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Existence results in Banach space for a nonlinear impulsive system

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Abstract

We deal with three important aspects of a generalized impulsive fractional order differential equation (DE) involving a nonlinear p-Laplacian operator: the existence of a solution, the uniqueness and the Hyers–Ulam stability. Our problem involves Caputo's fractional derivative. For these goals, we establish an equivalent fractional integral form of the problem and use a topological degree approach for the existence and uniqueness of the solution (EUS). Next, we check the stability of the suggested problem and then demonstrate the results via an illustrative example. In the literature, we could not find articles on the Hyers–Ulam stability of the impulsive fractional order DEs with ϕ_p operator.

MSC: 34B82; 26A33; 45N05

Keywords: Impulsive fractional differential equations; Caputo's fractional derivative; Existence and uniqueness of positive solution; Hyers–Ulam stability

1 Introduction

Nowadays, fractional calculus has drawn interest of experts working in numerous applied fields. Mathematical models by using fractional order DEs have been considered in control theory, viscoelastic theory, biology, fluid dynamics, hydrodynamics, image processing, signals, and computer networking. For details, we suggest [1–8].

Scientists are exploring different scientific aspects of fractional calculus. The existence and uniqueness of solutions (EUS) are popular studies. For this purpose, scientists are using numerous scientific approaches. Fixed point theorems are useful techniques for this study. Here, we highlight some recent and useful scientific results of well-known scientists in the field. For example, Agarwal et al. [9] provided a detailed description of the impulsive fractional DEs using Lyapunov functions and overviewed results for the stability in Caputo's sense. Wang et al. [10] studied the existence of solutions and applications for different classes of impulsive fractional DEs from the available literature. Hu et al. [11] considered a nonlinear system of fractional order DEs with p-Laplacian at resonance. Jafari et al. [12] studied EUS in Banach space for boundary value problems (BVPs) of fractional order DEs with p-Laplacian. Khan et al. [13] proved EUS and the Hyers–Ulam stability for a class of BVPs based on fractional order DEs and nonlinear p-Laplacian operator. Baleanu et al. [14] considered a hybrid system of fractional order DEs with p-Laplacian operator for the analysis of EUS and they given illustrated their results with examples. Khan et al. [15] proved several types of stability theorems for a fractional order DE with p-Laplacian involving two types of fractional order derivatives, that is, Caputo and Riemann–Liouville types. Raheem and Maqbul [16] studied oscillatory behavior for an impulsive fractional order DE with the help of inequality technique. Wang and Zhang [17] established EUS for an impulsive class of fractional DEs with Hadamard fractional order derivatives and presented applications. Cong and Tuan [18] considered a fractional order DE with delay for the study of EUS.

Recently, some authors considered fractional DEs with p-Laplacian operator for the existence and stability results. Here, we highlight some related contributions of the scientists. For instance, Hu et al. [11] considered a nonlinear system of fractional order DEs with p-Laplacian at resonance. Jafari et al. [12] studied EUS in Banach space for boundary value problems (BVPs) of fractional order DEs with p-Laplacian. Khan et al. [13] proved EUS and Hyers–Ulam stability for a class of BVPs based on fractional order DEs and a nonlinear p-Laplacian operator. Baleanu et al. [14] considered a hybrid system of fractional order DEs with a p-Laplacian operator for the analysis of EUS and they have illustrated their results with examples. Khan et al. [15] proved several types of stability theorems for a fractional order DE involving a nonlinear p-Laplacian operator. Abbas et al. [19] considered a partial functional DE for the analysis of the Hyers–Ulam stability and existence results and provided two examples for applications of their results.

Recently, Liu et al. [20] studied the following impulsive fractional differential equation for the existence and uniqueness of solutions and also provided an application:

$$\begin{aligned} &\mathcal{D}^{\gamma}_{0^+}\phi_p\big(\mathcal{D}^{\epsilon}_{0^+}u(t)\big) = \psi_1\big(t,u(t)\big), \quad t \in \mathcal{J}', \\ &\Delta u(t_k) = \mathcal{I}_k\big(u(t_k)\big), \qquad \Delta\phi_p\big(\mathcal{D}^{\epsilon}_{0^+}u(t_k)\big) = c_k, \quad k = 1,2,\ldots,m, \\ &u(0) = u_0, \qquad \mathcal{D}^{\epsilon}_{0^+}u(t)|_{t=0} = u_1, \end{aligned} \tag{1.1}$$

where $\psi_1(t,u(t))$, $\mathcal{I}_k(u(t))$ are continuous real valued functions. $\mathcal{D}^{\gamma}_{0^+}$, $\mathcal{D}^{\epsilon}_{0^+}$, are Caputo's fractional derivatives of order γ and ϵ where $0 < \epsilon, \gamma \leq 1$, $\epsilon + \gamma \leq 2$. $u_0, u_1 \in \mathbb{R}$, $\mathcal{J} = [0,1]$, $t_i \leq t_{i+1}$ for $i = 0,\ldots,m$, with $t_{m+1} = 1$, $\mathcal{J}' = \mathcal{J}\backslash\{t_1,\ldots,t_m\}$. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$, $u(t_k^-)$ are the right and left limits of $u(t)$ at $t = t_k$ $(k = 1,2,\ldots,m)$, respectively.

Feckan et al. [21] studied an impulsive FDE for the existence of solution of the kind:

$$\begin{aligned} &\mathcal{D}^{\epsilon}_{0,t}u(t) = \psi_1\big(t,u(t)\big), \quad t \in \mathcal{J}', \\ &u(0) = u_0, \qquad \Delta u(t_k) = \mathcal{I}_k\big(u\big(t_k^-\big)\big), \quad k = 1,2,\ldots,m, \end{aligned} \tag{1.2}$$

where $\psi_1(t,u(t))$, are continuous real valued functions, $\mathcal{I}_k : \mathbb{R} \to \mathbb{R}$. $\mathcal{D}^{\epsilon}_{0^+}$ is Caputo's fractional derivative of order $0 < \epsilon < 1$. $u_0 \in \mathbb{R}$, $\mathcal{J} = [0,1]$, $t_i \leq t_{i+1}$ for $i = 0,\ldots,m$, with $t_{m+1} = 1$, $\mathcal{J}' = \mathcal{J}\backslash\{t_1,\ldots,t_m\}$. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$, $u(t_k^-)$ are the right and left limits of $u(t)$ at $t = t_k$ $(k = 1,2,\ldots,m)$, respectively.

Inspired from the above research problems and the references given [22–26]. We apply a fixed point approach to examine EUS and Hyers–Ulam stability for the nonlinear impulsive fractional order DE:

$$\begin{cases} \mathcal{D}^{\gamma}[\phi_p[\mathcal{D}^{\epsilon}u(t)]] - \psi_1(t,u(t)) = 0, \\ \Delta[\phi_p[\mathcal{D}^{\epsilon}u(t_k)]] = b_k^*, \qquad \Delta[\phi_p[\mathcal{D}^{\epsilon}u(t_k)]]' = c_k^*, \qquad \Delta u(t_k) = I_k(u(t_k)), \\ \phi_p(\mathcal{D}^{\epsilon}u(0)) = 0, \qquad \phi_p(\mathcal{D}^{\epsilon}u(1)) = \lambda^*\phi_p'(\mathcal{D}^{\epsilon}u(1)), \qquad u(0) = 0, \end{cases} \tag{1.3}$$

where ψ_1 is a continuous function. The orders are $\epsilon \in (0,1]$, $\gamma \in (1,2]$. $\psi_1 \in \mathcal{L}[0,1]$ and $\mathcal{D}^\epsilon$, $\mathcal{D}^\gamma$ are Caputo's fractional derivatives. The p-Laplacian operator ϕ_p satisfies $\phi_p(r) = |r|^{p-2}r$ where $1/p + 1/q = 1$, ϕ_q is the inverse of ϕ_p. The problem (1.3) is more complicated and general than those considered earlier, given above (1.2).

Definition 1.1 The fractional integral for $\epsilon > 0$ of $\psi : (0,+\infty) \to \mathbb{R}$ is given by

$$\mathcal{I}^\epsilon \psi(t) = \frac{1}{\Gamma(\epsilon)} \int_0^t (t-s)^{\epsilon-1} \psi(s)\,ds,$$

provided that the integral on the right side is point-wise defined on the interval $(0,+\infty)$, where

$$\Gamma(\epsilon) = \int_0^{+\infty} e^{-s} s^{\epsilon-1}\,ds.$$

Definition 1.2 The fractional derivative in Caputo's sense of order $\epsilon > 0$, for continuous function $\psi(t) : (0,+\infty) \to \mathbb{R}$ is given by

$$\mathcal{D}^\epsilon \psi(t) = \frac{1}{\Gamma(k-\theta)} \int_0^t (t-s)^{k-\theta-1} \psi^{(k)}(s)\,ds,$$

for $n = [\theta] + 1$, where $[\theta]$ is used for the integer part of θ, defined on $(0,+\infty)$.

Lemma 1.1 *Let $\theta \in (n-1,n]$, $\psi \in C^{n-1}$, then*

$$\mathcal{I}^\theta \mathcal{D}^\theta \psi(t) = \psi(t) + k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1},$$

for the $k_i \in \mathbb{R}$ for $i = 0,1,2,\ldots,n-1$.

Lemma 1.2 ([13, 15, 27]) *For the ϕ_p-operator, the following hold true*:

(1) *If $1 < p \leq 2$, $\ell_1^* \ell_2^* > 0$ and $|\ell_1^*|, |\ell_2^*| \geq \rho > 0$, then*

$$\left|\phi_p(\ell_1^*) - \phi_p(\ell_2^*)\right| \leq (p-1)\rho^{p-2} \left|\ell_1^* - \ell_2^*\right|.$$

(2) *If $p > 2$, and $|\ell_1^*|, |\ell_2^*| \leq \rho^*$, then*

$$\left|\phi_p(\ell_1^*) - \phi_p(\ell_2^*)\right| \leq (p-1)\rho^{*p-2} \left|\ell_1^* - \ell_2^*\right|.$$

Paper organization: In this paper, we aim to study EUS and the Hyers–Ulam stability for BVPs of impulsive fractional order DEs with ϕ_p-operator (1.3) and to illustrate the results via an application. The paper is divided in five sections. The introduction part includes the basic and related literature in first section. The next section is reserved for the EUS for the impulsive fractional order DE with p-Laplacian operator (1.3). Hyers–Ulam stability is defined in the third section and it is verified that the impulsive fractional order DE (1.3) is Hyers–Ulam stable. The application of the results is given in Sect. 4. Finally, Sect. 5 is for the conclusion of the paper. The fractional order impulsive DE with p-Laplacian operator (1.3) is transferred into an integral equation with the help of a fractional integral operator and the properties of the p-Laplacian operator. With certain assumptions, the EUS for the (1.3) are proved and then the Hyers–Ulam stability is studied. An important example is also provided for application.

2 Results for EUS

Theorem 2.1 *Assume that* $\psi_1 \in \mathcal{C}[0,1]$ *be an integrable function satisfying* (1.3). *Then, for* $\alpha \in (1,2]$, $\gamma \in (0,1]$, *the solution of the following impulsive FDE with a p-Laplacian operator*:

$$\begin{cases} \mathcal{D}^{\gamma}[\phi_p[\mathcal{D}^{\epsilon}u(t)]] - \psi_1(t,u(t)) = 0, \\ \Delta[\phi_p[\mathcal{D}^{\epsilon}u(t_k)]] = b_k^*, \qquad \Delta[\phi_p[\mathcal{D}^{\epsilon}u(t_k)]]' = c_k^*, \qquad \Delta u(t_k) = I_k(u(t_k)), \\ \phi_p(\mathcal{D}^{\epsilon}u(0)) = 0, \qquad \phi_p(\mathcal{D}^{\epsilon}u(1)) = \lambda^*\phi_p'(\mathcal{D}^{\epsilon}u(1)), \qquad u(0) = 0, \end{cases} \tag{2.1}$$

is

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau \\ & + \sum_{k=1}^{m} I_k\big(u(t_k)\big), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{G}\big(u(t)\big) = {} & \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + \frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds \\ & + \sum_{k=1}^{m}\frac{(t-t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\ & + \frac{t}{1-\lambda^*}\left[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds\right. \\ & + \frac{\sum_{k=1}^{m}(\lambda^*-1+t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\ & - \frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds \\ & \left. - \frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + \sum_{k=1}^{m}(\lambda^*-1+t_k)c_k^*\right] \\ & + \sum_{k=1}^{m}b_k^* + \sum_{k=1}^{m}(t-t_k)c_k^*. \end{aligned} \tag{2.3}$$

Proof Let $u(t) = \phi_p[\mathcal{D}^{\epsilon}u(t)]$, then the problem (2.1) gets the form

$$\begin{cases} \mathcal{D}^{\gamma}[u(t)] - \psi_1(t,u(t)) = 0, \\ \Delta[u(t_k)] = b_k^*, \qquad \Delta[u(t_k)]' = c_k^*, \\ u(0) = 0, \qquad u(1) = \lambda^*u'(1). \end{cases} \tag{2.4}$$

We apply the $\mathcal{I}^\gamma$-operator on (2.4) and with the help of Lemma 1.1, we have

$$\begin{aligned} u(t) &= \mathcal{I}^\gamma\left[\psi_1\big(t,u(t)\big)\right] + c_1 + c_2 t \\ &= \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t}(t-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + c_1 + c_2 t. \end{aligned} \tag{2.5}$$

For $t \in [t_1, t_2]$, we have

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + c_3 + c_4(t-t_1). \tag{2.6}$$

From (2.5) and (2.6), we have

$$u\big(t_1^-\big) = \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + c_1 + c_2 t_1, \qquad u\big(t_1^+\big) = c_3. \tag{2.7}$$

With the help of (2.4), we have

$$\begin{aligned} \Delta u(t_1) &= u\big(t_1^+\big) - u\big(t_1^-\big) \\ &= c_3 - \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds - c_1 - c_2 t_1 = b_1^*. \end{aligned} \tag{2.8}$$

This implies

$$c_3 = \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + c_1 + c_2 t_1 + b_1^*. \tag{2.9}$$

For $t \in [t_0, t_1]$, (2.5) implies

$$u'(t) = \frac{1}{\Gamma(\gamma-1)}\int_{t_0}^{t}(t-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds + c_2. \tag{2.10}$$

Therefore

$$u'\big(t_1^-\big) = \frac{1}{\Gamma(\gamma-1)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds + c_2. \tag{2.11}$$

Similarly, we have $u'(t_1^+) = c_4$. Using the condition $\Delta[x(t_1)]' = c_1^*$, we have

$$c_4 = \frac{1}{\Gamma(\gamma-1)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds + c_2 + c_1^*. \tag{2.12}$$

With the help of (2.6), (2.9) and (2.12), we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + c_1 + c_2 t_1 + I_1\big(u(t_1)\big) \\ &\quad + (t-t_1)\left[\frac{1}{\Gamma(\gamma-1)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds + c_2 + c_1^*\right] \end{aligned}$$

$$= \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds + \frac{1}{\Gamma(\gamma)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \frac{(t-t_1)}{\Gamma(\gamma-1)}\int_{t_0}^{t_1}(t_1-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds + c_1 + c_2t + b_1^* + (t-t_1)c_1^*. \quad (2.13)$$

For $t \in [t_{k-1}, t_k]$, we have

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds + \frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \sum_{k=1}^{m}\frac{(t-t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_1-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds + c_1 + c_2t$$
$$+ \sum_{k=1}^{m}b_k^* + \sum_{k=1}^{m}(t-t_k)c_k^*. \quad (2.14)$$

With the help of (2.5), (2.13) and the initial condition in (2.4) we get $c_1 = 0$ and

$$c_2 = \frac{1}{1-\lambda^*}\Biggl[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \frac{\sum_{k=1}^{m}(\lambda^*-1+t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$- \frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds - \frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \sum_{k=1}^{m}(\lambda^*-1+t_k)c_k^*\Biggr]. \quad (2.15)$$

Ultimately, with the help of (2.14) and (2.15), we have

$$u(t) = \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds + \frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \sum_{k=1}^{m}\frac{(t-t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_1-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \frac{t}{1-\lambda^*}\Biggl[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$+ \frac{\sum_{k=1}^{m}(\lambda^*-1+t_k)}{\Gamma(\gamma-1)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-2}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$- \frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds$$
$$- \frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\psi_1\bigl(s,u(s)\bigr)\,ds + \sum_{k=1}^{m}(\lambda^*-1+t_k)c_k^*\Biggr]$$
$$+ \sum_{k=1}^{m}b_k^* + \sum_{k=1}^{m}(t-t_k)c_k^*. \quad (2.16)$$

With the help of (2.1) and (2.3), we have

$$D^{\epsilon}u(t) = \phi_q\big(\mathcal{G}\big(u(t)\big)\big). \tag{2.17}$$

Applying the integral operator I^{ϵ} and using Lemma 1.1 on (2.17), where $\epsilon \in (0,1]$ and $t \in [t_0, t_1]$, we have

$$u(t) = I^{\epsilon}\phi_q\big(\mathcal{G}\big(u(t)\big)\big) + d_1 = \frac{1}{\Gamma(\epsilon)}\int_{t_0}^{t}(t-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + d_1. \tag{2.18}$$

For $t \in [t_1, t_2]$, (2.18) implies

$$u(t) = \frac{1}{\Gamma(\epsilon)}\int_{t_1}^{t}(t-\tau)^{\epsilon-1}\phi_q(\mathcal{G}\big(u(\tau)\big)\,d\tau + d_2. \tag{2.19}$$

Thus

$$u\big(t_1^-\big) = \frac{1}{\Gamma(\epsilon)}\int_{t_0}^{t_1}(t_1-\tau)^{\epsilon-1}\phi_q(\mathcal{G}\big(u(\tau)\big)\,d\tau + d_1, \qquad u\big(t_1^+\big) = d_2. \tag{2.20}$$

With the help of the condition $\Delta u(t_1) = I_1(u(t_1))$ and (2.20), we get

$$d_2 = \frac{1}{\Gamma(\epsilon)}\int_{t_0}^{t_1}(t_1-\tau)^{\epsilon-1}\phi_q(\mathcal{G}\big(u(\tau)\big)\,d\tau + d_1 + I_1\big(u(t_1)\big). \tag{2.21}$$

From (2.19) and (2.21), we have

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\epsilon)}\int_{t_0}^{t}(t-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + \frac{1}{\Gamma(\epsilon)}\int_{t_0}^{t_1}(t_1-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau \\ & + d_1 + I\big(u(t_1)\big). \end{aligned} \tag{2.22}$$

For $t \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} u(t) = {} & \int_{t_{k-1}}^{t}\frac{(t-\tau)^{\epsilon-1}}{\Gamma(\epsilon)}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau \\ & + d_1 + \sum_{k=1}^{m} I_k\big(u(t_k)\big). \end{aligned} \tag{2.23}$$

With the initial condition $u(0) = 0$ and (2.18), we have $d_1 = 0$. Consequently, we have

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau \\ & + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + \sum_{k=1}^{m} I_k\big(u(t_k)\big). \end{aligned} \tag{2.24}$$

□

We define an operator $\mathcal{F}^*: \mathcal{B} \to \mathcal{B}$ by

$$\begin{aligned}\mathcal{F}^*\big(u\big((t)\big)\big) &= \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau \\ &\quad + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau + \sum_{k=1}^{m} I_k\big(u(t_k)\big),\end{aligned} \tag{2.25}$$

where $\mathcal{G}$ is defined in (2.3).

Theorem 2.2 *Assume that conditions* $(\mathcal{R}_1)$–$(\mathcal{R}_3)$ *hold true. Then the operator* $\mathcal{F}^*$ *is continuous and satisfies the inequality*

$$\begin{aligned}\big|\mathcal{F}^*u(t)\big| &\le \frac{m+1}{\Gamma(\gamma+1)}\phi_q\bigg(\bigg(\frac{1}{\Gamma(\gamma+1)} + \frac{m}{\Gamma(\gamma)} + \frac{1}{1-\lambda^*}\bigg[\frac{\lambda^*(1+m)}{\Gamma(\gamma)} + \frac{1+m}{\Gamma(1+\gamma)}\bigg]\bigg)\mathcal{M}_0 \\ &\quad + \frac{m\lambda^* C^*}{1-\lambda^*} + B^* + C^*\bigg) + m\mathcal{M}_1.\end{aligned}$$

Proof Consider a convergent sequence $\{u_n\} \to u$ in a bounded set $\rho_r = \{u \in \mathcal{B} : \|u\| \le r\}$. We show that $\|\mathcal{F}^*(u_n) - \mathcal{F}^*(u)\|$ goes to 0 as n approaches $+\infty$. For this purpose, with the help of Lemma 1.2, we have

$$\begin{aligned}&\big|\mathcal{F}^*(u_n) - \mathcal{F}^*(u)\big| \\ &\quad \le (q-1)m^{q-2}\int_{t_{k-1}}^{t}\frac{(t-\tau)^{\epsilon-1}}{\Gamma(\epsilon)}\bigg[\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{\tau}{1-\lambda^*}\bigg[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds\bigg]\bigg]\,d\tau \\ &\quad + (q-1)m^{q-2}\frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1} \\ &\quad \times\bigg[\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds \\ &\quad + \frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big) - \psi_1\big(s,u(s)\big)\big|\,ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big)-\psi_1\big(s,u(s)\big)\big|\,ds\\
&+\frac{\tau}{1-\lambda^*}\bigg[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big)-\psi_1\big(s,u(s)\big)\big|\,ds\\
&+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\big|\psi_1\big(s,u_n(s)\big)-\psi_1\big(s,u(s)\big)\big|\,ds\\
&+\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big)-\psi_1\big(s,u(s)\big)\big|\,ds\\
&+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\big|\psi_1\big(s,u_n(s)\big)-\psi_1\big(s,u(s)\big)\big|\,ds\bigg]\bigg]\,d\tau\\
&+\sum_{k=1}^{m}I_k\big(\big|u_n(t_k)-u(t_k)\big|\big).
\end{aligned}
\tag{2.26}
$$

By (2.26), and continuity of ψ_1, we have $|\mathcal{F}^*u_n-\mathcal{F}^*u|$ goes to 0 as n approaches $+\infty$. This implies that $\mathcal{F}^*$ is continuous. Next, consider a bounded subset $\mathcal{B}\subset\mathcal{L}$. Also, with the help of continuity of the functions f and $\mathcal{I}_k$, we have some constants, say, $\mathcal{M}_0,\mathcal{M}_1>0$, such that $|\psi_1(t,u(t))|\le\mathcal{M}_0$, $|\mathcal{I}_k(u(t_k))|\le\mathcal{M}_1$ for $k=1,2,\dots,m$ for $t\in\mathcal{J}=[0,1]$ and $u\in\mathcal{B}$. Then, for the boundedness of $\mathcal{F}^*$, we have

$$
\begin{aligned}
&\big|\mathcal{F}^*u(t)\big|\\
&\quad=\bigg|\frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds+\sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad+\frac{\tau}{1-\lambda^*}\bigg[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad-\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad-\frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds+\sum_{k=1}^{m}(\lambda^*-1+\tau_k)c_k^*\bigg]\\
&\qquad+\sum_{k=1}^{m}b_k^*+\sum_{k=1}^{m}(\tau-\tau_k)c_k^*)\,d\tau\\
&\qquad+\frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds\\
&\qquad+\sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
&+ \frac{t}{1-\lambda^*}\Bigg[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1 (1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\
&+ \frac{\sum_{k=1}^m(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\
&- \frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds \\
&- \frac{\sum_{k=}^m}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds + \sum_{k=1}^m(\lambda^*-1+\tau_k)c_k^*\Bigg] \\
&+ \sum_{k=1}^m b_k^* + \sum_{k=1}^m(\tau-\tau_k)c_k^*)\,d\tau + \sum_{k=1}^m I_k\big(u(t_k)\big)\Bigg| \qquad (2.28)\\
&\le \frac{1}{\Gamma(\epsilon+1)}\phi_q\Bigg(\Bigg(\frac{1}{\Gamma(\gamma+1)}+\frac{m}{\Gamma(\gamma+1)}+\frac{m}{\Gamma(\gamma)}+\frac{1}{1-\lambda^*}\Bigg[\frac{\lambda^*}{\Gamma(\gamma)}+\frac{m\lambda^*}{\Gamma(\gamma)}+\frac{1}{\Gamma(\gamma+1)} \\
&+ \frac{m}{\Gamma(\gamma+1)}\Bigg]\Bigg)\mathcal{M}_0+\frac{m\lambda^*C^*}{1-\lambda^*}+B^*+C^*\Bigg)+\frac{m}{\Gamma(\epsilon+1)}\phi_q\Bigg(\Bigg(\frac{1}{\Gamma(\gamma+1)}+\frac{m}{\Gamma(\gamma+1)} \\
&+ \frac{m}{\Gamma(\gamma)}+\frac{1}{1-\lambda^*}\Bigg[\frac{\lambda^*}{\Gamma(\gamma)}+\frac{m\lambda^*}{\Gamma(\gamma)}+\frac{1}{\Gamma(\gamma+1)}+\frac{m}{\Gamma(\gamma+1)}\Bigg]\Bigg)\mathcal{M}_0 \\
&+ \frac{m\lambda^*C^*}{1-\lambda^*}+B^*+C^*\Bigg)+m\mathcal{M}_1 \\
&= \frac{m+1}{\Gamma(\gamma+1)}\phi_q\Bigg(\Bigg(\frac{1}{\Gamma(\gamma+1)}+\frac{m}{\Gamma(\gamma)}+\frac{1}{1-\lambda^*}\Bigg[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\Bigg]\Bigg)\mathcal{M}_0 \\
&+ \frac{m\lambda^*C^*}{1-\lambda^*}+B^*+C^*\Bigg)+m\mathcal{M}_1.
\end{aligned}
$$

□

For the EUS theorems, we consider the assumptions:

- $(\mathcal{R}_1)$ $\psi_1(t,u(t)):\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ is a bounded continuous, that is, there exists some constant $\mathcal{M}_0>0$, such that $|\psi_1(t,u(t))|\le\mathcal{M}_0$.
- $(\mathcal{R}_2)$ For some constants $\mathcal{L}_{\psi_1}>0$, we have

$$\big|\psi_1\big(t,u(t)\big)-\psi_1\big(t,\bar{u}(t)\big)\big|\le\mathcal{L}_{\psi_1}\big|u(t)-\bar{u}(t)\big|. \qquad (2.29)$$

- $(\mathcal{R}_3)$ For some constant $\mathcal{L}>0$, we have

$$\big\|\mathcal{I}_k u(t_k)-\bar{\mathcal{I}}_k u(t_k)\big\|\le\mathcal{J}\|u-\bar{u}\|. \qquad (2.30)$$

Theorem 2.3 *With assumption $(\mathcal{R}_1)$, $\mathcal{F}^*:\mathcal{L}\to\mathcal{L}$ is compact and ξ-Lipschitz with constant zero.*

Proof With the help of Theorem 2.2, the operator $\mathcal{F}^*$ is bounded. Next, with the help of (2.3), (2.25), (2.28) and Lemma 1.2 for any $t_1,t_2\in[0,1]$, and we have

$$
\begin{aligned}
&\big|\mathcal{F}^*u(t_2)-\mathcal{F}^*u(t_1)\big| \\
&\le\bigg|\frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_2}(t_2-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau-\frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_1}(t_1-\tau)^{\epsilon-1}\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)\,d\tau\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\epsilon)}\left|\int_{t_{k-1}}^{t_2}(t_2-\tau)^{\epsilon-1}-\int_{t_{k-1}}^{t_1}(t_1-\tau)^{\epsilon-1}\right|\phi_q\left|\mathcal{G}\big(u(\tau)\big)\right|d\tau \\
&\le \frac{1}{\Gamma(\epsilon)}\left|\int_{t_{k-1}}^{t_2}(t-\tau)^{\epsilon-1}-\int_{t_{k-1}}^{t_1}(t_1-\tau)^{\epsilon-1}\right|\phi_q\left|\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds\right. \\
&\quad+\frac{\sum_{k=1}^m}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds+\sum_{k=1}^m\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\
&\quad+\frac{\tau}{1-\lambda^*}\left[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds\right. \\
&\quad+\frac{\sum_{k=1}^m(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\psi_1\big(s,u(s)\big)\,ds \\
&\quad-\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds-\frac{\sum_{k=1}^m}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\psi_1\big(s,u(s)\big)\,ds \\
&\quad\left.\left.+\sum_{k=1}^m\big(\lambda^*-1+\tau_k\big)c_k^*\right]+\sum_{k=1}^m b_k^*+\sum_{k=1}^m(t-t_k)c_k^*\right|d\tau \\
&\le \frac{1}{\Gamma(\epsilon)}\left|\int_0^{t_2}(t-\tau)^{\epsilon-1}-\int_0^{t_1}(t_1-\tau)^{\epsilon-1}\right|\phi_q\left[\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}\right.\right.\right. \\
&\quad\left.\left.\left.+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\big(k+\|u\|\big)+\frac{m\lambda^*C^*}{1-\lambda^*}+B^*+C^*\right]d\tau \\
&\le \frac{t_2-t_1}{\Gamma(\epsilon+1)}\left[\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\big(k+\|u\|\big)\right. \\
&\quad\left.+\frac{m\lambda^*C^*}{1-\lambda^*}+B^*+C^*\right]^{q-1}.
\end{aligned}
\tag{2.31}
$$

As $t_1 \to t_2$, (2.31) implies that $|\mathcal{F}^*u(t_2) - \mathcal{F}^*u(t_1)| \to 0$. Thus, $\mathcal{F}^*$ is equicontinuous and by Arzela–Ascoli's theorem, $\mathcal{F}^*$ is compact. Consequently, $\mathcal{F}^*$ is ξ-Lipschitz with constant 0. □

Theorem 2.4 *Assume that the conditions* $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ *are true. Then the impulsive fractional order DE with* ϕ_p (1.3) *has a unique solution with assumption* $\Delta^* < 1$ *for*

$$
\begin{aligned}
\Delta^* = {}&\left[(q-1)\mathcal{M}^{q-2}\frac{m+1}{\Gamma(\epsilon+1)}\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1}\right. \\
&\left.+m\mathcal{J}\right].
\end{aligned}
\tag{2.32}
$$

Proof With the assumption of conditions $(\mathcal{R}_1)$, $(\mathcal{R}_2)$, and Lemma 1.2, we have

$$
\begin{aligned}
&\big|\mathcal{F}^*u(t)-\mathcal{F}^*\bar{u}(t)\big| \\
&\quad\le \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\big|\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)-\phi_q\big(\mathcal{G}\big(\bar{u}(\tau)\big)\big)\big|\,d\tau \\
&\qquad+\frac{\sum_{k=1}^m}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\big|\phi_q\big(\mathcal{G}\big(u(\tau)\big)\big)-\phi_q\big(\mathcal{G}\big(\bar{u}(\tau)\big)\big)\big|\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\bigl|I_k\bigl(u(t_k)\bigr)-I_k\bigl(\bar{u}(t_k)\bigr)\bigr| \\
&\le (q-1)\mathcal{M}^{q-2}\frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\Biggl(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \qquad (2.33)\\
&+\sum_{k=1}^{m}\frac{(t-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(t_1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\tau}{1-\lambda^*}\biggl[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \qquad (2.34)\\
&+\frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds\biggr]\Biggr)\,d\tau \\
&+(q-1)\mathcal{M}^{q-2}\frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1} \\
&\times\Biggl(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\tau}{1-\lambda^*}\biggl[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds \\
&+\frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,\bar{u}(s)\bigr)\bigr|\,ds\biggr]\Biggr)\,d\tau \\
&+\sum_{k=1}^{m}\bigl|I_k\bigl(u(t_k)\bigr)-I_k\bigl(\bar{u}(t_k)\bigr)\bigr| \\
&\le\frac{(q-1)\mathcal{M}^{q-2}}{\Gamma(\epsilon+1)}\biggl[\biggl(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\biggl[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\biggr]\biggr)\mathcal{L}_{\psi_1}|u-\bar{u}|\biggr] \\
&+\frac{m(q-1)\mathcal{M}^{q-2}}{\Gamma(\epsilon+1)}\biggl[\biggl(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\biggl[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\biggr]\biggr)\mathcal{L}_{\psi_1}|u-\bar{u}|\biggr] \\
&+m\mathcal{J}|u-\bar{u}|
\end{aligned}
$$

$$\leq \left[(q-1)\mathcal{M}^{q-2}\frac{m+1}{\Gamma(\epsilon+1)}\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1} + m\mathcal{J}\right]\|u-\bar{u}\|.$$

With the assumption given in (2.32), and Banach's fixed point theorem, the operator $\mathcal{F}^*$ has a unique fixed point and is a unique solution of (1.3). □

3 Hyers–Ulam stability

Here, we prove that (1.3) is Hyers–Ulam stable. We give the following definition for the stability.

Definition 3.1 The integral equation (2.24) is said to be Hyers–Ulam stable if for some fixed constant value $\lambda^* > 0$,

$$\left|u(t) - \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q\bigl(\mathcal{G}\bigl(u(\tau)\bigr)\bigr)\,d\tau - \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\bigl(\mathcal{G}\bigl(u(\tau)\bigr)\bigr)\,d\tau - \sum_{k=1}^{m}I_k\bigl(u(t_k)\bigr)\right| \leq \lambda^*, \tag{3.1}$$

there exists a continuous function $u^*(t)$, satisfying the following equation:

$$u^*(t) = \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\phi_q\bigl(\mathcal{G}\bigl(u^*(\tau)\bigr)\bigr)\,d\tau + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\phi_q\bigl(\mathcal{G}\bigl(u^*(\tau)\bigr)\bigr)\,d\tau + \sum_{k=1}^{m}I_k\bigl(u^*(t_k)\bigr),$$

with

$$\bigl|u(t)-u^*(t)\bigr| \leq \mathcal{D}^*\lambda^*. \tag{3.2}$$

Theorem 3.2 *With the help of assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, *the impulsive fractional order DE* (1.3) *is Hyers–Ulam stable.*

Proof For the Hyers–Ulam stability of the integral equation (1.3), with assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ and Lemma 1.2, we have

$$\begin{aligned}\bigl|u(t)-u^*(t)\bigr| &\leq \frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1}\bigl|\phi_q\bigl(\mathcal{G}\bigl(u(\tau)\bigr)\bigr)-\phi_q\bigl(\mathcal{G}\bigl(u^*(\tau)\bigr)\bigr)\bigr|\,d\tau \\ &\quad + \frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1}\bigl|\phi_q\bigl(\mathcal{G}\bigl(u(\tau)\bigr)\bigr)-\phi_q\bigl(\mathcal{G}\bigl(u^*(\tau)\bigr)\bigr)\bigr|\,d\tau \\ &\quad + \sum_{k=1}^{m}\bigl|I_k\bigl(u(t_k)\bigr)-I_k\bigl(u^*(t_k)\bigr)\bigr|\end{aligned}$$

$$
\begin{aligned}
&\le (q-1)\mathcal{M}^{q-2}\frac{1}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t}(t-\tau)^{\epsilon-1} \\
&\quad\times\left(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right. \\
&\quad+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\sum_{k=1}^{m}\frac{(t-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(t_1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\frac{\tau}{1-\lambda^*}\left[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right. \\
&\quad+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \qquad (3.3) \\
&\quad\left.\left.+\frac{\sum_{k=}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right]\right)d\tau \\
&\quad+(q-1)\mathcal{M}^{q-2}\frac{\sum_{k=1}^{m}}{\Gamma(\epsilon)}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\epsilon-1} \\
&\quad\times\left(\frac{1}{\Gamma(\gamma)}\int_{\tau_1}^{\tau}(\tau-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right. \\
&\quad+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\sum_{k=1}^{m}\frac{(\tau-\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\frac{\tau}{1-\lambda^*}\left[\frac{\lambda^*}{\Gamma(\gamma-1)}\int_0^1(1-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right. \qquad (3.4) \\
&\quad+\frac{\sum_{k=1}^{m}(\lambda^*-1+\tau_k)}{\Gamma(\gamma-1)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-2}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds \\
&\quad\left.\left.+\frac{\sum_{k=1}^{m}}{\Gamma(\gamma)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\gamma-1}\bigl|\psi_1\bigl(s,u(s)\bigr)-\psi_1\bigl(s,u^*(s)\bigr)\bigr|\,ds\right]\right)d\tau \\
&\quad+\sum_{k=1}^{m}\bigl|I_k\bigl(u(t_k)\bigr)-I_k\bigl(u^*(t_k)\bigr)\bigr| \\
&\le\frac{(q-1)\mathcal{M}^{q-2}}{\Gamma(\epsilon+1)}\left[\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1}\left|u-u^*\right|\right] \\
&\quad+\frac{m(q-1)\mathcal{M}^{q-2}}{\Gamma(\epsilon+1)}\left[\left(\frac{1+m}{\Gamma(1+\gamma)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{1+m}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1}\left|u-u^*\right|\right] \\
&\quad+m\mathcal{J}\left|u-u^*\right|
\end{aligned}
$$

$$\leq \left[(q-1)\mathcal{M}^{q-2}\frac{1+m}{\Gamma(1+\epsilon)}\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{m+1}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1}\right.$$
$$\left.+ m\mathcal{J}\right]\|u-u^*\|.$$

□

4 Example/application

For the application of the theorems proved in Sects. 3 and 4, the following example is presented.

Example 4.1 Assume that $\psi_1(t,u(t)) = \frac{|u(t)|}{100+|u(t)|}$, for $t \in [0,1]$, $p = 3$. Consider the following impulsive fractional order DE as a particular example of (1.3):

$$\begin{cases} \mathcal{D}^{\gamma=1.5}[\phi_{p=3}[\mathcal{D}^{\epsilon=0.5}u(t)]] - \psi_1(t,u(t)) = 0, \\ \Delta[\phi_{p=3}[\mathcal{D}^{\epsilon=0.5}u(t_k)]] = \frac{1}{100}, \qquad \Delta[\phi_{p=3}[\mathcal{D}^{\epsilon=0.5}u(t_k)]]' = \frac{1}{100}, \\ \Delta u(t_k) = I_k(u(t_k)), \\ \phi_p(\mathcal{D}^{\epsilon=0.5}u(0)) = 0, \qquad \phi_p(\mathcal{D}^{\epsilon=0.5}u(1)) = \lambda^*\phi_p'(\mathcal{D}^{\epsilon=0.5}u(1)), \qquad u(0) = 0. \end{cases} \tag{4.1}$$

By a simple calculation, we have the values $\lambda^* = \mathcal{M}_0 = \mathcal{L}_{\psi_1} = b_k = c_k = \frac{1}{100}$, $m = 30$, $k = 1,2,\ldots,30$. The conditions $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ are satisfied and $\Delta^* < 1$ for

$$\Delta^* = \left[(q-1)\mathcal{M}^{q-2}\frac{m+1}{\Gamma(\epsilon+1)}\left(\frac{1+m}{\Gamma(\gamma+1)}+\frac{1}{1-\lambda^*}\left[\frac{\lambda^*(1+m)}{\Gamma(\gamma)}+\frac{m+1}{\Gamma(1+\gamma)}\right]\right)\mathcal{L}_{\psi_1}\right.$$
$$\left.+ m\mathcal{J}\right]. \tag{4.2}$$

Thus, with the help of Theorem 2.4, the impulsive fractional order DE (4.1), has a unique solution and is Hyers–Ulam stable.

5 Conclusion

In this paper, we have considered a generalized class of impulsive fractional order DEs for the study of EUS and Hyers–Ulam stability. For these goals of the paper, we converted the suggested FODE with impulsive effect and nonlinear ϕ_p operator to an equivalent fractional order integral system involving $\phi_q = \phi_p^{-1}$. Then, with the help of classical fixed point theorems, the existence results are obtained and stability is examined. For application, an illustrative example is given. For future work, we suggest to consider the problem with singularity and delay for different fractional order derivatives.

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