = (a_0, d_0, c_0, \Lambda_0, \sigma_0)$, then E_1 is a Bogdanov–Takens point of codimension 3, and system (2.1) localized at E_1 is topologically equivalent to*

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + Gx^3y + \mathcal{O}(|(x,y)|^3), \end{cases}$$

where $G < 0$.

Proof First of all, applying a linear transformation $T_1 : (x, y) \to (u, v)$, defined by $u = x - x_1$, $v = y - y_1$, we can reduce system (2.1) further to the form

$$\begin{cases} \frac{du}{dt} = p_{10}u + p_{01}v + \sum_{2\le i+j\le 4} p_{ij}u^iv^j + \mathcal{O}(|(u,v)|^5), \\ \frac{dv}{dt} = q_{10}u + q_{01}v + \sum_{2\le i+j\le 4} q_{ij}u^iv^j + \mathcal{O}(|(u,v)|^5), \end{cases} \tag{3.5}$$

where

$$p_{10} = -d - \frac{y_1(ay_1+1)}{y_4^2+c}, \qquad p_{01} = \sigma - \frac{-y_1^2 + 2acy_1 + c}{(y_1^2+c)^2}x_1, \qquad p_{11} = -\frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2},$$

$$p_{02} = -\frac{x_1(ac^2 - 3acy_1^2 - 3cy_1 + y_1^3)}{(c+y_1^2)^3}, \qquad p_{12} = \frac{-ac^2 + 3acy_1^2 + 3cy_1 - y_1^3}{(c+y_1^2)^3},$$

$$p_{03} = \frac{x_1(4ac^2y_1 - 4acy_1^3 + c^2 - 6cy_1^2 + y_1^4)}{(c+y_1^2)^4},$$

$$p_{13} = \frac{v^3(4ac^2y_1 - 4acy_1^3 + c^2 - 6cy_1^2 + y_1^4)}{(c+y_1^2)^4},$$

$$p_{04} = \frac{x_1(ac^3 - 10ac^2y_1^2 + 5acy_1^4 - 5c^2y_1 + 10cy_1^3 - y_1^5)}{(c+y_1^2)^5},$$

$$q_{10} = \frac{y_1(ay_1+1)}{y_4^2+c}, \qquad q_{01} = -1 + \frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}x_1, \qquad q_{11} = \frac{2acy_1 + c - y_1^2}{(c+y_1^2)^2},$$

$$q_{02} = \frac{x_1(ac^2 - 3acy_1^2 - 3cy_1 + y_1^3)}{(c+y_1^2)^3}, \qquad q_{12} = \frac{(ac^2 - 3acy_1^2 - 3cy_1 + y_1^3)}{(c+y_1^2)^3},$$

$$q_{03} = \frac{x_1(4ac^2y_1 - 4acy_1^3 + c^2 - 6cy_1^2 + y_1^4)}{(c+y_1^2)^4},$$

$$q_{13} = \frac{(-4ac^2y_1 + 4acy_1^3 - c^2 + 6cy_1^2 - y_1^4)}{(c+y_1^2)^4},$$

$$q_{04} = \frac{x_1(ac^3 - 10ac^2y_1^2 + 5acy_1^4 - 5c^2y_1 + 10cy_1^3 - y_1^5)}{(c + y_1^2)^5},$$

$$p_{20} = p_{21} = p_{22} = p_{30} = p_{31} = p_{40} = q_{20} = q_{21} = q_{22} = q_{30} = q_{31} = q_{40} = 0.$$

Another transformation $T_2 : (u, v) \to (x, y)$, defined by $x = v$, $y = q_{10}u - p_{10}v$, reduces system (3.5) to

$$\begin{cases} \frac{dx}{dt} = y + \sum_{2\le i+j\le 4} a_{ij}x^iy^j + \mathcal{O}(|(x,y)|^5), \\ \frac{dy}{dt} = \sum_{2\le i+j\le 4} b_{ij}x^iy^j + \mathcal{O}(|(x,y)|^5), \end{cases}$$

where

$$a_{11} = \frac{q_{11}}{q_{10}}, \qquad a_{20} = q_{02} + \frac{p_{10}q_{11}}{q_{10}}, \qquad a_{21} = \frac{q_{12}}{q_{10}}, \qquad a_{30} = \frac{q_{03}q_{10} + p_{10}q_{12}}{q_{10}},$$

$$a_{40} = q_{04}, \qquad b_{11} = p_{11} - \frac{p_{10}q_{11}}{q_{10}}, \qquad b_{20} = p_{10}p_{11} - p_{10}q_{02} + p_{02}q_{10} - \frac{p_{10}^2q_{11}}{q_{10}},$$

$$b_{21} = p_{12} - \frac{p_{10}q_{12}}{q_{10}}, \qquad b_{30} = p_{10}(p_{12} - q_{03}) + p_{03}q_{10} - \frac{p_{10}^2q_{12}}{q_{10}},$$

$$b_{40} = -p_{10}q_{04} + p_{04}q_{10},$$

and the other coefficients are equal to zero.

Then using the near-identity transformation $T_3 : (x, y) \to (u, v)$, defined by $u = x + \frac{1}{2}(a_{11} + b_{02})x^2 - a_{02}xy$, $v = y + a_{20}x^2 - b_{20}xy$, and parameters $(a_0, d_0, c_0, \Lambda_0, \sigma_0)$ make $b_{11} + 2a_{20} = 0$, so we obtain

$$\begin{cases} \frac{du}{dt} = v + \sum_{3\le i+j\le 4} d_{ij}u^iv^j + \mathcal{O}(|(u,v)|^5), \\ \frac{dv}{dt} = b_{20}u^2 + \sum_{3\le i+j\le 4} e_{ij}x^iy^j + \mathcal{O}(|(u,v)|^5), \end{cases} \tag{3.6}$$

where

$$d_{30} = a_{30} + a_{20}(a_{11} + b_{02}) - a_{02}b_{20}, \qquad d_{21} = -a_{02}(a_{20} - b_{11}) + a_{21} + a_{11}(a_{11} + b_{02}),$$

$$d_{12} = 2a_{12}, \qquad d_{13} = a_{13} + a_{03}(a_{11} + 4b_{02}) - a_{02}(3a_{02}b_{02} + b03), \qquad d_{03} = -a_{02}^2 + a_{03},$$

$$d_{40} = a_{40} - (a_{11} + b_{02})\left(a_{11}a_{20} + \frac{1}{2}a_{30} - \frac{3}{2}a_{02}b_{20}\right) - \frac{3}{2}a_{20}(a_{11} + b_{02})^2 - a_{20}a_{21} - a_{02}b_{30},$$

$$d_{31} = a_{31} - a_{11}^3 + a_{02}\bigl(2a_{30} + 3a_{20}b_{02} + a_{11}(4a_{20} + b_{11}) - b_{21} - 3a_{02}b_{20}\bigr) + b_{02}\bigl(a_{21} - a_{11}^2\bigr) - 2a_{12}a_{20},$$

$$d_{22} = a_{22} - 3a_{03}a_{20} + \frac{1}{2}a_{12}(a_{11} + 5b_{02}) + a_{02}^2(a_{20} - 2b_{11}) + a_{02}\bigl(a_{21} + 2a_{11}(a_{11} + b_{02}) - b_{12}\bigr),$$

$$d_{04} = -a_{02}a_{03} + a_{04}, \qquad e_{30} = b_{30} - a_{20}b_{11} - (a_{11} + 2b_{02})b_{20},$$

$$e_{21} = a_{11}a_{20} - \frac{1}{2}(a_{11} + b_{02})b_{11} + 2a_{02}b_{20} + b_{21},$$

$$e_{12} = -b_{02}(a_{11} + b_{02}) + a_{02}(4a_{20} + b_{11}) + b_{12},$$
$$e_{03} = a_{02}a_{20} + b_{03}, \qquad e_{04} = b_{04} + a_{03}a_{20},$$
$$\begin{aligned} e_{40} &= a_{20}\big(2(a_{30} - a_{11}a_{20} - a_{02}b_{20}) - a_{20}b_{02} - b_{21}\big) \\ &\quad + \frac{3}{2}(a_{11} + b_{02})(a_{20}b_{11} + b_{02}b_{20} - b_{30}) + \frac{5}{4}(a_{11} + b_{02})^2 b_{20} - b_{02}b_{30} + b_{40}, \\ e_{31} &= a_{20}\big(2a_{21} - 2b_{12} + 3b_{02}^2\big) - b_{02}(a_{30} + b_{21}) + \frac{b_{02}^2 b_{11}}{2} + \frac{1}{2}a_{11}^2(-2a_{20} + b_{11}) \\ &\quad + a_{11}\big(b_{02}(3a_{20} + b_{11}) - 4a_{02}b_{20} - b_{21}\big) \\ &\quad - a_{02}\big(a_{20}(6a_{20} + 4b_{11}) + 5b_{02}b_{20} - 3b_{30}\big) + b_{31}, \\ e_{22} &= \frac{1}{2}\big(a_{20}(4a_{12} - 6b_{03}) + \big(a_{11}^2 - 2a_{21} + 14a_{02}a_{20} + b_{12} - 3b_{02}^2\big)b_{02} \\ &\quad + (6a_{02}b_{20} + 4b_{21} - b_{02}b_{11})a_{02} + 2b_{22} - a_{11}\big(2b_{02}^2 + 3a_{02}b_{11} + b_{12}\big)\big), \end{aligned}$$
$$e_{13} = 2a_{03}a_{20} - a_{12}b_{02} + 2b_{02}b_{03} + a_{02}^2(4a_{20} + b_{11}) + a_{02}\big(-b_{02}(a_{11} + 4b_{02}) + b_{12}\big) + b_{13}.$$

We perform a near-identity smooth change of coordinates,

$$\begin{aligned} x &= u - \left(\frac{1}{3}d_{21} + \frac{1}{6}e_{12}\right)u^3 - \left(\frac{1}{2}d_{12} + \frac{1}{2}e_{03}\right)u^2 v, \\ y &= v + d_{30}u^3 + d_{03}v^3 - \frac{1}{2}e_{12}u^2 v - e_{03}uv^2. \end{aligned}$$

Then system (3.6) becomes

$$\begin{cases} \frac{dx}{dt} = y + d_{40}x^4 + d_{04}y^4 + g_{31}x^3 y + d_{13}xy^3 + d_{22}x^2y^2 + \mathcal{O}(|(x,y)|^5), \\ \frac{dy}{dt} = b_{20}x^2 + e_{30}x^3 + f_{40}x^4 + y(f_{21}x^2 + f_{31}x^3) + f_{13}xy^3 + e_{04}y^4 \\ \qquad + f_{22}x^2y^2 + \mathcal{O}(|(x,y)|^5), \end{cases} \tag{3.7}$$

where

$$g_{31} = \left(d_{31} - \frac{e_{11}}{2}(d_{12} + e_{03})\right), \qquad f_{21} = 3d_{30} + e_{21}, \qquad f_{40} = \frac{1}{6}\big(b_{20}(4d_{21} - e_{12}) + 6e_{40}\big),$$
$$f_{31} = b_{20}d_{12} - b_{20}e_{03} + e_{31}, \qquad f_{13} = e_{13}, \qquad f_{22} = 3b_{20}d_{03} + e_{22}.$$

In order to kill the non-resonant cubic terms of system (3.7), we let

$$\begin{aligned} u &= x + \frac{1}{12}(d_{04} + d_{13} + e_{04} + f_{13} - f_{22} - 3g_{31})x^4 + d_{04}y^4 \\ &\quad + \frac{1}{6}(-2d_{22} + d_{04} + d_{13} + e_{04} + f_{13})x^3 y, \\ v &= y + d_{40}x^4 + \frac{1}{3}(d_{04} + d_{13} + e_{04} + f_{13} - f_{22})x^3 y \\ &\quad + (d_{04} + d_{13})xy^3 - \frac{1}{2}(d_{04} + d_{13} + e_{04} + f_{13})x^2y^2, \end{aligned}$$

system (3.7) becomes

$$\begin{cases} \frac{du}{dt} = v + \mathcal{O}(|(u,v)|^5), \\ \frac{dv}{dt} = b_{20}u^2 + e_{30}u^3 + f_{40}u^4 + (f_{21}u^2 + (4d_{40} + f_{31})u^3)v + \mathcal{O}(|(u,v)|^5). \end{cases} \tag{3.8}$$

Finally, we set $x = b_{20}u$, $y = b_{20}(v + \mathcal{O}(|(u,v)|^5))$, system (3.8) becomes

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + \frac{e_{30}}{b_{20}^2}x^3 + \frac{f_{40}}{b_{20}^3}x^4 + y(\frac{f_{21}}{b_{20}^2}x^2 + \frac{4d_{40}+f_{31}}{b_{20}^3}x^3) + \mathcal{O}(|(x,y)|^5). \end{cases}$$

According to Lemma 3.8 the above system is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + Gx^3y + \mathcal{O}(|(x,y)|^3), \end{cases}$$

where $G = \frac{b_{20}(4d_{40}+f_{31})-e_{30}f_{21}}{b_{20}^4}$. Computing the coefficients b_{20}, d_{40}, f_{31}, e_{30}, f_{21} with the condition $(a,d,c,\Lambda,\sigma) = (a_0,d_0,c_0,\Lambda_0,\sigma_0)$ and straightforward calculation lead to $G = -1.061119 \times 10^6 < 0$. Thus, E_1 is a cusp type of Bogdanov–Takens singularity with codimension 3. □

Remark 3.12 The authors in [13, 18] proved their epidemic model with saturated incidence rate undergoes a Bogdanov–Takens bifurcation of codimension 2. When we consider the incidence of a combination of the saturated incidence rate and a non-monotonic incidence, the codimension of Bogdanov–Takens bifurcation can grow up to 3.

Remark 3.13 Xiao and Ruan (see [11]) showed that either the number of infective individuals tends to zero as time evolves or the disease persists. The authors in [13, 18] proved that their epidemic model with saturated incidence rate undergoes a Bogdanov–Takens bifurcation of codimension 2. When we consider the incidence of a combination of the saturated incidence rate and a non-monotonic incidence, the codimension of Bogdanov–Takens bifurcation can grow up to 3.

4 Conclusions

In this paper, by combining qualitative and bifurcation analyses we study an SIS epidemic model with the incidence rate $\frac{aI^2}{c+I^2} + \frac{bI}{c+I^2}$, which is a combination of the saturated incidence rate studied in [13, 18], describing the inhibition effect from the behavioral change and the non-monotonic incidence studied by Ruan in [11], interpreting the "psychological" effect. In Sect. 2, we give a full-scale analysis for the types and stability of the equilibria E_i ($i = 0,1,2,3,4,5$). We prove that for system (2.1) there can occur backward bifurcation and the backward bifurcation will disappear if $a = 0$. At equilibrium E_i ($i = 3,5$), a degenerate Hopf bifurcation arises under certain conditions. When the critical condition Ψ_i ($i = 3,5$) satisfied, we calculate the Liapunov value of the weak focus and obtain the maximal multiplicity of the weak focus is two, indicating that there exist at most two limit cycles around E_i ($i = 3,5$). In Fig. 6 and Fig. 8, we give the phase portraits corresponding to equilibrium E_3 and E_5 exhibiting a unique limit cycle and adding a new limit cycle after a small perturbation of the parameters Λ and c. In Sect. 3.3, we proved that the model exhibits Bogdanov–Takens bifurcation of codimension 2 and codimension 3, under certain conditions. If the parameter $a = 0$, the model can just have a Bogdanov–Takens bifurcation of codimension 2, shown in [13].

In reality, we show that the model exhibits multi-stable states. This interesting phenomenon indicates that the initial states of an epidemic can determine the final states of an epidemic to go extinct or not. Moreover, the periodical oscillations signify that the trend of the disease may be affected by the behavior of the susceptible and the effect of psychology of the disease.

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