Zhan Advances in Difference Equations (2019) 2019:27
https://doi.org/10.1186/s13662-019-1969-8
Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On anisotropic parabolic equations with a nonlinear convection term depending on the spatial variable

Huashui Zhan[1,2*]

*Correspondence:
huashuizhan@163.com
[1]School of Applied Mathematics, Xiamen University of Technology, Xiamen, China
[2]Fujian Engineering and Research Center of Rural Sewage Treatment and Water Safety, Xiamen, China

Abstract
Consider an anisotropic parabolic equation with a nonlinear convection term depending on the spatial variable. If the diffusion coefficients are degenerate, in general, the boundary trace cannot be defined for the weak solution. The existence and the uniqueness of weak solution are researched without the boundary value condition. Moreover, a general method to prove stability of weak solutions independent of the boundary value condition is introduced for the first time.

MSC: 35L65; 35L85; 35K92

Keywords: Anisotropic parabolic equation; Nonlinear convection term; Boundary value condition

1 Introduction

In this paper, the anisotropic parabolic equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right) + \sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.1}$$

is considered, where Ω is a bounded domain in $\mathbb{R}^N$ with a C^2 smooth boundary $\partial\Omega$, $p_i > 1$, $Q_T = \Omega \times (0,T)$, $a_i(x) \in C^1(\overline{\Omega})$, $b_i(\cdot,x,t) \in C(\overline{Q_T})$.

Equation (1.1) arises in the mathematical modeling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, and processes of filtration in glaciology [1–3]. A particular case of Eq. (1.1) is the usual non-Newtonian fluid equation,

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \tag{1.2}$$

which has been researched far and widely, one can refer to [4–6] and the references therein. In recent years, there are more and more mathematicians interested in the anisotropic parabolic equations

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,\nabla u), \quad (x,t) \in Q_T,$$

one can refer to [7–14].

In this paper, we suppose that

$$a_i(x)|_{x\in\Omega} > 0, \qquad a_i(x)|_{x\in\partial\Omega} = 0, \quad i = 1,2,\dots,N, \tag{1.3}$$

then Eq. (1.1) is always degenerate on the boundary. To study the well-posedness of the solutions of Eq. (1.1), the initial value

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

is always indispensable. Moreover, the usual boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.5}$$

may be invalid. This is due to the fact that the weak solution of Eq. (1.1) may lack the enough regularity to be defined the trace on the boundary [15]. Accordingly, one has tried to study the uniqueness of weak solution only depending on the initial value condition (1.4) [16, 17]. In fact, for a degenerate parabolic equation, that the boundary value (1.5) may be overdetermined is well known, one can refer to [18–27]. But how to impose a suitable boundary value condition instead of (1.5) has been a difficult and interesting unsolved problem for a long time.

Inspired by [15–27], we may conjecture that the degeneracy of $a_i(x)$ on the boundary may take the place of the usual boundary value condition (1.5). In other words, the stability of weak solutions can be proved without the condition (1.5). Comparing with our previous work [16, 17], not only the anisotropic case is more complicated than the isotropic case, but also the nonlinear convection term $\sum_{i=1}^{N} \frac{\partial b_i(u,x,t)}{\partial x_i}$ adds difficulties. We employ some special techniques to overcome these difficulties. Moreover, we will introduce a general method to study the stability of weak solutions for a parabolic equation without the boundary value condition.

2 Definitions and main results

We denote

$$p_- = \min\{p_1,p_2,\dots,p_{N-1},p_N\}, \quad p_- > 1,$$
$$p_+ = \max\{p_1,p_2,\dots,p_{N-1},p_N\}.$$

In the first place, we introduce definition of weak solutions.

Definition 2.1 A function $u(x,t)$ is said to be a weak solution of Eq. (1.1) with the initial value (1.4), if

$$u \in L^\infty(Q_T), \qquad a_i(x)|u_{x_i}|^{p_i} \in L^1(Q_T), \qquad u_t \in L^2(Q_T), \tag{2.1}$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} u_t\varphi\,dx\,dt + \sum_{i=1}^{N}\iint_{Q_T}\bigl[a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\cdot\varphi_{x_i} + b_i(u,x,t)\cdot\varphi_{x_i}\bigr]\,dx\,dt = 0. \tag{2.2}$$

The initial value is satisfied in the following sense:

$$\lim_{t\to 0}\int_{\Omega}\left|u(x,t)-u_0(x)\right|dx=0. \tag{2.3}$$

Definition 2.2 The function $u(x,t)$ is said to be the weak solution of Eq. (1.1) with the initial boundary values (1.4)–(1.5) if u satisfies Definition 2.1, and the boundary value condition (1.5) is satisfied in the sense of trace.

Theorem 2.3 *If $p_- > 2$, $a_i(x)\in C^1(\overline{\Omega})$ satisfies (1.3), $b_i(s,x,t)$ is a C^1 function on $\mathbb{R}\times\overline{\Omega}\times[0,T]$,*

$$u_0\in L^{\infty}(\Omega),\qquad |u_{0x_i}|\in L^{p_i}(\Omega),\quad i=1,2,\ldots,N, \tag{2.4}$$

either

$$\int_{\Omega}a_i^{-\frac{2}{p_i-2}}(x)\,dx<\infty, \tag{2.5}$$

or

$$\left|b_{is}(s,x,t)\right|\le ca^{\frac{1}{p_i}}(x), \tag{2.6}$$

then Eq. (1.1) with initial value (1.4) has a weak solution.

Theorem 2.4 *Let $p_- > 2$, for every $1\le i\le N$, either condition (2.5) be true, or $\int_{\Omega}a_i^{-\frac{1}{p_i-1}}(x)\,dx<\infty$ and condition (2.6) be true, $a_i(x)\in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ be a C^1 function on $\mathbb{R}\times\overline{\Omega}\times[0,T]$. Then the initial boundary value problem (1.1)–(1.4)–(1.5) has a solution.*

If $b_i\equiv 0$, then only if $p_- > 1$ and $\int_{\Omega}a_i^{-\frac{1}{p_i-1}}(x)\,dx<\infty$, Theorem 2.3 and Theorem 2.4 are still true. However, if $b_i\equiv 0$ is not valid, when $p_- > 1$, then it is difficult to prove that $u_t\in L^2(Q_T)$. If we do not require $u_t\in L^2(Q_T)$, in other words, if we admit u_t belonging to another kind of Banach space, then the conditions (2.5) and (2.6) may not be necessary, one can refer to our previous work [28]. Moreover, the condition (2.6) (also the condition (2.9)) reflects that there are some relationships between the diffusion coefficient and the convection term. At least, one of our motivations on condition (2.6) (also the condition (2.9)) initially comes from the study of a model of strong degenerate parabolic equation arising in mathematical finance [29], which has the form

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{2}\frac{\partial b_i(u)}{\partial x_i},\quad \text{in } Q_T=\Omega\times(0,T),$$

and satisfies

$$\left|b_i'(s)\right|\le a^{ii}(s),\quad i=1,2,\ldots,N,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with the smooth boundary $\partial\Omega$. From this, one can see that there are some relationships between the diffusion coefficient and the convection term.

Since we mainly are concerned about how the degeneracy of the coefficient $a_i(x)$ affects the uniqueness or the stability of weak solutions, we have no intention to make a deep research on the existence. The main results of this paper are the following stability theorems.

Theorem 2.5 *Let $p_- > 1$, for $1 \le i \le N$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$ and $b_i(s,x,t)$ be a Lipchitz function $\mathbb{R} \times \overline{\Omega} \times [0,T]$. If u and v are two solutions of Eq. (1.1) with the same homogeneous boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.7}$$

and with different initial values $u_0(x)$ and $v_0(x)$, respectively, then

$$\int_\Omega \bigl|u(x,t) - v(x,t)\bigr|\,dx \le \int_\Omega \bigl|u_0(x) - v_0(x)\bigr|\,dx, \quad t \in [0,T). \tag{2.8}$$

Roughly speaking, the condition $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$ can guarantee that the boundary value condition (1.5) is true in the sense of trace. If this condition is invalid, for example,

$$\int_\Omega a_1^{-\frac{1}{p_1-1}}(x)\,dx < \infty$$

and

$$\int_\Omega a_2^{-\frac{1}{p_2-1}}(x)\,dx = \infty,$$

that whether Theorems 2.4–2.5 are true or not is an open problem. Fortunately, by adding some restrictions on $a_i(x)$ and $b_i(s,x,t)$, we are able to prove the following stability of weak solutions without any boundary value condition, no matter whether $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$ or not.

Theorem 2.6 *Let $p_- > 1$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R} \times \overline{\Omega} \times [0,T]$. Let u and v be two solutions of (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. If $b_i(s,x,t)$ satisfies*

$$\bigl|b_i(u,x,t) - b_i(v,x,t)\bigr| \le c a_i^{\frac{1}{p_i}} |u - v|, \quad i = 1,2,\ldots,N, \tag{2.9}$$

and, for η small enough,

$$\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} \le c, i = 1,2,\ldots,N, \tag{2.10}$$

then the stability (2.8) is true.

Here, $\Omega_\eta = \{x \in \Omega : (\prod_{j=1}^{N} a_j(x)) > \eta\}$.

Comparing Theorem 2.6 with Theorem 2.5, we find that, in some cases, the degeneracy of $a_i(x)$ on the boundary can take the place of the usual boundary value condition (1.5). Even, for some given kind of the weak solutions, the condition (2.10) may not be necessary. For example, we have the following result.

Theorem 2.7 *Let $p_- > 1$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R} \times \overline{\Omega} \times [0,T]$. Let u and v be two solutions of (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, and for η small enough,*

$$\begin{aligned} &\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}} \le c, \\ &\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}} \le c, \quad i = 1,2,\ldots,N. \end{aligned} \tag{2.11}$$

If $b_i(s,x,t)$ satisfies (2.9), then the stability (2.8) is true.

However, for some weak solutions, condition (2.9) may not be necessary. In fact, if the convection term is independent of the diffusion coefficient, we have the following result.

Theorem 2.8 *Let $p_- > 1$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R} \times \overline{\Omega} \times [0,T]$. If u and v are two solutions of Eq. (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then, for any $\Omega_1 \subset\subset \Omega$,*

$$\int_{\Omega_1} \left|u(x,t) - v(x,t)\right|^2 dx \le c(\Omega_1) \int_{\Omega} \left|u_0(x) - v_0(x)\right|^2 dx, \tag{2.12}$$

which implies that the uniqueness of weak solution is true.

Actually, by the general method introduced in the last section of this paper, many kinds of stability theorems of weak solutions can be found.

3 The weak solutions dependent on the initial value

We consider the following regularized problem:

$$u_{\varepsilon t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left(a_i(x) + \varepsilon\right)|u_{\varepsilon x_i}|^{p_i-2} u_{\varepsilon x_i}\right) - \sum_{i=1}^{N} \frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i} = 0, \quad (x,t) \in Q_T, \tag{3.1}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.2}$$

$$u_\varepsilon(x,0) = u_{\varepsilon 0}(x), \quad x \in \Omega. \tag{3.3}$$

Here, $u_{\varepsilon 0} \in C_0^\infty(\Omega)$, $|u_{\varepsilon 0}|_{L^\infty(\Omega)} \le |u_0|_{L^\infty(\Omega)}$, $|\nabla u_{\varepsilon 0}|$ converges to $|\nabla u_0(x)|$ in $L^{p_+}(\Omega)$. It is well known that the above problem has an unique weak solution $u_\varepsilon \in L^\infty(0,T; W_0^{1,\vec{p}}(a_i(x), \Omega))$ [5, 30].

By the maximum principle [5], there is a constant c only dependent on $\|u_0\|_{L^\infty(\Omega)}$ but independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c.$$

Multiplying (3.1) by u_ε and integrating it over Q_T, then

$$\begin{aligned}
&\frac{1}{2}\int_\Omega u_\varepsilon^2\,dx + \sum_{i=1}^N \iint_{Q_T}\big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt + \iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\,dt \\
&\quad= \frac{1}{2}\int_\Omega u_0^2\,dx. \qquad (3.4)
\end{aligned}$$

If $\int_\Omega a_i^{-\frac{2}{p_i-2}}(x)\,dx < \infty$, we know that $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$, then

$$\begin{aligned}
\left|\int_\Omega \frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\right| &\le \int_\Omega \left|\frac{\partial b_i(s,x,t)}{\partial s}\bigg|_{s=u_\varepsilon} u_{\varepsilon x_i}\right||u_\varepsilon|\,dx \\
&\le c\int_\Omega \left|\frac{\partial b_i(s,x,t)}{\partial s}\bigg|_{s=u_\varepsilon} u_{\varepsilon x_i}\right|dx \\
&\le \frac{1}{2}\int_\Omega a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx + \frac{c}{2}\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx \\
&\le \frac{1}{2}\int_\Omega a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx + c.
\end{aligned}$$

If the condition (2.6) is true, then

$$\begin{aligned}
\left|\int_\Omega \frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}u_\varepsilon\,dx\right| &\le c\int_\Omega \left|\frac{\partial b_i(s,x,t)}{\partial s}\bigg|_{s=u_\varepsilon} u_{\varepsilon x_i}\right|dx \\
&\le \frac{1}{2}\int_\Omega a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx + c,
\end{aligned}$$

clearly. Accordingly, by (3.4), we have

$$\int_\Omega u_\varepsilon^2\,dx + \sum_{i=1}^N \iint_{Q_T}\big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt \le c. \qquad (3.5)$$

For any $\Omega_1 \subset\subset \Omega$, since $p_- = \min\{p_i\} > 2$, $a_i(x)$ satisfies (1.3),

$$a_i(x) \ge c(\Omega_1) > 0, \quad i = 1,2,\ldots,N,$$

by (3.5),

$$\begin{aligned}
\int_0^T\int_{\Omega_1}|\nabla u_\varepsilon|^2\,dx\,dt &\le c\left(\int_0^T\int_{\Omega_1}|\nabla u_\varepsilon|^{p_-}\,dx\,dt\right)^{\frac{2}{p_-}} \\
&\le c(\Omega_1)\sum_{i=1}^N\left(\int_0^T\int_{\Omega_1}a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt\right)^{\frac{2}{p_i}} \\
&\le c(\Omega_1)\sum_{i=1}^N\left(\int_0^T\int_{\Omega}a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt\right)^{\frac{2}{p_i}} \\
&\le c, \qquad (3.6)
\end{aligned}$$

where $c(\Omega_1)$ represents the constant depending upon the compact subset Ω_1, but it may be different from one to another.

Multiplying (2.5) by $u_{\varepsilon t}$, integrating it over Q_T, it yields

$$\begin{aligned}
&\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt \\
&\quad= \sum_{i=1}^{N} \iint_{Q_T} \frac{\partial}{\partial x_i}\big(\big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\big)u_{\varepsilon t}\,dx\,dt \\
&\qquad+ \sum_{i=1}^{N} \iint_{Q_T} u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt. \qquad (3.7)
\end{aligned}$$

Noticing that

$$|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}u_{\varepsilon x_i t} = \frac{1}{2}\frac{d}{dt}\int_0^{|u_{\varepsilon x_i}|^2} s^{\frac{p_i-2}{2}}\,ds,$$

then

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial}{\partial x_i}\big(\big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\big)u_{\varepsilon t}\,dx\,dt \\
&\quad= -\iint_{Q_T} \big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}u_{\varepsilon x_i t}\,dx\,dt \\
&\quad= -\frac{1}{2}\iint_{Q_T} \big(a_i(x)+\varepsilon\big)\frac{d}{dt}\int_0^{|u_{\varepsilon x_i}|^2} s^{\frac{p_i-2}{2}}\,ds\,dx\,dt. \qquad (3.8)
\end{aligned}$$

If $\int_\Omega a_i^{-\frac{2}{p_i-2}}(x)\,dx < \infty$,

$$\begin{aligned}
\iint_{Q_T} u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt &\le \iint_{Q_T} \big|b_{iu}(u_\varepsilon,x,t)\big||u_{\varepsilon x_i}||u_{\varepsilon t}|\,dx\,dt \\
&\quad+ \iint_{Q_T} \big|b_{ix_i}(u_\varepsilon,x,t)\big||u_{\varepsilon t}|\,dx\,dt \\
&\le \frac{1}{2}\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + c\iint_{Q_T} |u_{\varepsilon x_i}|^2\,dx\,dt + c, \qquad (3.9)
\end{aligned}$$

by the Hölder inequality

$$\begin{aligned}
\iint_{Q_T} |u_{\varepsilon x_i}|^2\,dx\,dt &= c\iint_{Q_T} a^{-\frac{2}{p_i}}a^{\frac{2}{p_i}}|u_{\varepsilon x_i}|^2\,dx\,dt \\
&\le c\left(\iint_{Q_T} a^{-\frac{2}{p_i-2}}\,dx\,dt\right)^{\frac{p_i-2}{p_i}}\left(\iint_{Q_T} a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt\right)^{\frac{2}{p_i}} \\
&\le c. \qquad (3.10)
\end{aligned}$$

If $|b_{is}(s,x,t)| \leq c a_i^{\frac{2}{p_i}}(x)$, $p_i \geq 2$, then by the Young inequality

$$\begin{aligned}
\iint_{Q_T} u_{\varepsilon t} \frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt &\leq \iint_{Q_T} \big|b_{iu}(u_\varepsilon,x,t)\big||u_{\varepsilon x_i}||u_{\varepsilon t}|\,dx\,dt \\
&\quad + \iint_{Q_T} \big|b_{ix_i}(u_\varepsilon,x,t)\big||u_{\varepsilon t}|\,dx\,dt \\
&\leq \iint_{Q_T} a_i^{\frac{2}{p_i}}(x)|u_{\varepsilon x_i}|^2\,dx\,dt + \frac{1}{2}\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + c \\
&\leq c\iint_{Q_T} a_i(x)|u_{\varepsilon x_i}|^{p_i}\,dx\,dt + \frac{1}{2}\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + c.
\end{aligned}$$

Combining (3.7)–(3.10), we have

$$\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + \sum_{i=1}^{N}\iint_{Q_T} \big(a_i(x)+\varepsilon\big)\frac{d}{dt}\int_0^{|u_{\varepsilon x_i}|^2} s^{\frac{p_i-2}{2}}\,ds\,dx\,dt \leq c,$$

by the above inequality, we have

$$\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt \leq c + c\sum_{i=1}^{N}\int_{\Omega} \big(a_i(x)+\varepsilon\big)|u_{\varepsilon 0x_i}|^{p_i}\,dx \leq c. \tag{3.11}$$

Now, by (3.4), (3.5), (3.6) and (3.11), there exist a function u and an n-dimensional vector function $\overrightarrow{\zeta} = (\zeta_1,\ldots,\zeta_n)$ satisfying $u_\varepsilon \to u$ a.e. in Q_T, and

$$\begin{aligned}
& u \in L^\infty(Q_T), \qquad |\zeta_i| \in L^{\frac{p_i}{p_i-1}}(Q_T), \\
& u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T), \\
& b_i(u_\varepsilon,x,t) \to b_i(u,x,t), \quad \text{a.e. in } Q_T, \\
& u_{\varepsilon x_i} \rightharpoonup u_{x_i}, \quad \text{in } L^{p_i}_{\text{loc}}(Q_T), \\
& a_i(x)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i} \rightharpoonup \zeta_i, \quad \text{in } L^{\frac{p_i}{p_i-1}}(Q_T).
\end{aligned} \tag{3.12}$$

It is easy to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^{N}\iint_{Q_T} \big(a_i(x)+\varepsilon\big)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\varphi_{x_i}\,dx\,dt \\
&\quad = \lim_{\varepsilon\to 0}\sum_{i=1}^{N}\iint_{Q_T} a_i(x)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\varphi_{x_i}\,dx\,dt \\
&\quad = \iint_{Q_T} \overrightarrow{\zeta}\cdot\nabla\varphi\,dx\,dt,
\end{aligned} \tag{3.13}$$

for any $\varphi \in C_0^1(Q_T)$.

Now, we will prove that

$$\sum_{i=1}^{N}\iint_{Q_T} a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\varphi_{1x_i}\,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta}\cdot\nabla\varphi_1\,dx\,dt, \tag{3.14}$$

for any given function $\varphi_1 \in C_0^1(Q_T)$. In detail, we notice that, for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} \left[\frac{\partial u_\varepsilon}{\partial t}\varphi + \sum_{i=1}^N \bigl(a_i(x)+\varepsilon\bigr)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\varphi_{x_i} + \sum_{i=1}^N b_i(u_\varepsilon,x,t)\varphi_{x_i} \right] dx\,dt = 0. \tag{3.15}$$

Let $\varepsilon \to 0$. Then

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t}\varphi + \sum_{i=1}^N \zeta_i \varphi_{x_i} + \sum_{i=1}^N b_i(u,x,t)\varphi_{x_i} \right] dx\,dt = 0. \tag{3.16}$$

Let $0 \le \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ on $\operatorname{supp}\varphi_1$. Let $v \in L^\infty(Q_T)$, $a_i(x)|v_{x_i}|^{p_i} \in L^1(Q_T)$. One has

$$\iint_{Q_T} \psi a_i(x)\bigl(|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\bigr)(u_{\varepsilon x_i} - v_{x_i})\,dx\,dt \ge 0. \tag{3.17}$$

By choosing $\varphi = \psi u_\varepsilon$ in (3.15),

$$\begin{aligned} &\iint_{Q_T} \left[\frac{\partial u_\varepsilon}{\partial t}\psi u_\varepsilon + \sum_{i=1}^N \bigl(a_i(x)+\varepsilon\bigr)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}(\psi u_\varepsilon)_{x_i} + \sum_{i=1}^N b_i(u_\varepsilon,x,t)(\psi u_\varepsilon)_{x_i} \right] dx\,dt \\ &\quad = 0. \end{aligned} \tag{3.18}$$

By (3.17)–(3.18), we have

$$\begin{aligned} &\frac{1}{2}\iint_{Q_T} \psi_t u_\varepsilon^2\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} \bigl(a_i(x)+\varepsilon\bigr)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\psi_{x_i}u_\varepsilon\,dx\,dt \\ &\qquad - \sum_{i=1}^N \iint_{Q_T} \bigl(a_i(x)+\varepsilon\bigr)|v_{x_i}|^{p_i-2}v_{x_i}(u_{\varepsilon x_i} - v_{x_i})\psi\,dx\,dt \\ &\qquad - \sum_{i=1}^N \iint_{Q_T} \bigl(a_i(x)+\varepsilon\bigr)|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}v_{x_i}\psi\,dx\,dt \\ &\qquad - \sum_{i=1}^N \iint_{Q_T} b_i(u_\varepsilon,x,t)(u_{\varepsilon x_i}\psi + u_\varepsilon\psi_{x_i})\,dx\,dt \\ &\quad \ge 0. \end{aligned} \tag{3.19}$$

Let $\varepsilon \to 0$. Then

$$\begin{aligned} &\frac{1}{2}\iint_{Q_T} \psi_t u^2\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} u\zeta_i\psi_{x_i}\,dx\,dt \\ &\qquad - \sum_{i=1}^N \iint_{Q_T} a_i(x)|v_{x_i}|^{p_i-2}v_{x_i}(u_{x_i} - v_{x_i})\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} a_i(x)\zeta_{x_i}v_{x_i}\psi\,dx\,dt \\ &\qquad - \sum_{i=1}^N \iint_{Q_T} b_i(u,x,t)(u_{x_i}\psi + u\psi_{x_i})\,dx\,dt \\ &\quad \ge 0. \end{aligned} \tag{3.20}$$

Let $\varphi = \psi u$ in (3.16). We get

$$\frac{1}{2}\iint_{Q_T} u^2\psi_t\,dx\,dt - \sum_{i=1}^{N}\iint_{Q_T}\psi\zeta_i u_{x_i}\,dx\,dt - \sum_{i=1}^{N}\iint_{Q_T} u\zeta_i\psi_{x_i}\,dx\,dt - \sum_{i=1}^{N}\iint_{Q_T} b_i(u,x,t)(u_{x_i}\psi + u\psi_{x_i})\,dx\,dt = 0. \tag{3.21}$$

Thus

$$\sum_{i=1}^{N}\iint_{Q_T}\psi\bigl(\zeta_i - a_i(x)|v_{x_i}|^{p_i-2}v_{x_i}\bigr)(u_{x_i}-v_{x_i})\,dx\,dt \geq 0. \tag{3.22}$$

Let $v = u - \lambda\varphi_1$, $\lambda > 0$. Then

$$\sum_{i=1}^{N}\iint_{Q_T}\psi\bigl(\zeta_i - a_i(x)\bigl|(u-\lambda\varphi_1)_{x_i}\bigr|^{p_i-2}\bigr)\varphi_{1x_i}\,dx\,dt \geq 0.$$

If $\lambda \to 0$, then

$$\sum_{i=1}^{N}\iint_{Q_T}\psi\bigl(\zeta_i - a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\bigr)\varphi_{1x_i}\,dx\,dt \geq 0.$$

Moreover, if $\lambda < 0$, similarly we can get

$$\sum_{i=1}^{N}\iint_{Q_T}\psi\bigl(\zeta_i - a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\bigr)\varphi_{1x_i}\,dx\,dt \leq 0.$$

Thus

$$\sum_{i=1}^{N}\iint_{Q_T}\psi\bigl(\zeta_i - a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\bigr)\varphi_{1x_i}\,dx\,dt = 0.$$

Noticing that $\psi = 1$ on suppφ_1, then (3.14) holds.

At last, we are able to prove (2.3) as in [31], then u is a solution of Eq. (1.1) with the initial value (1.4) in the sense of Definition 2.1. Thus we have Theorem 2.3.

Now, by a similar method as in [32], we can prove the following.

Lemma 3.1 *If* $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$, *$u$ is a weak solution of Eq.* (1.1) *with the initial condition* (1.4). *Then, for any given* $t \in [0,T)$,

$$\int_\Omega |u_{x_i}|\,dx \leq c, \quad i = 1,2,\ldots,N. \tag{3.23}$$

For simplicity, we omit the details of the proof of Lemma 3.1 here. By (3.23) and the fact $\iint_{Q_T}|u_t|\,dx\,dt \leq c$, we know that $u \in BV(Q_T)$, $C_0^\infty(Q_T)$ is dense in $BV(Q_T)$ and the trace of u on the boundary $\partial\Omega$ can be defined in the traditional way. By Theorem 2.3 and Lemma 3.1, we clearly have Theorem 2.4.

4 The stability of the initial boundary value problem

In order to prove the stability of the weak solutions, for small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+.$$

Obviously, $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{aligned} &h_\eta(s) \geq 0, \qquad \left|sh_\eta(s)\right| \leq 1, \qquad \left|S_\eta(s)\right| \leq 1; \\ &\lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \qquad \lim_{\eta\to 0} sS'_\eta(s) = 0. \end{aligned} \tag{4.1}$$

Clearly, if we denote $H_\eta(s) = \int_0^s S_\eta(\tau)\,d\tau$, then we have

$$\lim_{\eta\to 0} H_\eta(s) = |s|, \quad s \in (-\infty, +\infty). \tag{4.2}$$

Lemma 4.1 *Let $p_- > 1$, for $1 \leq i \leq N$, $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$ and*

$$\left|b_i(u,x,t) - b_i(v,x,t)\right| \leq c a_i^{\frac{1}{p_i}} |u - v|. \tag{4.3}$$

If u and v are two solutions of Eq. (1.1) with the same homogeneous value condition

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{4.4}$$

and with different initial values $u_0(x)$ and $v_0(x)$, respectively, then

$$\int_\Omega \left|u(x,t) - v(x,t)\right| dx \leq \int_\Omega \left|u_0(x) - v_0(x)\right| dx, \quad t \in [0,T).$$

Proof Let u and v be two weak solutions of Eq. (1.1). Since $\int_\Omega a_i^{-\frac{1}{p_i-1}}(x)\,dx < \infty$, by Lemma 3.1, $u, v \in BV(Q_T)$ we can choose $\varphi = \chi_{[\tau,s]} S_\eta(u - v)$ as the test function. Here $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s] \subset (0,T)$. Then

$$\begin{aligned} &\int_\tau^s \int_\Omega S_\eta(u - v)\frac{\partial(u - v)}{\partial t}\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u - v)_{x_i} h_\eta(u - v)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \left[b_i(u,x,t) - b_i(v,x,t)\right](u - v)_{x_i} h_\eta(u - v)\,dx\,dt \\ &= 0. \end{aligned} \tag{4.5}$$

As usual, one has

$$\int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u - v)_{x_i} h_\eta(u - v)\,dx\,dt \geq 0. \tag{4.6}$$

Since $\iint_{Q_T} |u_t|\,dx\,dt \le c$, $\iint_{Q_T} |v_t|\,dx\,dt \le c$, using the dominated convergence theorem, one has

$$\begin{aligned}
&\lim_{\eta\to 0}\int_\tau^s \int_\Omega S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&= \lim_{\eta\to 0}\int_\Omega \big[H_\eta(u-v)(x,s) - H_\eta(u-v)(x,\tau)\big]\,dx \\
&= \int_\Omega |u-v|(x,s)\,dx - \int_\Omega |u-v|(x,\tau)\,dx, \qquad (4.7)
\end{aligned}$$

where $H_\eta(u-v)(x,s) = H_\eta(u(x,s)-v(x,s))$.

Moreover, since $b_i(s,x,t)$ satisfies the condition (4.3), one has

$$\begin{aligned}
&\lim_{\eta\to 0}\sum_{i=1}^N \left|\int_\tau^s \int_\Omega \big[b_i(u,x,t) - b_i(v,x,t)\big](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt\right| \\
&\le c\lim_{\eta\to 0}\sum_{i=1}^N \int_\tau^s \int_\Omega \big|h_\eta(u-v)(u-v)a_i^{\frac{1}{p_i}}(u-v)_{x_i}\phi_\eta\big|\,dx\,dt \\
&\le c\lim_{\eta\to 0}\sum_{i=1}^N \left(\int_\tau^s \int_\Omega a_i\big(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\big)\,dx\,dt\right)^{\frac{1}{p_i}} \\
&\quad\cdot \left(\int_\tau^s \int_\Omega \big|(u-v)h_\eta(u-v)\big|^{\frac{p_i}{p_i-1}}\,dx\,dt\right)^{\frac{p_i-1}{p_i}} \\
&= 0. \qquad (4.8)
\end{aligned}$$

Now, let $\eta\to 0$ in (4.5). By (4.6)–(4.8), one has

$$\int_\Omega \big|u(x,s) - v(x,s)\big|\,dx \le \int_\Omega \big|u(x,\tau) - v(x,\tau)\big|\,dx.$$

Let $\tau\to 0$. Then

$$\int_\Omega \big|u(x,s) - v(x,s)\big|\,dx \le \int_\Omega \big|u_0(x) - v_0(x)\big|\,dx.$$

Lemma 4.1 is proved. □

In fact, the condition (4.3) in Lemma 4.1 is not the optimal. Without the condition (4.3), we have Theorem 2.5.

Proof of Theorem 2.5 From the above proof of Lemma 4.1, we only need to prove that

$$\lim_{\eta\to 0}\left|\int_\Omega \big[b_i(u,x,t) - b_i(v,x,t)\big](u-v)_{x_i}h_\eta(u-v)\,dx\right| = 0, \qquad (4.9)$$

without the condition (4.3). In detail, we have

$$\begin{aligned}
&\int_\Omega \big[b_i(u,x,t) - b_i(v,x,t)\big](u-v)_{x_i}h_\eta(u-v)\,dx \\
&= \int_{\{\Omega:|u-v|<\eta\}} \big[b_i(u,x,t) - b_i(v,x,t)\big](u-v)_{x_i}h_\eta(u-v)\,dx.
\end{aligned}$$

If the set $\{\Omega : |u-v| = 0\}$ has zero a measure, then

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_{\{\Omega:|u-v|<\eta\}}\big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}h_\eta(u-v)\,dx\right| \\
&\quad\le c\left(\int_{\{\Omega:|u-v|=0\}}\big(a_i^{\frac{1}{p_i}}|u_{x_i}-v_{x_i}|\big)^{p_i}\,dx\right)^{\frac{1}{p_i}}\left(\int_\Omega a_i^{-\frac{1}{p_i-1}}\,dx\right)^{\frac{p_i-1}{p_i}} \\
&\quad= 0.
\end{aligned}$$

If the set $\{\Omega : |u-v| = 0\}$ only has a positive measure, then by, $a_i^{-\frac{1}{p_i-1}} \in L^1(\Omega)$,

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_{\{\Omega:|u-v|<\eta\}}\big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}h_\eta(u-v)\,dx\right| \\
&\quad\le c\left(\int_\Omega\big(a_i^{\frac{1}{p_i}}|u_{x_i}-v_{x_i}|\big)^{p_i}\,dx\right)^{\frac{1}{p_i}}\left(\int_{\{\Omega:|u-v|=0\}} a_i^{-\frac{1}{p_i-1}}\,dx\right)^{\frac{p_i-1}{p_i}} \\
&\quad\le c\left(\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\big)\,dx\right)^{\frac{1}{p_i}}\left(\int_{\{\Omega:|u-v|=0\}} a_i^{-\frac{1}{p_i-1}}\,dx\right)^{\frac{p_i-1}{p_i}} \\
&\quad= 0.
\end{aligned}$$

Thus, we have the conclusion. □

5 The global stability without the boundary value condition

Proof of Theorem 2.6 Let u and v be two weak solutions of Eq. (1.1) with the initial values $u_0(x)$, $v_0(x)$, respectively.

Let $\Omega_\eta = \{x \in \Omega : \prod_{i=1}^N a_i(x) > \eta\}$, and

$$\phi_\eta(x) = \begin{cases} 1, & \text{if } x \in \Omega_\eta, \\ \frac{1}{\eta}\prod_{i=1}^N a_i(x), & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases} \tag{5.1}$$

Let us recall

$$J(x) = \begin{cases} k\exp[\frac{-1}{1-|x|^2}], & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

where k is a constant such that $\int_{\mathbb{R}^N} J(x)\,dx = 1$. The usual mollifier is defined as

$$J_\varepsilon(x) = \frac{1}{\varepsilon^N}J\left(\frac{x}{\varepsilon}\right)$$

for small $\varepsilon > 0$. Let

$$f_\varepsilon(x) = J_\varepsilon f(x) = J_\varepsilon * f(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y)f(y)\,dy,$$

for any $f(x) \in L^1_{\mathrm{loc}}(\overline{\Omega})$.

Let $\phi_{\eta\varepsilon}(x)$ be the mollified function of $\phi_\eta(x)$. We can choose $\chi_{[\tau,s]}\phi_{\eta\varepsilon}(x)S_\eta(u-v)$ as the test function. By the process of taking the limit, $\varepsilon \to 0$, we can choose $\chi_{[\tau,s]}\phi_\eta(x)S_\eta(u-v)$ as the test function finally. Then

$$\begin{aligned}
&\int_\tau^s \int_\Omega \phi_\eta S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}\nabla v\big)(u_{x_i}-v_{x_i})h_\eta(u-v)\phi_\eta(x)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\big)(u-v)S_\eta(u-v)\phi_{\eta x_i}\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \big[b_i(u,x,t)-b_i(v,x,t)\big]\phi_{\eta x_i}S_\eta(u-v)\,dx\,dt \\
&\quad + \sum_{i=1}^N \int_\tau^s \int_\Omega \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt \\
&= 0.
\end{aligned} \tag{5.2}$$

Let us observe every term on the left-hand side of (5.2).

For the first term, using the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{\eta\to 0}\int_\tau^s \int_\Omega \phi_\eta(x)S_\eta(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&= \lim_{\eta\to 0}\int_\tau^s \int_\Omega \frac{\partial[\phi_\eta(x)H_\eta(u-v)]}{\partial t}\,dx\,dt \\
&= \lim_{\eta\to 0}\int_\Omega \phi_\eta(x)\big[H_\eta(u-v)(x,s)-H_\eta(u-v)(x,\tau)\big]\,dx \\
&= \int_\Omega |u-v|(x,s)\,dx - \int_\Omega |u-v|(x,\tau)\,dx.
\end{aligned} \tag{5.3}$$

For the second term, we have

$$\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\big)(u_{x_i}-v_{x_i})h_\eta(u-v)\phi_\eta(x)\,dx \ge 0. \tag{5.4}$$

For the third term, obviously, $\phi_{\eta x_i} = \frac{1}{\eta}(\prod_{j=1}^N a_j(x))_{x_i}$ when $x \in \Omega \setminus \Omega_\eta$, in the other places, it is identical to zero. By the condition (2.10), we have

$$\begin{aligned}
&\left|\int_\Omega a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\big)\phi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&= \left|\int_{\Omega\setminus\Omega_\eta} a_i(x)\big(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\big)\phi_{\eta x_i}S_\eta(u-v)\,dx\right| \\
&\le \frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} a_i(x)\big(|u_{x_i}|^{p_i-1} + |v_{x_i}|^{p_i-1}\big)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i} S_\eta(u-v)\right|\,dx
\end{aligned}$$

$$\begin{aligned}
&\leq c\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}\left(a_i(x)|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\right)dx\right)^{\frac{p_i-1}{p_i}}\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\\
&\leq c\left[\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}+\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}\right]\\
&\quad\cdot\left[\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\right]\\
&\leq c\left[\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}+\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}\right].
\end{aligned}\tag{5.5}$$

Then

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\tau^s\int_\Omega a_i(x)\left(|u_{x_i}|^{p-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\phi_{\eta x_i}S_\eta(u-v)\,dx\,dt\right|\\
&\leq c\lim_{\eta\to 0}\int_\tau^s\left[\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|u_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}+\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)|v_{x_i}|^{p_i}\,dx\right)^{\frac{p_i-1}{p_i}}\right]dt\\
&=0.
\end{aligned}\tag{5.6}$$

For the fourth term, since $b_i(s,x,t)$ satisfies the condition (2.9), we have

$$\lim_{\eta\to 0}\sum_{i=1}^N\int_\tau^s\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\phi_\eta h_\eta(u-v)\,dx\,dt=0,\tag{5.7}$$

as before.

Finally, for the fifth term, by the condition (2.10), we have

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\Omega\left[b_i(u,x,t)-b_i(v,x,t)\right]\phi_{\eta x_i}S_\eta(u-v)\,dx\right|\\
&=\lim_{\eta\to 0}\left|\int_{\Omega\setminus\Omega_\eta}\left[b_i(u,x,t)-b_i(v,x,t)\right]\phi_{\eta x_i}S_\eta(u-v)\,dx\right|\\
&\leq\lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta}a_i^{\frac{1}{p_i}}\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|\left|S_\eta(u-v)(u-v)\right|dx\\
&\leq\lim_{\eta\to 0}\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta}a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\left(\int_\Omega\left|S_\eta(u-v)(u-v)\right|^{\frac{p_i}{p_i-1}}dx\right)^{\frac{p_i-1}{p_i}}\\
&\leq c\lim_{\eta\to 0}\left(\int_\Omega\left|S_\eta(u-v)(u-v)\right|^{\frac{p_i}{p_i-1}}dx\right)^{\frac{p_i-1}{p_i}}\\
&\leq c\left(\int_\Omega|u-v|\,dx\right)^{\frac{p_i-1}{p_i}},
\end{aligned}\tag{5.8}$$

since $u,v\in L^\infty(Q_T)$.

Now, let $\eta \to 0$ in (5.2). Then

$$\int_\Omega \left|u(x,s)-v(x,s)\right| dx \le \int_\Omega \left|u(x,\tau)-v(x,\tau)\right| dx + c\left(\int_0^t \int_\Omega |u-v|\,dx\,dt\right)^l, \tag{5.9}$$

where $l < 1$.

Let $\kappa(s) = \int_\Omega |u(x,s)-v(x,s)|\,dx$. Without loss of the generality, we may assume that there exists $\tau \in [0,T)$, $\kappa(\tau) > 0$. Then, for any $s > \tau$, $\int_\tau^s k(t)\,dt > 0$. If we denote

$$\tau_0 = \max\bigl\{t \in [\tau,s], \kappa(t) > 0\bigr\}, \qquad \int_\tau^{\tau_0} k(t)\,dt = c_1,$$

then $\tau < \tau_0 \le s$, and

$$\int_\tau^s k(t)\,dt \ge \int_\tau^{\tau_0} k(t)\,dt = c_1.$$

By $u, v \in L^\infty(Q_T)$, there exists a constant $C > 0$ such that

$$\frac{c(\int_\tau^s k(t)\,dt)^l}{\int_\tau^s k(t)\,dt} \le \frac{c(\int_\tau^s k(t)\,dt)^l}{c_1} \le C = C(c,c_1,T,q). \tag{5.10}$$

By (5.9) and (5.10), we have

$$\kappa(s) - \kappa(\tau) \le (C+c)\int_\tau^s k(t)\,dt,$$

using the Gronwall inequality, we easily get

$$\int_\Omega \left|u(x,s)-v(x,s)\right| dx \le c\int_\Omega \left|u(x,\tau)-v(x,\tau)\right| dx.$$

then, by the arbitrariness of τ,

$$\int_\Omega \left|u(x,s)-v(x,s)\right| dx \le c\int_\Omega \left|u_0(x)-v_0(x)\right| dx. \qquad \square$$

Proof of Theorem 2.7 Similar to the proof of Theorem 2.6, we have (5.1)–(5.4). Now, by the condition (2.11), we have

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\Omega a_i(x)\bigl(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\bigr)\phi_{\eta x_i} S_\eta(u-v)\,dx\right| \\
&\quad = \lim_{\eta\to 0}\left|\int_{\Omega\setminus\Omega_\eta} a_i(x)\bigl(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\bigr)\phi_{\eta x_i} S_\eta(u-v)\,dx\right| \\
&\quad \le \lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} a_i(x)\bigl(|u_{x_i}|^{p_i-1} + |v_{x_i}|^{p_i-1}\bigr)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i} S_\eta(u-v)\right| dx \\
&\quad \le c\lim_{\eta\to 0}\frac{1}{\eta}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)\bigl(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\bigr)\,dx\right)^{\frac{p_i-1}{p_i}}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}}
\end{aligned}$$

$$\begin{aligned}
&\leq c \lim_{\eta\to 0}\left(\int_{\Omega\setminus\Omega_\eta} a_i(x)\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}}\\
&= 0.
\end{aligned} \tag{5.11}$$

Last but not least, since $a_i(x) \in C^1(\overline{\Omega})$, $a_i(x) = 0$ when $x \in \partial\Omega$, we have

$$a_i^{\frac{1}{p_i}}(x)\left(\prod_{j=1}^N a_j(x)\right)_{x_i} = 0, \quad x \in \partial\Omega. \tag{5.12}$$

According to the definition of Ω_η, we have

$$\begin{aligned}
&\lim_{\eta\to 0}\left|\int_\Omega \left[b_i(u,x,t) - b_i(v,x,t)\right]\phi_{\eta x_i} S_\eta(u-v)\, dx\right|\\
&= \lim_{\eta\to 0}\left|\int_{\Omega\setminus\Omega_\eta} \left[b_i(u,x,t) - b_i(v,x,t)\right]\phi_{\eta x_i} S_\eta(u-v)\, dx\right|\\
&\leq \lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} a_i^{\frac{1}{p_i}}\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right|\left|S_\eta(u-v)(u-v)\right| dx\\
&\leq c \lim_{\eta\to 0}\frac{1}{\eta}\int_{\Omega\setminus\Omega_\eta} a_i^{\frac{1}{p_i}}\left|\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\right| dx\\
&\leq c \max_{x\in\partial\Omega} a_i^{\frac{1}{p_i}}(x)\left(\prod_{j=1}^N a_j(x)\right)_{x_i}\\
&= 0.
\end{aligned} \tag{5.13}$$

Now, let $\eta \to 0$ in (5.2). Then

$$\int_\Omega \left|u(x,s) - v(x,s)\right| dx \leq \int_\Omega \left|u(x,\tau) - v(x,\tau)\right| dx.$$

By the arbitrariness of τ,

$$\int_\Omega \left|u(x,s) - v(x,s)\right| dx \leq \int_\Omega \left|u_0(x) - v_0(x)\right| dx. \qquad \square$$

6 The uniqueness of the solution

Theorem 2.6 and Theorem 2.7 both imply that the uniqueness of the weak solution is true, their proofs are based on the condition (2.9). Actually, without the condition (2.9), we still can prove the uniqueness of the solution without any boundary value condition.

Theorem 6.1 *Let $p_- > 1$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ be a Lipschitz function on $\mathbb{R} \times \overline{\Omega} \times [0,T]$. If u and v are two solutions of Eq. (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, then there exists a positive constant $\beta_j \geq 2$ such that*

$$\int_\Omega \left(\prod_{j=1}^N a_j^{\beta_j}(x)\right)\left|u(x,t) - v(x,t)\right|^2 dx \leq c\int_\Omega \left(\prod_{j=1}^N a_j^{\beta_j}(x)\right)\left|u_0(x) - v_0(x)\right|^2 dx. \tag{6.1}$$

In particular, for any small enough constant $\delta > 0$,

$$\int_{\Omega_\delta} \left|u(x,t) - v(x,t)\right|^2 dx \le c(\delta, \beta_j) \int_{\Omega} \left|u_0(x) - v_0(x)\right|^2 dx, \tag{6.2}$$

where $\Omega_\delta = \{x \in \Omega : \prod_{j=1}^{N} a_j^{\beta_j}(x) > \delta\}$.

Proof Let u and v be two solutions of Eq. (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. By the process of taking the limit, we may choose $\varphi = \chi_{[\tau,s]} \prod_{j=1}^{N} a_j^{\beta_j}(u - v)$ as a test function. Denoting that $Q_{\tau s} = \Omega \times [\tau, s]$, then

$$\begin{aligned} &\iint_{Q_{\tau s}} (u-v) \prod_{j=1}^{N} a_j^{\beta_j} \frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &= -\sum_{i=1}^{N} \iint_{Q_{\tau s}} a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right) \left[(u-v)\prod_{j=1}^{N} a_j^{\beta_j}\right]_{x_i} dx\,dt \\ &\quad - \sum_{i=1}^{N} \iint_{Q_{\tau s}} \left[b_i(u,x,t) - b_i(v,x,t)\right]\left[(u-v)\prod_{j=1}^{N} a_j^{\beta_j}\right]_{x_i} dx\,dt. \end{aligned} \tag{6.3}$$

In the first place, we have

$$\iint_{Q_{\tau s}} a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i} \prod_{j=1}^{N} a_j^{\beta_j}\,dx\,dt \ge 0 \tag{6.4}$$

and

$$\begin{aligned} &\left|\iint_{Q_{\tau s}} (u-v)a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\left(\prod_{j=1}^{N} a_j^{\beta_j}\right)_{x_i} dx\,dt\right| \\ &\le \iint_{Q_{\tau s}} |u-v|a_i(x)\left(|u_{x_i}|^{p_i-1} + |v_{x_i}|^{p_i-1}\right)\left|\left(\prod_{j=1}^{N} a_j^{\beta_j}\right)_{x_i}\right| dx\,dt \\ &\le c\left(\int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\right) dx\,dt\right)^{\frac{p_i-1}{p_i}} \\ &\quad \cdot \left(\int_\tau^s \int_\Omega a_i(x)\left|\left(\prod_{j=1}^{N} a_j^{\beta_j}\right)_{x_i}\right|^{p_i} |u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}} \\ &\le c\left(\int_\tau^s \int_\Omega a_i(x)\left(|u_{x_i}|^{p_i} + |v_{x_i}|^{p_i}\right) dx\,dt\right)^{\frac{p_i-1}{p_i}} \\ &\quad \cdot \left(\int_\tau^s \int_\Omega a_i(x)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|\right)^{p_i} |u-v|^{p_i}\,dx\,dt)^{\frac{1}{p_i}} \\ &\le c\left(\int_\tau^s \int_\Omega a_i(x)\prod_{j=1}^{N}\left|a_j^{\beta_j-1}a_{jx_i}\right|^{p_i}|u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}}. \end{aligned} \tag{6.5}$$

Here, we have used the fact that $|a_{x_i}| \leq c$. Now, we choose $\beta_i \geq 2$. If $p_i \geq 2$,

$$\left(\int_\tau^s \int_\Omega a_i(x) \prod_{j=1}^N \left|a_j^{\beta_j-1} a_{jx_i}\right|^{p_i} |u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}} \leq c\left(\int_\tau^s \int_\Omega \prod_{j=1}^N a_j^{\beta_j} |u-v|^2\,dx\,dt\right)^{\frac{1}{p_i}}. \tag{6.6}$$

If $1 < p_i < 2$, by the Hölder inequality

$$\left(\int_\tau^s \int_\Omega a_i(x) \prod_{j=1}^N \left|a_j^{\beta_j-1} a_{jx_i}\right|^{p_i} |u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}} \leq c\left(\int_\tau^s \int_\Omega \prod_{j=1}^N a_j^{\beta_j} |u-v|^2\,dx\,dt\right)^{\frac{1}{2}}. \tag{6.7}$$

Combining (6.5)–(6.7), we obtain

$$\left|\iint_{Q_{\tau s}} (u-v)a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i} - |v_{x_i}|^{p_i-2}v_{x_i}\right)\left(\prod_{j=1}^N a_j^{\beta_j}\right)_{x_i} dx\,dt\right| \leq c\left(\int_\tau^s \int_\Omega \prod_{j=1}^N a_j^{\beta_j} |u-v|^2\,dx\,dt\right)^{l}, \tag{6.8}$$

where $l < 1$.

In the second place,

$$\begin{aligned}
&\iint_{Q_{\tau s}} \left[b_i(u,x,t) - b_i(v,x,t)\right]\left[(u-v)\prod_{j=1}^N a_j^{\beta_j}\right]_{x_i} dx\,dt \\
&= \iint_{Q_{\tau s}} \left[b_i(u,x,t) - b_i(v,x,t)\right](u-v)\left(\prod_{j=1}^N a_j^{\beta_j}\right)_{x_i} dx\,dt \\
&\quad + \iint_{Q_s} \left[b_i(u,x,t) - b_i(v,x,t)\right](u-v)_{x_i} \prod_{j=1}^N a_j^{\beta_j}\,dx\,dt.
\end{aligned} \tag{6.9}$$

For the first term on the right-hand side of (6.7), since $\beta_j \geq 2$, $|a_{jx_i}| \leq c$, by the Hölder inequality,

$$\begin{aligned}
&\iint_{Q_{\tau s}} \left[b_i(u,x,t) - b_i(v,x,t)\right](u-v)\left(\prod_{j=1}^N a_j^{\beta_j}\right)_{x_i} dx\,dt \\
&= \int_\tau^s \int_\Omega \left[b_i(u,x,t) - b_i(v,x,t)\right](u-v)\sum_{k=1}^N\left(\beta_k a_k^{\beta_k-1} a_{kx_i} \prod_{j=1, j\neq k}^N a_j^{\beta_j}\right) dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leq c\int_{\tau}^{s}\int_{\Omega}|u-v|\sum_{k=1}^{N}\left(\beta_k a_k^{\beta_k-1}a_{kx_i}\prod_{j=1,j\neq k}^{N}a_j^{\beta_j}\right)dx\,dt\\
&\leq c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}}. \qquad (6.10)
\end{aligned}
$$

For the second term on the right-hand side of (6.9), since $\beta_i \geq 1$, denoting $p_i' = \frac{p_i}{p_i-1}$ as usual, we have

$$
\left(\beta_i-\frac{1}{p_i}\right)p_i' \geq \beta_i.
$$

By this inequality, we have

$$
\begin{aligned}
&\left|\iint_{Q_{\tau s}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr](u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}\,dx\,dt\right|\\
&\leq c\left(\int_{\tau}^{s}\int_{\Omega}a_i^{(\beta_i-\frac{1}{p_i})p_i'}\left(\prod_{j=1,j\neq i}^{N}a_j^{\beta_j}\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\right)^{p_i'}dx\,dt\right)^{\frac{1}{p_i'}}\\
&\quad\cdot\left(\int_{\tau}^{s}\int_{\Omega}a_i\bigl(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\bigr)\,dx\,dt\right)^{\frac{1}{p_i}}\\
&\leq c\left(\int_{\tau}^{s}\int_{\Omega}a_i^{(\beta_i-\frac{1}{p_i})p_i'}\left(\prod_{j=1,j\neq i}^{N}a_j^{\beta_j}\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\right)^{p_i'}dx\,dt\right)^{\frac{1}{p_i'}}\\
&\leq c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_i'}\,dx\,dt\right)^{\frac{1}{p_i'}}. \qquad (6.11)
\end{aligned}
$$

If $p_i > 2$, then $1 < p_i' < 2$. By the Hölder inequality,

$$
\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_i'}\,dx\,dt\right)^{\frac{1}{p_i'}} \leq c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{2}\,dx\,dt\right)^{\frac{1}{2}}. \qquad (6.12)
$$

If $1 < p_i \leq 2$, then $p_i' \geq 2$,

$$
\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{p_i'}\,dx\,dt\right)^{\frac{1}{p_i'}} \leq c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^{2}\,dx\,dt\right)^{\frac{1}{p_i'}}. \qquad (6.13)
$$

Combining (6.11)–(6.13), we have

$$
\begin{aligned}
&\left|\iint_{Q_{\tau s}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr](u-v)_{x_i}\prod_{j=1}^{N}a_j^{\beta_j}\,dx\,dt\right|\\
&\leq c\left(\int_{\tau}^{s}\int_{\Omega}\prod_{j=1}^{N}a_j^{\beta_j}|u-v|^2\,dx\,dt\right)^{l}, \qquad (6.14)
\end{aligned}
$$

where $l < 1$.

Moreover,

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u-v)\prod_{j=1}^{N} a_j^{\beta_j}\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad= \iint_{Q_{\tau s}} (u-v)\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}\frac{\partial[\sqrt{\prod_{j=1}^{N} a_j^{\beta_j}}(u-v)]}{\partial t}\,dx\,dt \\
&\quad= \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl[u(x,s)-v(x,s)\bigr]^2\,dx - \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl[u(x,\tau)-v(x,\tau)\bigr]^2\,dx. \qquad (6.15)
\end{aligned}$$

According to (6.3), (6.4), (6.8), (6.10), (6.14) and (6.15), we have

$$\begin{aligned}
&\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl[u(x,s)-v(x,s)\bigr]^2\,dx - \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl[u(x,\tau)-v(x,\tau)\bigr]^2\,dx \\
&\quad\le c\left(\int_0^s\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl|u(x,t)-v(x,t)\bigr|^2\,dx\,dt\right)^l, \qquad (6.16)
\end{aligned}$$

where $l<1$. By (6.16), we easily show that

$$\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl|u(x,\tau)-v(x,\tau)\bigr|^2\,dx \le \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl|u(x,\tau)-v(x,\tau)\bigr|^2\,dx. \qquad (6.17)$$

Thus, by the arbitrariness of τ, we have

$$\int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl|u(x,s)-v(x,s)\bigr|^2\,dx \le \int_\Omega \prod_{j=1}^{N} a_j^{\beta_j}\bigl|u_0(x)-v_0(x)\bigr|^2\,dx. \qquad (6.18)$$

By (6.18), we clearly have (6.1) and (6.2). The proof is complete. □

By this theorem, Theorem 2.8 is true.

7 The general method to prove the stability of weak solutions

We can generalize the method used in Sect. 6 to prove various kinds of stability of weak solutions.

Let $\chi(x)$ be a $C^1(\overline{\Omega})$ function satisfying

$$\chi(x)=0, \quad \text{if } x\in\partial\Omega; \qquad \chi(x)>0, \quad \text{if } x\in\Omega. \qquad (7.1)$$

Theorem 7.1 *Let $p_- \ge 2$, $a_i(x)\in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ is bounded when s is bounded and $(x,t)\in\Omega\times[0,T)$. If there exist constants $0<\sigma_i<1$, $0<\delta_i<1$, and there exists $\chi(x)$ satisfying (7.1) and*

$$\int_\Omega \left|\frac{(\chi(x))_{x_i}}{\chi^{\frac{\delta_i}{2}}(x)}\right|^{\frac{2}{2-\delta_i}}\,dx \le c, \qquad (7.2)$$

$$\int_{\Omega}\left|\left(\chi(x)\right)^{p_i'-\frac{\sigma_i}{2}}a_i^{-\frac{1}{p_i-1}}(x)\right|^{\frac{2}{2-\sigma_i}}dx\le c. \tag{7.3}$$

Let u and v are be solutions of Eq. (1.1) *with the initial values $u_0(x)$ and $v_0(x)$, respectively. Then, for any $\Omega_1\subset\subset\Omega$,*

$$\int_{\Omega_1}\left|u(x,t)-v(x,t)\right|^2dx\le c(\Omega_1)\int_{\Omega}\left|u_0(x)-v_0(x)\right|^2dx.$$

Here, $p_i'=\frac{p_i}{p_i-1}$ as usual.

Proof Let u and v be two solutions of Eq. (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. By the process of taking the limit, we may choose $\varphi=\chi_{[\tau,s]}\chi(x)(u-v)$ as a test function. Denoting $Q_{\tau s}=\Omega\times[\tau,s]$, then

$$\begin{aligned}&\iint_{Q_{\tau s}}(u-v)\chi(x)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&\quad=-\sum_{i=1}^{N}\iint_{Q_{\tau s}}a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\left[(u-v)\chi(x)\right]_{x_i}dx\,dt\\&\qquad-\sum_{i=1}^{N}\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right]\left[(u-v)\chi(x)\right]_{x_i}dx\,dt.\end{aligned} \tag{7.4}$$

In the first place, we have

$$\iint_{Q_{\tau s}}a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i}\chi(x)\,dx\,dt\ge 0, \tag{7.5}$$

and using (7.2) we deduce that

$$\begin{aligned}&\left|\iint_{Q_{\tau s}}(u-v)a_i(x)\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\left(\chi(x)\right)_{x_i}dx\,dt\right|\\&\quad\le\iint_{Q_{\tau s}}|u-v|a_i(x)\left(|u_{x_i}|^{p_i-1}+|v_{x_i}|^{p_i-1}\right)\left|\left(\chi(x)\right)_{x_i}\right|dx\,dt\\&\quad\le c\left(\int_\tau^s\int_\Omega a_i(x)\left(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\right)dx\,dt\right)^{\frac{p_i-1}{p_i}}\\&\qquad\cdot\left(\int_\tau^s\int_\Omega a_i(x)\left|\left(\chi(x)\right)_{x_i}\right|^{p_i}|u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}}\\&\quad\le c\left(\int_\tau^s\int_\Omega a_i(x)\left|\left(\chi(x)\right)_{x_i}\right|^{p_i}|u-v|^{p_i}\,dx\,dt\right)^{\frac{1}{p_i}}\\&\quad=c\left(\int_\tau^s\int_\Omega a_i(x)\frac{|(\chi(x))_{x_i}|^{p_i}}{\chi^{\frac{\delta_i}{2}}(x)}\chi^{\frac{\delta_i}{2}}|u-v|^{\delta_i}|u-v|^{p_i-\delta_i}\,dx\,dt\right)^{\frac{1}{p_i}}\\&\quad\le c\left(\int_\tau^s\int_\Omega\left(\frac{|(\chi(x))_{x_i}|^{p_i}}{\chi^{\frac{\delta_i}{2}}(x)}\right)^{\frac{2}{2-\delta_i}}dx\,dt\right)^{\frac{1}{p_i}\frac{2-\delta_i}{2}}\end{aligned}$$

$$\cdot \left(\int_\tau^s \int_\Omega \chi(x)|u-v|^2\,dx\right)^{\frac{\delta_i}{2}\frac{1}{p_i}}$$

$$\leq c\left(\int_\tau^s \int_\Omega \chi(x)|u-v|^2\,dx\right)^{\frac{\delta_i}{2}\frac{1}{p_i}}. \tag{7.6}$$

In the second place,

$$\begin{aligned}
&\iint_{Q_{\tau s}} \big[b_i(u,x,t)-b_i(v,x,t)\big]\big[(u-v)\chi(x)\big]_{x_i}\,dx\,dt \\
&= \iint_{Q_{\tau s}} \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)\big(\chi(x)\big)_{x_i}\,dx\,dt \\
&\quad + \iint_{Q_s} \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}\chi(x)\,dx\,dt.
\end{aligned} \tag{7.7}$$

For the first term on the right-hand side of (7.7), since $b_i(u,x,t)$ and $b_i(v,x,t)$ are bounded when $u \in L^\infty(Q_T)$, $v \in L^\infty(Q_T)$, and by that $p_- \geq 2$ implies $p_i' - \frac{2\delta_i}{(2-\delta_i)(p_i-1)} \geq 0$, by the Hölder inequality, using (7.2), we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)\big(\chi(x)\big)_{x_i}\,dx\,dt\right| \\
&\leq c\left[\iint_{Q_{\tau s}} \left(\frac{|(\chi(x))_{x_i}|^{\frac{2}{2-\delta_i}}}{(\chi(x))^{\frac{\delta_i}{(2-\delta_i)p_i}}}\right)^{p_i} dx\right]^{\frac{1}{p_i}} \left(\iint_{Q_{\tau s}} \big[\chi^{\frac{\delta_i}{(2-\delta_i)p_i}}|u-v|\big]^{p_i'}\,dx\right)^{\frac{1}{p_i'}} \\
&\leq c\left(\iint_{Q_{\tau s}} \big[\chi^{\frac{\delta_i}{(2-\delta_i)p_i}}|u-v|\big]^{p_i'}\,dx\right)^{\frac{1}{p_i'}} \\
&= c\left(\iint_{Q_{\tau s}} \big[\chi^{\frac{\delta_i}{(2-\delta_i)(p_i-1)}}|u-v|\big]^{\frac{2\delta_i}{(2-\delta_i)(p_i-1)}}|u-v|^{p_i'-\frac{2\delta_i}{(2-\delta_i)(p_i-1)}}\,dx\right)^{\frac{1}{p_i'}} \\
&\leq c\left(\iint_{Q_{\tau s}} \chi(x)|u-v|^2\,dx\right)^{\frac{\delta_i}{(2-\delta_i)p_i}}.
\end{aligned} \tag{7.8}$$

For the second term on the right-hand side of (7.7), by this inequality and the condition (7.3), we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} \big[b_i(u,x,t)-b_i(v,x,t)\big](u-v)_{x_i}\chi(x)\,dx\,dt\right| \\
&= \left|\iint_{Q_{\tau s}} \big(a_i(x)\big)^{\frac{1}{p_i}}(u-v)_{x_i}\chi(x)\big(a_i(x)\big)^{-\frac{1}{p_i}}(u-v)\,dx\,dt\right| \\
&\leq c\left(\int_\tau^s \int_\Omega \big|\chi(x)a_i^{-\frac{1}{p_i}}(u-v)\big|^{p_i'}\,dx\,dt\right)^{\frac{1}{p_i'}} \left(\int_\tau^s \int_\Omega a_i(x)\big(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i}\big)\,dx\,dt\right)^{\frac{1}{p_i}} \\
&\leq c\left(\int_\tau^s \int_\Omega \big|\chi(x)a_i^{-\frac{1}{p_i}}(u-v)\big|^{p_i'}\,dx\,dt\right)^{\frac{1}{p_i'}} \\
&= c\left(\int_\tau^s \int_\Omega |\chi(x)|^{p_i'-\frac{\sigma_i}{2}}a_i^{-\frac{1}{p_i-1}}|\chi(x)|^{\frac{\sigma_i}{2}}|u-v|^{\sigma_i}(u-v)|^{p_i'-\sigma_i}\,dx\,dt\right)^{\frac{1}{p_i'}}
\end{aligned}$$

$$\leq c\left(\int_\tau^s \int_\Omega \left(\chi^{p_i'-\frac{\sigma_i}{2}} a_i^{-\frac{1}{p_i-1}}\right)^{\frac{2}{2-\sigma_i}} dx\,dt\right)^{\frac{2-\sigma_i}{2p_i'}} \left(\int_\tau^s \int_\Omega \chi(x)|u-v|^2\,dx\,dt\right)^{\frac{\sigma_i}{2p_i'}}$$

$$\leq c\left(\int_\tau^s \int_\Omega \chi(x)|u-v|^2\,dx\,dt\right)^{\frac{\sigma_i}{2p_i'}}. \tag{7.9}$$

Moreover,

$$\iint_{Q_{\tau s}} (u-v)\chi(x)\frac{\partial(u-v)}{\partial t}\,dx\,dt$$
$$= \iint_{Q_{\tau s}} (u-v)\sqrt{\chi(x)}\frac{\partial[\sqrt{\chi(x)}(u-v)]}{\partial t}\,dx\,dt$$
$$= \int_\Omega \chi(x)\left[u(x,s)-v(x,s)\right]^2 dx - \int_\Omega \chi(x)\left[u(x,\tau)-v(x,\tau)\right]^2 dx. \tag{7.10}$$

According to (7.4)–(7.10), we have

$$\int_\Omega \chi(x)\left[u(x,s)-v(x,s)\right]^2 dx - \int_\Omega \chi(x)\left[u(x,\tau)-v(x,\tau)\right]^2 dx$$
$$\leq c\left(\int_\tau^s \int_\Omega \chi(x)\left|u(x,t)-v(x,t)\right|^2 dx\,dt\right)^l, \tag{7.11}$$

where $l<1$. By (7.11), we easily can show that

$$\int_\Omega \chi(x)\left|u(x,\tau)-v(x,\tau)\right|^2 dx \leq \int_\Omega \chi(x)\left|u(x,\tau)-v(x,\tau)\right|^2 dx. \tag{7.12}$$

By the arbitrariness of τ, then

$$\int_\Omega \chi(x)\left|u(x,s)-v(x,s)\right|^2 dx \leq \int_\Omega \chi(x)\left|u_0(x)-v_0(x)\right|^2 dx. \tag{7.13}$$

Since (7.1), by (7.13), the inequality (7.3) is true clearly.

One can see that the condition $p_- \geq 2$ is only used to estimate (7.8). We are sure that it can weakened to $p_- > 1$. For example, if there exists constant $\gamma_i > 0$ such that

$$\left|b_i(u,x,t)\right| \leq c\left|\chi(x)\right|^{\gamma_i}, \tag{7.14}$$

then we obtain

$$\left|\iint_{Q_{\tau s}} \left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\chi(x)\,dx\,dt\right|$$
$$\leq c\left(\iint_{Q_{\tau s}} \chi(x)|u-v|^2\,dx\,dt\right)^l, \tag{7.15}$$

where $l<1$. Thus, we still have the conclusion of Theorem 7.1. □

However, we are not ready to discuss how to weaken the condition $p_- \geq 2$ again in what follows. We prefer to explain the importance of Theorem 7.1. That is, if we choose various kinds of functions $\chi(x)$, we can obtain the corresponding stability theorems. Let us give some examples.

If we choose $\chi(x) = \prod_{i=1}^{N} a_i^{\beta}(x)$, we have a similar conclusion to Theorem 2.8. By the process of taking the limit, we can choose $\chi(x) = d^{\alpha}(x)$, where $\alpha > 0$ is a constant, $d(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary. Then we have the following theorem.

Theorem 7.2 *Let $p_- \geq 2$, $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.3), $b_i(s,x,t)$ is bounded when s is bounded and $(x,t) \in \Omega \times [0,T)$. Assume $\alpha > 1$, and assume there exist constants $0 < \sigma_i < 1$,*

$$\int_{\Omega} \left| \left(d^{\alpha}(x)\right)^{p_i' - \frac{\sigma_i}{2}} a_i^{-\frac{1}{p_i - 1}}(x) \right|^{\frac{2}{2-\sigma_i}} dx \leq c. \tag{7.16}$$

Let u and v are be solutions of Eq. (1.1) with the initial values $u_0(x)$ and $v_0(x)$, respectively. Then, for any $\Omega_1 \subset\subset \Omega$,

$$\int_{\Omega_1} \left| u(x,t) - v(x,t) \right|^2 dx \leq c(\Omega_1) \int_{\Omega} \left| u_0(x) - v_0(x) \right|^2 dx.$$

Proof Since $\chi(x) = d^{\alpha}(x)$, $\alpha > 1$, for any $0 < \sigma_i < 1$, it is not difficult to show the inequality (7.1) is true. Then we have the conclusion. □

As long as one wants, one can choose other types of the functions $\chi(x)$, e.g. $\chi(x) = \sum_{i=1}^{N} a_i(x)$, $\chi(x) = e^{a_i(x)} - 1$ for any given $i \in \{1, 2, \ldots, N\}$, or $\chi(x) = \max\{a_i(x)\}$, to obtain the corresponding stability theorems.

8 Conclusion

The anisotropic parabolic equations considered in this paper arise from many applied fields such as non-Newtonian fluid theory, reaction–diffusion problems. If the convection term depends on the diffusion coefficient which is degenerate on the boundary, then the stability of weak solutions may be proved without any boundary value condition. If the convection term is independent of the diffusion coefficient, the uniqueness of the weak solution is still true only if the convection function $b_i(u,x,t)$ is bounded when $|u| \leq c$. Moreover, a general method to prove the stability of the weak solutions without the boundary value condition is introduced for the first time in this paper. We believe such a method can be used in many kinds of parabolic equations, especially those lacking the regularity for the trace on the boundary to be defined.

Acknowledgements
The author would like to thank everyone for help.

Funding
The paper is supported by Natural Science Foundation of Fujian province, supported by the Open Research Fund Program form Fujian Engineering and Research Center of Rural Sewage Treatment and Water Safety, supported by Science Foundation of Xiamen University of Technology, China.

Availability of data and materials
Not applicable.

Competing interests
The author declares to have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 July 2018 Accepted: 15 January 2019 Published online: 25 January 2019

References

1. Antontsev, S.V., Diaz, J.I., Shmarev, S.: Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics. Progress in Nonlinear Differential Equations and Their Applications, vol. 48. Birkhauser Boston, Boston (2002)
2. Diaz, J., Padial, J.: Uniqueness and existence of a solution in $BV_t(q)$ space to a doubly nonlinear parabolic problem. Publ. Mat. **40**(2), 527–560 (1996)
3. Diaz, J., Thelin, F.: On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM J. Math. Anal. **25**(4), 1085–1111 (1994)
4. DiBenedetto, E.: Degenerate Parabolic Equations. Springer, New York (1993)
5. Wu, Z., Zhao, J., Yin, J., Li, H.: Nonlinear Diffusion Equations. Word Scientific, Singapore (2001)
6. Lee, K., Petrosyan, A., Vazquez, J.L.: Large time geometric properties of solutions of the evolution p-Laplacian equation. J. Differ. Equ. **229**, 389–411 (2006)
7. Rădulescu, V.D., Stăncut, I.-L.: Combined concave-convex effects in anisotropic elliptic equations with variable exponents. Nonlinear Differ. Equ. Appl. **22**(3), 391–410 (2015)
8. Antontsev, S.V., Shmarev, S.: Existence and uniqueness for doubly nonlinear parabolic equations with nonstandard growth conditions. Differ. Equ. Appl. **4**(1), 67–94 (2012)
9. Bahrouni, A., Radulescu, V.D., Repovs, D.D.: A weighted anisotropic variant of the Caffarelli–Kohn–Nirenberg inequality and applications. Nonlinearity **31**(4), 1516–1534 (2018)
10. Barbu, L., Enache, C.: Maximum principles, Liouville-type theorems and symmetry results for a general class of quasilinear anisotropic equations. Adv. Nonlinear Anal. **5**(4), 395–405 (2016)
11. Ho, K., Sim, I.: A-priori bounds and existence for solutions of weighted elliptic equations with a convection term. Adv. Nonlinear Anal. **6**(4), 427–445 (2017)
12. Radulescu, V.D., Repovs, D.D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2015)
13. Stancut, I.-L., Stircu, I.D.: Eigenvalue problems for anisotropic equations involving a potential on Orlicz–Sobolev type spaces. Opusc. Math. **36**(1), 81–101 (2016)
14. Zhang, Q., Radulescu, V.D.: Double phase anisotropic variational problems and combined effects of reaction and absorption terms. J. Math. Pures Appl. **118**(9), 159–203 (2018)
15. Yin, J., Wang, C.: Evolutionary weighted p-Laplacian with boundary degeneracy. J. Differ. Equ. **237**, 421–445 (2007)
16. Zhan, H., Feng, Z.: Solutions of evolutionary $p(x)$-Laplacian equation based on the weighted variable exponent space. Z. Angew. Math. Phys. **68**, 134 (2017)
17. Zhan, H.: Existence of solutions to an evolution p-Laplacian equation with a nonlinear gradient term. Electron. J. Differ. Equ. **2017**, 311 (2017)
18. Kobayasi, K., Ohwa, H.: Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle. J. Differ. Equ. **252**, 137–167 (2012)
19. Li, Y., Wang, Q.: Homogeneous Dirichlet problems for quasilinear anisotropic degenerate parabolic–hyperbolic equations. J. Differ. Equ. **252**, 4719–4741 (2012)
20. Lions, P.L., Perthame, B., Tadmor, E.: A kinetic formation of multidimensional conservation laws and related equations. J. Am. Math. Soc. **7**, 169–191 (1994)
21. Escobedo, M., Vazquez, J.L., Zuazua, E.: Entropy solutions for diffusion–convection equations with partial diffusivity. Trans. Am. Math. Soc. **343**, 829–842 (1994)
22. Guarguaglini, F.R., Milišić, V., Terracina, A.: A discrete BGK approximation for strongly degenerate parabolic problems with boundary conditions. J. Differ. Equ. **202**, 183–207 (2004)
23. Málek, J., Necas, J., Rokyta, M., Ruzicka, M.: Weak and Measure-Valued Solutions to Evolutionary PDEs. Applied Mathematics and Mathematical Computation, vol. 13. Chapman & Hall, London (1996)
24. Mascia, C., Porretta, A., Terracina, A.: Nonhomogeneous Dirichlet problems for degenerate parabolic–hyperbolic equations. Arch. Ration. Mech. Anal. **163**(2), 87–124 (2002)
25. Vallet, G.: Dirichlet problem for a degenerated hyperbolic–parabolic equation. Adv. Math. Sci. Appl. **15**, 423–450 (2005)
26. Zhan, H.: The solution of a hyperbolic–parabolic mixed-type equation on half-space domain. J. Differ. Equ. **259**, 1449–1481 (2015)
27. Zhan, H., Feng, Z.: A hyperbolic–parabolic mixed type equation with non-homogeneous boundary condition. J. Differ. Equ. **264**, 7384–7411 (2018)
28. Zhan, H.: The uniqueness of a nonlinear diffusion equation related to the p-Laplacian. J. Inequal. Appl. **2018**, 7 (2018)
29. Antonelli, F., Barucci, E., Mancino, M.E.: A comparison result for FBSDE with applications to decisions theory. Math. Methods Oper. Res. **54**, 407–423 (2001)
30. Antontsev, S., Shmarev, S.: Anisotropic parabolic equations with variable nonlinearity. Publ. Mat. **53**, 355–399 (2009)
31. Zhan, H.: The solution of convection–diffusion equation. Chin. Ann. Math. **34**(2), 235–256 (2013) (in Chinese)
32. Zhan, H.: Infiltration equation with degeneracy on the boundary. Acta Appl. Math. **153**(1), 147–161 (2018)