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The Kamenev type interval oscillation criteria of mixed nonlinear impulsive differential equations under variable delay effects

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Abstract

In this paper, a class of mixed nonlinear impulsive differential equations is studied. When the delay $\sigma(t)$ is variable, each given interval is divided into two parts on which the quotients of $x(t-\sigma(t))$ and $x(t)$ are estimated. Then, by introducing binary auxiliary functions and using the Riccati transformation, several Kamenev type interval oscillation criteria are established. The well-known results obtained by Liu and Xu (Appl. Math. Comput. 215:283–291, 2009) for $\sigma(t)=0$ and by Guo et al. (Abstr. Appl. Anal. 2012:351709, 2012) for $\sigma(t)=\sigma_0$ ($\sigma_0 \geq 0$) are developed. Moreover, an example illustrating the effectiveness and non-emptiness of our results is also given.

Keywords: Interval oscillation; Impulsive differential equation; Variable delay; Interval delay function

1 Introduction

We consider the following mixed nonlinear impulsive differential equations with variable delay:

$$
\begin{aligned}
&\big(r(t)\Phi_\alpha\big(x'(t)\big)\big)' + p_0(t)\Phi_\alpha\big(x(t)\big) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\big(x\big(t-\sigma(t)\big)\big) = f(t), \qquad t \geq t_0, t \neq \tau_k, \\
&x\big(\tau_k^+\big) = a_k x(\tau_k), \qquad x'\big(\tau_k^+\big) = b_k x'(\tau_k), \quad k = 1,2,\ldots,
\end{aligned}
\tag{1}
$$

where $\Phi_*(s) = |s|^{*-1}s$, $\{\tau_k\}$ denotes the impulse moments, $0 \leq t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$, $\{a_k\}$ and $\{b_k\}$ are real constant sequences and $b_k \geq a_k > 0$ for $k = 1,2,\ldots$, $\sigma(t) \in C([t_0,\infty))$ and there exists a nonnegative constant σ_0 such that $0 \leq \sigma(t) \leq \sigma_0$ for all $t \geq t_0$, $r(t) \in C^1([t_0,\infty),(0,\infty))$ is nondecreasing.

For some particular cases of (1), many authors have devoted work to the interval oscillation problem (see [3–13]). Particularly, when $\alpha = 1$, $a_k = b_k = 1$ and $\sigma(t) = 0$, (1) reduces to the mixed type Emden–Fowler equation

$$
\big(r(t)x'(t)\big)' + p_0(t)x(t) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\big(x(t)\big) = f(t), \tag{2}
$$

which was given much attention due to the effect of modeling the growth of bacteria population with competitive species. For example, in [14] and [15], the authors established interval oscillation theorems for (2) which improved the well-known criteria of [16] and [17]. For additional studies of Emden–Fowler differential equations, see [18–20].

In [1], the authors considered (2) with impulse effects,

$$\begin{aligned} &\bigl(r(t)x'(t)\bigr)' + p_0(t)x(t) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\bigl(x(t)\bigr) = f(t), \quad t \ge t_0, t \neq \tau_k, \\ &x\bigl(\tau_k^+\bigr) = a_k x(\tau_k), \qquad x'\bigl(\tau_k^+\bigr) = b_k x'(\tau_k), \quad k = 1,2,\ldots, \end{aligned} \tag{3}$$

and established some interval oscillation results which extended those of [14, 15] and [21].

When $\sigma(t) = 0$, (1) becomes the following impulse equations without delay:

$$\begin{aligned} &\bigl(r(t)\Phi_\alpha\bigl(x'(t)\bigr)\bigr)' + p_0(t)\Phi_\alpha\bigl(x(t)\bigr) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\bigl(x(t)\bigr) = f(t), \quad t \ge t_0, t \neq \tau_k, \\ &x\bigl(\tau_k^+\bigr) = a_k x(\tau_k), \qquad x'\bigl(\tau_k^+\bigr) = b_k x'(\tau_k), \quad k = 1,2,\ldots. \end{aligned} \tag{4}$$

In [22], Özbekler and Zafer investigated (4). They considered the coefficients $p_i(t)$ ($i = 1,2,\ldots,n$) satisfying two cases: (i) $p_i(t) \ge 0$ for $i = 1,2,\ldots,n$ and (ii) $p_i(t) \ge 0$ for $i = 1,2,\ldots,m$; $p_i(t)$ are allowed to be negative for $i = m+1,\ldots,n$ and obtained several interval oscillation results which recovered the early ones in [8] and [14].

When $\sigma(t)$ is a nonnegative constant, i.e., $\sigma(t) = \sigma_0$ ($\sigma_0 \ge 0$), by idea of [23], Guo et al. [2] studied (1) and developed the results of [1, 22, 24].

Recently, in [25], the authors studied (1) with the assumption of delay $\sigma(t)$ being variable. They used Riccati transformation and univariate ω functions to obtain some generalized interval oscillation results.

In this paper, we continue the discussion of the interval oscillation of (1). Unlike the methods of [22, 25], we introduce a binary auxiliary function, divide each given interval into two parts and then estimate the quotients of $x(t-\sigma(t))$ and $x(t)$. Due to the considered delay being variable, the results obtained here are the development of some well-known ones, such as in [1] and [2]. Moreover, we also give an example to illustrate the effectiveness and non-emptiness of our results.

2 Main results

First, we define a functional space $C_-(I,\mathbb{R})$ as follows:

$$\begin{aligned} C_-(I,\mathbb{R}) := \bigl\{&y : I \to \mathbb{R} \mid I \text{ is a real interval}, y \text{ is continuous on } I \setminus \{t_i\} \text{ and} \\ &y\bigl(t_i^-\bigr) = y(t_i), i \in \mathbb{N}\bigr\}. \end{aligned}$$

In the following, we always assume:

(A) the exponents satisfy $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$; $f(t), p_i(t) \in C_-([t_0,\infty),\mathbb{R})$, $i = 0,1,\ldots,n$; $\tau_{k+1} - \tau_k > \sigma_0$ for all $k = 1,2,\ldots$.

Let $k(s) = \max\{i : t_0 < \tau_i < s\}$. For any given intervals $[c_j,d_j]$ ($j = 1,2$), we suppose that $k(c_j) < k(d_j)$ ($j = 1,2$), then there exist impulse moments $\tau_{k(c_j)+1},\ldots,\tau_{k(d_j)}$ in $[c_j,d_j]$ ($j = 1,2$) and we have the following cases to consider.

(S_1) $\tau_{k(c_j)} + \sigma_0 < c_j$ and $\tau_{k(d_j)} + \sigma_0 < d_j$;
(S_2) $\tau_{k(c_j)} + \sigma_0 < c_j$ and $\tau_{k(d_j)} + \sigma_0 > d_j$;
(S_3) $\tau_{k(c_j)} + \sigma_0 > c_j$ and $\tau_{k(d_j)} + \sigma_0 > d_j$;
(S_4) $\tau_{k(c_j)} + \sigma_0 > c_j$ and $\tau_{k(d_j)} + \sigma_0 < d_j$.

We further assume that there exist points $\delta_j \in (c_j, d_j) \setminus \{\tau_k\}$ $(j = 1, 2)$ which divide intervals $[c_j, d_j]$ into two parts $[c_j, \delta_j]$ and $[\delta_j, d_j]$. In view of whether or not there are impulsive moments of $x(t)$ in $[c_j, \delta_j]$ and $[\delta_j, d_j]$, we should consider the following cases.

$(\bar{S}_1)$ $k(c_j) < k(\delta_j) < k(d_j)$;
$(\bar{S}_2)$ $k(c_j) = k(\delta_j) < k(d_j)$;
$(\bar{S}_3)$ $k(c_j) < k(\delta_j) = k(d_j)$.

We define a interval delay function ([12]):

$$D_k(t) = t - \tau_k - \sigma(t), \quad t \in (\tau_k, \tau_{k+1}], k = 1, 2, \dots,$$

and we assume there is a point $t_k \in (\tau_k, \tau_{k+1}]$ such that $D_k(t_k) = 0$, $D_k(t) < 0$ for $t \in (\tau_k, t_k)$ and $D_k(t) > 0$ for $t \in (t_k, \tau_{k+1}]$.

Moreover, for the relationship of the division point δ_j and the zero point $t_{k(\delta_j)}$ of $D_{k(\delta_j)}$ on $[\tau_{k(\delta_j)}, \tau_{k(\delta_j)+1}]$ we should have

$(\bar{\bar{S}}_1)$ $t_{k(\delta_j)} < \delta_j$;
$(\bar{\bar{S}}_2)$ $t_{k(\delta_j)} > \delta_j$; or
$(\bar{\bar{S}}_3)$ $t_{k(\delta_j)} = \delta_j$.

We only consider the case of combination of (S_1) with $(\bar{S}_1)$ and $(\bar{\bar{S}}_1)$. For the other cases, the discussion will be omitted here.

Lemma 2.1 *Assume that, for any $T \geq t_0$, there exist $T < c_1 - \sigma_0 < c_1 < \delta_1 < d_1$ and*

$$f(t) \leq 0, \qquad p_i(t) \geq 0, \quad t \in [c_1 - \sigma_0, d_1] \setminus \{\tau_k\}, i = 0, 1, 2, \dots, n, \tag{5}$$

and t_k is a zero point of $D_k(t_k)$ in $(\tau_k, \tau_{k+1}]$. If $x(t)$ is a positive solution of (1), then the ratio $x(t - \sigma(t))/x(t)$ will be estimated as follows:

(a) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_k-\sigma(t)}{t-\tau_k}$, $t \in (t_k, \tau_{k+1}]$ *for* $k = k(c_1) + 1, \dots, k(d_1) - 1$, $k \neq k(\delta_1)$;
(b) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_k}{b_k(t+\sigma(t)-\tau_k)}$, $t \in (\tau_k, t_k]$ *for* $k = k(c_1) + 1, \dots, k(d_1)$;
(c) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_{k(\delta_1)}-\sigma(t)}{t-\tau_{k(\delta_1)}}$, $t \in (t_{k(\delta_1)}, \delta_1]$;
(d) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_{k(d_1)}-\sigma(t)}{t-\tau_{k(d_1)}}$, $t \in (t_{k(d_1)}, d_1]$;
(e) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_{k(\delta_1)}-\sigma(t)}{t-\tau_{k(\delta_1)}}$, $t \in (\delta_1, \tau_{k(\delta_1)+1}]$;
(f) $\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}$, $t \in [c_1, \tau_{k(c_1)+1}]$.

Proof From (1), (5) and (A), we obtain, for $t \in [c_1, d_1] \setminus \{\tau_k\}$,

$$\big(r(t)\Phi_\alpha\big(x'(t)\big)\big)' = f(t) - p_0(t)\Phi_\alpha\big(x(t)\big) - \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}\big(x\big(t - \sigma(t)\big)\big) \leq 0.$$

Hence $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on the interval $[c_1, d_1] \setminus \{\tau_k\}$. Next, we give the proof of case (a) only. For the other cases, the proof is similar and will be omitted.

If $t_k < t \le \tau_{k+1}$, from $D_k(t) > 0$, we know $(t-\sigma(t), t) \subset (\tau_k, \tau_{k+1}]$. Thus there is no impulse moment in $(t-\sigma(t), t)$. Therefore, for any $s \in (t-\sigma(t), t)$, there exists a $\xi_k \in (\tau_k, s)$ such that $x(s) - x(\tau_k^+) = x'(\xi_k)(s-\tau_k)$. Since $x(\tau_k^+) > 0$, $r(s)$ is nondecreasing, $\Phi_\alpha(\cdot)$ is an increasing function and $r(t)\Phi_\alpha(x'(t))$ is nonincreasing on $(\tau_k, \tau_{k+1}]$, we have

$$\begin{aligned}\Phi_\alpha\bigl(x(s)\bigr) &\ge \frac{r(\xi_k)}{r(s)}\Phi_\alpha\bigl(x(s)\bigr) > \frac{r(\xi_k)}{r(s)}\Phi_\alpha\bigl(x'(\xi_k)(s-\tau_k)\bigr) = \frac{r(\xi_k)\Phi_\alpha(x'(\xi_k))}{r(s)}(s-\tau_k)^\alpha \\ &\ge \frac{r(s)\Phi_\alpha(x'(s))}{r(s)}(s-\tau_k)^\alpha = \Phi_\alpha\bigl(x'(s)(s-\tau_k)\bigr).\end{aligned}$$

Therefore,

$$\frac{x'(s)}{x(s)} < \frac{1}{s-\tau_k}.$$

Integrating both sides of the above inequality from $t-\sigma(t)$ to t, we obtain

$$\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_k-\sigma(t)}{t-\tau_k}, \quad t \in (t_k, \tau_{k+1}].$$

The proof is completed. □

Lemma 2.2 *Assume that, for any $T \ge t_0$, there exist $T < c_2 - \sigma_0 < c_2 < \delta_2 < d_2$ and*

$$f(t) \ge 0, \qquad p_i(t) \ge 0, \quad t \in [c_2-\sigma_0, d_2] \setminus \{\tau_k\}, \quad i = 0,1,2,\dots,n, \tag{6}$$

and t_k is a zero point of $D_k(t_k)$ in $(\tau_k, \tau_{k+1}]$. If $x(t)$ is a negative solution of (1), *then estimations* (a)–(f) *in Lemma* 2.1 *are correct with the replacement of c_1, d_1 and δ_1 by c_2, d_2 and δ_2, respectively.*

The proof of Lemma 2.2 is similar to that of Lemma 2.1 and will be omitted.

Lemma 2.3 *Assume that for any $T \ge t_0$ there exists $T < c_1 - \sigma_0 < c_1 < d_1$ and* (5) *holds. If $x(t)$ is a positive solution of* (1) *and $u(t)$ is defined by*

$$u(t) := \frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}, \quad t \in [c_1, d_1], \tag{7}$$

then we have the following estimations of $u(t)$:

(g) $u(\tau_{k+1}) \le \frac{\widetilde{r}}{(\tau_{k+1}-\tau_k)^\alpha}$, $\tau_{k+1} \in [c_1, d_1]$, $k = k(c_1)+1, \dots, k(d_1)-1$, $k \ne k(\delta_1)$;

(h) $u(\tau_{k(c_1)+1}) \le \frac{\widetilde{r}}{(\tau_{k(c_1)+1}-c_1)^\alpha}$, $\tau_{k(c_1)+1} \in [c_1, d_1]$;

(i) $u(\tau_{k(\delta_1)+1}) \le \frac{\widetilde{r}}{(\tau_{k(\delta_1)+1}-\delta_1)^\alpha}$, $\tau_{k(\delta_1)+1} \in [c_1, d_1]$,

where $\widetilde{r} = \max_{t\in[c_1,d_1]\cup[c_2,d_2]}\{r(t)\}$.

Proof For $t \in (\tau_k, \tau_{k+1}] \subset [c_1, d_1]$, $k = k(c_1)+1, \dots, k(d_1)-1$, there exists $\varsigma_k \in (\tau_k, t)$ such that

$$x(t) - x\bigl(\tau_k^+\bigr) = x'(\varsigma_k)(t-\tau_k).$$

In view of $x(\tau_k^+) > 0$ and the monotone properties of $\Phi_\alpha(\cdot)$, $r(t)\Phi_\alpha(x'(t))$ and $r(t)$, we obtain

$$\Phi_\alpha\bigl(x(t)\bigr) > \Phi_\alpha\bigl(x'(\varsigma_k)\bigr)\Phi_\alpha(t-\tau_k) \geq \frac{r(t)}{r(\varsigma_k)}\Phi_\alpha\bigl(x'(t)\bigr)(t-\tau_{k-1})^\alpha.$$

That is,

$$\frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))} < \frac{r(\varsigma_k)}{(t-\tau_k)^\alpha}.$$

Letting $t \to \tau_{k+1}^-$, we obtain conclusion (g). Using a similar analysis on $(c_1, \tau_{k(c_1)+1}]$ and $(\delta_1, \tau_{k(\delta_1)+1}]$, we can get (h) and (i). The proof is completed. □

Lemma 2.4 *Assume that, for any $T \geq t_0$, there exist $c_2, d_2, \delta_2 \notin \{\tau_k\}$ such that $T < c_2 - \sigma_0 < c_2 < \delta_2 < d_2$ and* (6) *hold. If $x(t)$ is a negative solution of* (1) *and $u(t)$ is defined by*

$$u(t) := \frac{r(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))}, \quad t \in [c_2, d_2], \tag{8}$$

then the estimations (g)–(i) *in Lemma* 2.3 *are correct with the replacement of c_1, d_1 and δ_1 by c_2, d_2 and δ_2, respectively.*

The proof of Lemma 2.4 is similar to that of Lemma 2.3 and will be omitted.

Lemma 2.5 (cf. Lemma 1.1 in [22]) *Let $\{\beta_1, \dots, \beta_n\}$ be the n-tuple satisfying* (A). *Then there exists an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

$$\begin{aligned} &\text{(i)} \quad \sum_{i=1}^n \beta_i \eta_i = \alpha, \quad \textit{and} \\ &\text{(ii)} \quad \sum_{i=1}^n \eta_i = \lambda < 1, \quad 0 < \eta_i < 1. \end{aligned} \tag{9}$$

In the following we will establish Kamenev type interval oscillation criteria for (1) by the idea of Philos [26]. For the research of Kamenev/Philos-type oscillation criteria for differential equations, see [27–31].

Let $E = \{(t,s) : t_0 \leq s \leq t\}$, $H_1, H_2 \in C^1(E, \mathbb{R})$. Then a pair of functions H_1, H_2 is said to belong to a function set $\mathscr{H}$, denoted by $(H_1, H_2) \in \mathscr{H}$, if there exist $h_1, h_2 \in L_{\mathrm{loc}}(E, \mathbb{R})$ satisfying the following conditions:

(C_1) $H_1(t,t) = H_2(t,t) = 0$, $H_1(t,s) > 0$, $H_2(t,s) > 0$ for $t > s$;

(C_2) $\frac{\partial}{\partial t}H_1(t,s) = h_1(t,s)H_1(t,s)$, $\frac{\partial}{\partial s}H_2(t,s) = h_2(t,s)H_2(t,s)$.

For convenience in the expression below, we also use the following notation:

$$\int_{[c,d]} := \int_c^{\tau_{k(c)+1}} + \sum_{k=k(c)+1}^{k(d)-1} \left(\int_{\tau_k}^{t_k} + \int_{t_k}^{\tau_{k+1}} \right) + \int_{\tau_{k(d)}}^{t_{k(d)}} + \int_{t_{k(d)}}^{d}.$$

Lemma 2.6 *Assume that the conditions of Lemma 2.1 hold. Let $x(t)$ be a positive solution of (1) and $u(t)$ be defined by (7). Then, for any $(H_1, H_2) \in \mathscr{H}$, we have*

$$\begin{aligned} &\int_{[c_1,\delta_1]} \psi(t)H_1(t,c_1)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t \\ &\quad + \int_{c_1}^{\delta_1} H_1(t,c_1)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_1(t,c_1)\right|^{1+\alpha}\right]\mathrm{d}t \\ &\leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_1(\tau_i,c_1)u(\tau_i) - H_1(\delta_1,c_1)u(\delta_1) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\int_{[\delta_1,d_1]} \psi(t)H_2(d_1,t)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t \\ &\quad + \int_{\delta_1}^{d_1} H_2(d_1,t)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_2(d_1,t)\right|^{1+\alpha}\right]\mathrm{d}t \\ &\leq \sum_{i=k(\delta_1)+1}^{k(d_1)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_2(d_1,\tau_i)u(\tau_i) + H_2(d_1,\delta_1)u(\delta_1), \end{aligned} \tag{11}$$

where $\psi(t) = \eta_0^{-\eta_0}|f(t)|^{\eta_0}\prod_{i=1}^n \eta_i^{-\eta_i}(p_i(t))^{\eta_i}$ with $\eta_0 = 1 - \sum_{i=1}^n \eta_i$ and $\eta_1, \eta_2, \dots, \eta_n$ are positive constants satisfying conditions of Lemma 2.5.

Proof Differentiating $u(t)$ and in view of (1), we obtain, for $t \neq \tau_k$,

$$\begin{aligned} u'(t) = &-\left[\sum_{i=1}^n p_i(t)\Phi_{\beta_i-\alpha}\bigl(x\bigl(t-\sigma(t)\bigr)\bigr) + \frac{|f(t)|}{\Phi_\alpha(x(t-\sigma(t)))}\right]\frac{\Phi_\alpha(x(t-\sigma(t)))}{\Phi_\alpha(x(t))} \\ &- \frac{\alpha}{r^{1/\alpha}(t)}\left|u(t)\right|^{1+1/\alpha} - p_0(t). \end{aligned} \tag{12}$$

Let

$$v_0 = \eta_0^{-1}\frac{|f(t)|}{\Phi_\alpha(x(t-\sigma(t)))}, \qquad v_i = \eta_i^{-1}p_i(t)\Phi_{\beta_i-\alpha}\bigl(x\bigl(t-\sigma(t)\bigr)\bigr), \quad i = 1,2,\dots,n,$$

where $\eta_1, \eta_2, \dots, \eta_n$ are chosen to satisfy conditions of Lemma 2.5 with $\eta_0 = 1 - \sum_{i=1}^n \eta_i$ for given $\beta_1, \dots, \beta_n$ and α. Employing the arithmetic–geometric mean inequality (see [32])

$$\sum_{i=0}^n \eta_i v_i \geq \prod_{i=0}^n v_i^{\eta_i},$$

we have, from (12),

$$u'(t) \leq -\psi(t)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} - \frac{\alpha}{r^{1/\alpha}(t)}\left|u(t)\right|^{1+1/\alpha} - p_0(t), \tag{13}$$

where

$$\psi(t) = \eta_0^{-\eta_0}\left|f(t)\right|^{\eta_0}\prod_{i=1}^n \eta_i^{-\eta_i}\bigl(p_i(t)\bigr)^{\eta_i}.$$

Multiplying both sides of (13) by $H_1(t,c_1)$ and integrating it from c_1 to δ_1, we have

$$\begin{aligned}\int_{[c_1,\delta_1]} H_1(t,c_1)u'(t)\,\mathrm{d}t \leq {}& -\int_{[c_1,\delta_1]} \psi(t)H_1(t,c_1)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t \\ & - \alpha\int_{[c_1,\delta_1]} H_1(t,c_1)\frac{|u(t)|^{1+1/\alpha}}{r^{1/\alpha}(t)}\,\mathrm{d}t \\ & - \int_{c_1}^{\delta_1} H_1(t,c_1)p_0(t)\,\mathrm{d}t. \end{aligned} \tag{14}$$

Noticing that the impulse moments $\tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(\delta_1)}$ are in $[c_1,\delta_1]$ and using the integration by parts formula on the left-hand side of the above inequality, we obtain

$$\begin{aligned}\int_{[c_1,\delta_1]} H_1(t,c_1)u'(t)\,\mathrm{d}t = {}& \sum_{i=k(c_1)+1}^{k(\delta_1)} \left(1-\frac{b_i^\alpha}{a_i^\alpha}\right)H_1(\tau_i,c_1)u(\tau_i) + H_1(\delta_1,c_1)u(\delta_1) \\ & - \int_{[c_1,\delta_1]} H_1(t,c_1)h_1(t,c_1)u(t)\,\mathrm{d}t. \end{aligned} \tag{15}$$

Substituting (15) into (14), we obtain

$$\begin{aligned}&\int_{[c_1,\delta_1]} \psi(t)H_1(t,c_1)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t \\ &\quad \leq \sum_{i=k(c_1)+1}^{k(\delta_1)} \left(\frac{b_i^\alpha}{a_i^\alpha}-1\right)H_1(\tau_i,c_1)u(\tau_i) - H_1(\delta_1,c_1)u(\delta_1) \\ &\qquad - \int_{c_1}^{\delta_1} p_0(t)H_1(t,c_1)\,\mathrm{d}t + \int_{[c_1,\delta_1]} H_1(t,c_1)V\big(u(t)\big)\,\mathrm{d}t, \end{aligned}$$

where $V(u(t)) = [|h_1(t,c_1)||u(t)| - \frac{\alpha}{r^{1/\alpha}(t)}|u(t)|^{1+1/\alpha}]$. We easily see that

$$V\big(u(t)\big) \leq \sup_{u\in\mathbb{R}} V\big(u(t)\big) = \frac{r(t)}{(1+\alpha)^{1+\alpha}}\big|h_1(t,c_1)\big|^{1+\alpha}.$$

Thus, we obtain (10).

Multiplying both sides of (13) by $H_2(d_1,t)$ and using a similar analysis to the above, we can obtain (11). The proof is completed. □

Lemma 2.7 *Assume that the conditions of Lemma 2.2 hold. Let $x(t)$ be a negative solution of (1) and $u(t)$ be defined by (8). Then for any $(H_1,H_2)\in\mathscr{H}$ we have*

$$\begin{aligned}&\int_{[c_2,\delta_2]} \psi(t)H_1(t,c_2)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t \\ &\qquad + \int_{c_2}^{\delta_2} H_1(t,c_2)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\big|h_1(t,c_2)\big|^{1+\alpha}\right]\mathrm{d}t \\ &\quad \leq \sum_{i=k(c_2)+1}^{k(\delta_2)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha}H_1(\tau_i,c_2)u(\tau_i) - H_1(\delta_2,c_2)u(\delta_2) \end{aligned} \tag{16}$$

and

$$\int_{[\delta_2,d_2]} \psi(t)H_2(d_2,t)\frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)}\,\mathrm{d}t + \int_{\delta_2}^{d_2} H_2(d_2,t)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_2(d_2,t)\right|^{1+\alpha}\right]\mathrm{d}t \le \sum_{i=k(\delta_2)+1}^{k(d_2)} \frac{b_i^\alpha - a_i^\alpha}{a_i^\alpha} H_2(d_1,\tau_i)u(\tau_i) - H_2(d_2,\delta_2)u(\delta_2), \tag{17}$$

where $\psi(t)$ is defined as in Lemma 2.6.

The proof of Lemma 2.7 is similar to that of Lemma 2.6 and will be omitted.

For two constants $\nu_1,\nu_2 \notin \{\tau_k\}$ with $\nu_1 < \nu_2$ and $k(\nu_1) < k(\nu_2)$, using function $\varphi \in C([\nu_1,\nu_2],\mathbb{R})$ and function $\phi \in C_-([\nu_1,\nu_2],\mathbb{R})$, we define a functional $Q : C([\nu_1,\nu_2],\mathbb{R}) \to \mathbb{R}$ by

$$Q_{\nu_1}^{\nu_2}[\varphi] := \frac{\widetilde{r}(b_{k(\nu_1)+1}^\alpha - a_{k(\nu_1)+1}^\alpha)\varphi(\tau_{k(\nu_1)+1})}{a_{k(\nu_1)+1}^\alpha(\tau_{k(\nu_1)+1} - \nu_1)^\alpha} + \sum_{k=k(\nu_1)+2}^{k(\nu_2)} \frac{\widetilde{r}(b_k^\alpha - a_k^\alpha)\varphi(\tau_k)}{a_k^\alpha(\tau_k - \tau_{k-1})^\alpha}, \tag{18}$$

where $\sum_m^n = 0$ if $m > n$, and a functional $L : C_-([\nu_1,\nu_2],\mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned} L_{\nu_1}^{\nu_2}[\phi] := {} & \int_{\nu_1}^{\tau_{k(\nu_1)+1}} \phi(t)\frac{(t-\tau_{k(\nu_1)}-\sigma(t))^\alpha}{(t-\tau_{k(\nu_1)})^\alpha}\,\mathrm{d}t \\ & + \sum_{k=k(\nu_1)+1}^{k(\nu_2)-1}\left[\int_{\tau_k}^{t_k} \phi(t)\frac{(t-\tau_k)^\alpha}{b_k^\alpha(t-\tau_k+\sigma(t))^\alpha}\,\mathrm{d}t + \int_{t_k}^{\tau_{k+1}} \phi(t)\frac{(t-\tau_k-\sigma(t))^\alpha}{(t-\tau_k)^\alpha}\,\mathrm{d}t\right] \\ & + \int_{\tau_{k(\nu_2)}}^{t_{k(\nu_2)}} \phi(t)\frac{(t-\tau_{k(\nu_2)})^\alpha}{b_{k(\nu_2)}^\alpha(t-\tau_{k(\nu_2)}+\sigma(t))^\alpha}\,\mathrm{d}t \\ & + \int_{t_{k(\nu_2)}}^{\nu_2} \phi(t)\frac{(t-\tau_{k(\nu_2)}-\sigma(t))^\alpha}{(t-\tau_{k(\nu_2)})^\alpha}\,\mathrm{d}t, \end{aligned} \tag{19}$$

where t_k are zero points of $D_k(t)$ on $[\tau_k,\tau_{k+1}]$ for $k = k(\nu_1)+1,\dots,k(\nu_2)$.

For convenience in the expression below, we define, for $j = 1,2$,

$$\Pi_{c_j}^{\delta_j}\left[H_1(t,c_j)\right] := L_{c_j}^{\delta_j}\left[\psi(t)H_1(t,c_j)\right] + \int_{c_j}^{\delta_j} H_1(t,c_j)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_1(t,c_j)\right|^{1+\alpha}\right]\mathrm{d}t$$

and

$$\Pi_{\delta_j}^{d_j}\left[H_2(d_j,t)\right] := L_{\delta_j}^{d_j}\left[\psi(t)H_2(d_j,t)\right] + \int_{\delta_j}^{d_j} H_2(d_j,t)\left[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_2(d_j,t)\right|^{1+\alpha}\right]\mathrm{d}t,$$

where $\psi(t) = \eta_0^{-\eta_0}|f(t)|^{\eta_0}\prod_{i=1}^n \eta_i^{-\eta_i}(p_i(t))^{\eta_i}$.

Theorem 2.1 *Assume that, for any $T \geq t_0$, there exist $T < c_1 - \sigma_0 < c_1 < d_1 \leq c_2 - \sigma_0 < c_2 < d_2$ and (5) and (6) hold. If there exists a pair of $(H_1, H_2) \in \mathscr{H}$ such that*

$$\frac{\Pi_{c_j}^{\delta_j}[H_1(t,c_j)]}{H_1(\delta_j,c_j)} + \frac{\Pi_{\delta_j}^{d_j}[H_2(d_j,t)]}{H_2(d_j,\delta_j)} > \frac{Q_{c_j}^{\delta_j}[H_1(\cdot,c_j)]}{H_1(\delta_j,c_j)} + \frac{Q_{\delta_j}^{d_j}[H_2(d_j,\cdot)]}{H_2(d_j,\delta_j)}, \quad j = 1,2, \tag{20}$$

then (1) is oscillatory.

Proof Assume, to the contrary, that $x(t)$ is a nonoscillatory solution of (1). If $x(t)$ is a positive solution, we choose the interval $[c_1, d_1]$ to consider.

From Lemma 2.6, we obtain (10) and (11). Applying the estimation (a)–(f) into the left side, meanwhile (g)–(i) into the right side, of (10) and (11), we get

$$\Pi_{c_1}^{\delta_1}\bigl[H_1(t,c_1)\bigr] \leq Q_{c_1}^{\delta_1}\bigl[H_1(\cdot,c_1)\bigr] - H_1(\delta_1,c_1)u(\delta_1) \tag{21}$$

and

$$\Pi_{\delta_1}^{d_1}\bigl[H_2(d_1,t)\bigr] \leq Q_{\delta_1}^{d_1}\bigl[H_2(d_1,\cdot)\bigr] + H_2(d_1,\delta_1)u(\delta_1). \tag{22}$$

Dividing (21) and (22) by $H_1(\delta_1, c_1)$ and $H_2(d_1, \delta_1)$, respectively, and adding them, we get

$$\frac{\Pi_{c_1}^{\delta_1}[H_1(t,c_1)]}{H_1(\delta_1,c_1)} + \frac{\Pi_{\delta_1}^{d_1}[H_2(d_1,t)]}{H_2(d_1,\delta_1)} \leq \frac{Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)]}{H_1(\delta_1,c_1)} + \frac{Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]}{H_2(d_1,\delta_1)},$$

which contradicts (20) for $j = 1$.

If $x(t)$ is a negative solution of (1), we choose interval $[c_2, d_2]$ and can get a contradiction to (20) for $j = 2$. The details will be omitted.

The proof is complete. □

Remark 2.1 When $\sigma(t) = 0$, i.e., the delay disappears and $\alpha = 1$ in (1), Theorem 2.1 reduces to Theorem 2.2 in [1].

Remark 2.2 When $\sigma(t) = \sigma_0$, i.e., the delay is constant, Theorem 2.1 reduces to Theorem 2.8 in [2].

In Eq. (19), zero points t_k of $D_k(t)$ appear at upper limit (or lower limit) of integrals. However, these zero points are generally not easy to solve from $D_k(t) = 0$, which will lead to difficult in the calculation of (19). To overcome this difficulty we need to re-estimate $x(t-\sigma(t))/x(t)$ on $(t_k, \tau_{k+1}]$, (τ_k, t_k), $(t_{k(d_j)}, d_j)$ and $(\tau_{k(d_j)}, t_{k(d_j)})$ in Lemma 2.1 and Lemma 2.2.

If $x(t)$ is a positive solution of (1), from (a) in Lemma 2.1, we have, for $k = k(c_1) + 1, \ldots, k(d_1) - 1$, $k \neq k(\delta_1)$,

$$\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_k-\sigma(t)}{t-\tau_k} > \frac{t-\tau_k-\sigma(t)}{t}, \quad t \in (t_k, \tau_{k+1}]. \tag{23}$$

From (b) in Lemma 2.1, we have, for $k = k(c_1) + 1, \ldots, k(d_1)$,

$$\frac{x(t-\sigma(t))}{x(t)} > \frac{t-\tau_k}{b_k(t+\sigma(t)-\tau_k)} > 0 > \frac{t-\tau_k-\sigma(t)}{t}, \quad t \in (\tau_k, t_k]. \tag{24}$$

Combining (23) with (24), we obtain estimation of $x^\alpha(t-\sigma(t))/x^\alpha(t)$ on $(\tau_k,\tau_{k+1}]$ for $k = k(c_1)+1,\dots,k(d_1)-1$, $k\neq k(\delta_1)$,

$$(\bar{\text{a}})\quad \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \Phi_\alpha\left(\frac{t-\tau_k-\sigma(t)}{t}\right),\quad t\in(\tau_k,\tau_{k+1}].$$

Similarly, from (b) and (c) in Lemma 2.1, we have

$$(\bar{\text{b}})\quad \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \Phi_\alpha\left(\frac{t-\tau_{k(\delta_1)}-\sigma(t)}{t}\right),\quad t\in(\tau_{k(\delta_1)},t_{k(\delta_1)}]\cup(t_{k(\delta_1)},\delta_1],$$

from (b) and (d) in Lemma 2.1, we have

$$(\bar{\text{c}})\quad \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \Phi_\alpha\left(\frac{t-\tau_{k(d_1)}-\sigma(t)}{t}\right),\quad t\in(\tau_{k(d_1)},t_{k(d_1)}]\cup(t_{k(d_1)},d_1],$$

and from (e) and (f) in Lemma 2.1, we have

$$(\bar{\text{d}})\quad \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \Phi_\alpha\left(\frac{t-\tau_{k(\delta_1)}-\sigma(t)}{t-\tau_{k(\delta_1)}}\right),\quad t\in[\delta_1,\tau_{k(\delta_1)+1}],$$

and

$$(\bar{\text{e}})\quad \frac{x^\alpha(t-\sigma(t))}{x^\alpha(t)} > \Phi_\alpha\left(\frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}\right),\quad t\in[c_1,\tau_{k(c_1)+1}].$$

If $x(t)$ is a negative solution of (1), from Lemma 2.2, we can get similar estimations to the above for $t\in[c_2,d_2]$.

For convenience, we define functional $\widetilde{L}: C_-([c_j,d_j],\mathbb{R})\to\mathbb{R}$, for $j=1,2$, by

$$\begin{aligned}\widetilde{L}_{c_j}^{\delta_j}[\phi] :={}& \int_{c_j}^{\tau_{k(c_j)+1}}\phi(t)\Phi_\alpha\left(\frac{t-\tau_{k(c_j)}-\sigma(t)}{t-\tau_{k(c_j)}}\right)\mathrm{d}t \\ &+\sum_{k=k(c_j)+1}^{k(\delta_j)-1}\int_{\tau_k}^{\tau_{k+1}}\phi(t)\Phi_\alpha\left(\frac{t-\tau_k-\sigma(t)}{t}\right)\mathrm{d}t \\ &+\int_{\tau_{k(\delta_j)}}^{\delta_j}\phi(t)\Phi_\alpha\left(\frac{t-\tau_{k(\delta_j)}-\sigma(t)}{t}\right)\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\widetilde{L}_{\delta_j}^{d_j}[\phi] :={}& \int_{\delta_j}^{\tau_{k(\delta_j)+1}}\phi(t)\Phi_\alpha\left(\frac{t-\tau_{k(\delta_j)}-\sigma(t)}{t-\tau_{k(\delta_j)}}\right)\mathrm{d}t \\ &+\sum_{k=k(\delta_j)+1}^{k(d_j)-1}\int_{\tau_k}^{\tau_{k+1}}\phi(t)\Phi_\alpha\left(\frac{t-\tau_k-\sigma(t)}{t}\right)\mathrm{d}t \\ &+\int_{\tau_{k(\delta_j)}}^{d_j}\phi(t)\Phi_\alpha\left(\frac{t-\tau_{k(\delta_j)}-\sigma(t)}{t}\right)\mathrm{d}t.\end{aligned}$$

Further, we define, for $j = 1, 2$,

$$\widetilde{\Pi}_{c_j}^{\delta_j}\big[H_1(t,c_j)\big] := \widetilde{L}_{c_j}^{\delta_j}\big[\psi(t)H_1(t,c_j)\big] + \int_{c_j}^{\delta_j} H_1(t,c_j)\bigg[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\big|h_1(t,c_j)\big|^{1+\alpha}\bigg]\,\mathrm{d}t$$

and

$$\widetilde{\Pi}_{c_j}^{\delta_j}\big[H_2(d_j,t)\big] := \widetilde{L}_{c_j}^{\delta_j}\big[\psi(t)H_2(d_j,t)\big] + \int_{c_j}^{\delta_j} H_2(d_j,t)\bigg[p_0(t) - \frac{r(t)}{(1+\alpha)^{1+\alpha}}\big|h_2(d_j,t)\big|^{1+\alpha}\bigg]\,\mathrm{d}t,$$

where $\psi(t) = \eta_0^{-\eta_0}|f(t)|^{\eta_0}\prod_{i=1}^{n}\eta_i^{-\eta_i}(p_i(t))^{\eta_i}$.

Using similar proof method to that of Theorem 2.1 and applying estimations ($\bar{\text{a}}$)–($\bar{\text{e}}$), we can obtain following theorem.

Theorem 2.2 *Assume that, for any $T \geq t_0$, there exist $T < c_1 - \sigma_0 < c_1 < d_1 \leq c_2 - \sigma_0 < c_2 < d_2$ and (5) and (6) hold. If there exists a pair of $(H_1, H_2) \in \mathscr{H}$ such that*

$$\frac{\widetilde{\Pi}_{c_j}^{\delta_j}[H_1(t,c_j)]}{H_1(\delta_j,c_j)} + \frac{\widetilde{\Pi}_{\delta_j}^{d_j}[H_2(d_j,t)]}{H_2(d_j,\delta_j)} > \frac{Q_{c_j}^{\delta_j}[H_1(\cdot,c_j)]}{H_1(\delta_j,c_j)} + \frac{Q_{\delta_j}^{d_j}[H_2(d_j,\cdot)]}{H_2(d_j,\delta_j)}, \quad j = 1,2, \tag{25}$$

then (1) is oscillatory.

3 Example

In this section, we give an example to illustrate the effectiveness and non-emptiness of our results.

Example 3.1 Consider the following equation:

$$\begin{aligned} &x''(t) + \mu_1 p_1(t)\Phi_{\frac{5}{2}}\big(x\big(t-\sigma(t)\big)\big) + \mu_2 p_2(t)\Phi_{\frac{1}{2}}\big(x\big(t-\sigma(t)\big)\big) = f(t), \quad t \neq \tau_k, \\ &x\big(\tau_k^+\big) = a_k x(\tau_k), \qquad x'\big(\tau_k^+\big) = b_k x'(\tau_k), \quad k = 1,2,\ldots, \end{aligned} \tag{26}$$

where $\Phi_*(s) = |s|^{*-1}s$, $\sigma(t) = \frac{1}{3}\sin^2(\pi t)$, μ_1, μ_2 are positive constants and τ_k: $\tau_{n,1} = 8n + \frac{3}{2}$, $\tau_{n,2} = 8n + \frac{5}{2}$, $\tau_{n,3} = 8n + \frac{11}{2}$, $\tau_{n,4} = 8n + \frac{13}{2}$, $n \in \mathbb{N}$.

Let

$$p_1(t) = p_2(t) = \begin{cases} (t-8n), & t \in [8n, 8n+3], \\ 3, & t \in [8n+3, 8n+5], \\ (8n+8-t), & t \in [8n+5, 8n+8], \end{cases}$$

and

$$f(t) = \begin{cases} (t-8n)(t-8n-4)^3, & t \in [8n, 8n+4], \\ (t-8n-4)^3(8n+8-t), & t \in [8n+4, 8n+8]. \end{cases}$$

For any $t_0 > 0$, we choose n large enough such that $t_0 < 8n$ and let $[c_1, d_1] = [8n+1, 8n+3]$, $[c_2, d_2] = [8n+5, 8n+7]$, $\delta_1 = 8n+2$ and $\delta_2 = 8n+6$. We see that there has a zero point of

$D_k(t)$ on each interval of $[c_1,\delta_1]$, $[\delta_1,d_1]$, $[c_2,\delta_2]$ and $[\delta_2,d_1]$. By approximate calculation, we get $t_1 \approx 8n+1.709$, $t_2 \approx 8n+2.710$, $t_3 \approx 8n+5.709$ and $t_4 \approx 8n+6.710$. Moreover, from conditions $\alpha=1$, $\beta_1=5/2$ and $\beta_2=1/2$, we can choose $\eta_1=1/3$, $\eta_1=1/3$ and $\eta_0=1-\eta_1-\eta_2=1/3$. So, the conditions of Lemma 2.5 are satisfied.

Letting $H_1(t,s)=H_2(t,s)=(t-s)^2$ and $h_1(t,s)=-h_2(t,s)=\frac{2}{t-s}$. By simple calculation, we have, for $t\in[c_1,\delta_1]$,

$$\int_{c_1}^{\delta_1} H_1(t,c_1)\left[p_0(t)-\frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_1(t,c_1)\right|^{1+\alpha}\right]\mathrm{d}t = \int_{8n+1}^{8n+2}(t-8n-1)^2\left(0-\frac{2^2}{2^2(t-8n-1)^2}\right)\mathrm{d}t=-1.$$

Let

$$\begin{aligned}\phi_1(t) &:= \psi(t)H_1(t,c_1)=\eta_0^{-\eta_0}\left|f(t)\right|^{\eta_0}\prod_{i=1}^{2}\eta_i^{-\eta_i}\left(p_i(t)\right)^{\eta_i}(t-c_1)^2\\ &= 3\sqrt[3]{\mu_1\mu_2}(t-8n)(t-8n-4)(t-8n-1)^2.\end{aligned}$$

Then

$$\begin{aligned}L_{c_1}^{\delta_1}\left[\phi_1(t)\right] &= \int_{8n+1}^{8n+\frac{3}{2}}\phi_1(t)\frac{t-8n+\frac{3}{2}-\frac{1}{3}\sin^2(\pi t)}{t-8n+\frac{3}{2}}\,\mathrm{d}t\\
&\quad+\int_{8n+\frac{3}{2}}^{t_1}\phi_1(t)\frac{t-8n-\frac{3}{2}}{b_{n,1}(t-8n-\frac{3}{2}+\frac{1}{3}\sin^2(\pi t))}\,\mathrm{d}t\\
&\quad+\int_{t_1}^{8n+2}\phi_1(t)\frac{t-8n-\frac{3}{2}-\frac{1}{3}\sin^2(\pi t)}{t-8n-\frac{3}{2}}\,\mathrm{d}t\\
&= 3\sqrt[3]{\mu_1\mu_2}\left(\int_1^{\frac{3}{2}}\frac{t(4-t)(t-1)^2(t+\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t+\frac{3}{2}}\,\mathrm{d}t\right.\\
&\quad+\int_{\frac{3}{2}}^{1.709}\frac{t(4-t)(t-1)^2(t-\frac{3}{2})}{b_{n,1}(t-\frac{3}{2}+\frac{1}{3}\sin^2(\pi t))}\,\mathrm{d}t\\
&\quad\left.+\int_{1.709}^{2}\frac{t(4-t)(t-1)^2(t-\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{3}{2}}\,\mathrm{d}t\right)\\
&\approx 3\sqrt[3]{\mu_1\mu_2}\left(2.373+\frac{0.2551}{b_{n,1}}\right).\end{aligned}$$

Therefore,

$$\Pi_{c_1}^{\delta_1}\left[H_1(t,c_1)\right]=3\sqrt[3]{\mu_1\mu_2}\left(2.373+\frac{0.2551}{b_{n,1}}\right)-1.$$

Similarly, for $t\in[\delta_1,d_1]$, we have

$$\phi_2(t):=\psi(t)H_2(d_1,t)=3\sqrt[3]{\mu_1\mu_2}(t-8n)(8n+4-t)(8n+3-t)^2,$$

$$\int_{\delta_1}^{d_1}H_2(d_1,t)\left[p_0(t)-\frac{r(t)}{(1+\alpha)^{1+\alpha}}\left|h_2(d_1,t)\right|^{1+\alpha}\right]\mathrm{d}t=-1,$$

and

$$\begin{aligned} L_{\delta_1}^{d_1}\big[\phi_2(t)\big] &= 3\sqrt[3]{\mu_1\mu_2}\bigg(\int_2^{\frac{5}{2}} \frac{t(4-t)(3-t)^2(t-\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{3}{2}}\,\mathrm{d}t \\ &\quad + \int_{\frac{5}{2}}^{2.71} \frac{t(4-t)(3-t)^2(t-\frac{5}{2})}{b_{n,2}(t-\frac{5}{2}+\frac{1}{3}\sin^2(\pi t))}\,\mathrm{d}t \\ &\quad + \int_{2.71}^{3} \frac{t(4-t)(3-t)^2(t-\frac{5}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{5}{2}}\,\mathrm{d}t\bigg) \\ &\approx 3\sqrt[3]{\mu_1\mu_2}\bigg(2.964+\frac{0.078}{b_{n,2}}\bigg). \end{aligned}$$

Therefore,

$$\varPi_{\delta_1}^{d_1}\big[H_2(d_1,t)\big] = 3\sqrt[3]{\mu_1\mu_2}\bigg(2.964+\frac{0.078}{b_{n,2}}\bigg) - 1.$$

Since

$$H_1(\delta_1,c_1) = (\delta_1-c_1)^2 = 1, \qquad H_2(d_1,\delta_1) = (d_1-\delta_1)^2 = 1,$$

the left-hand side of inequality (20) is

$$\frac{\varPi_{c_1}^{\delta_1}[H_1(t,c_1)]}{H_1(\delta_1,c_1)} + \frac{\varPi_{\delta_1}^{d_1}[H_2(d_1,t)]}{H_2(d_1,\delta_1)} \approx 3\sqrt[3]{\mu_1\mu_2}\bigg(5.337+\frac{0.255}{b_{n,1}}+\frac{0.078}{b_{n,2}}\bigg) - 2.$$

Because $\widetilde{r}_1 = \widetilde{r}_2 = 1$, $\tau_{k(c_1)+1} = \tau_{k(\delta_1)} = \tau_{n,1} = 8n + \frac{3}{2} \in (c_1,\delta_1)$ and $\tau_{k(\delta_1)+1} = \tau_{k(d_1)} = \tau_{n,2} = 8n + \frac{5}{2} \in (\delta_1, d_1)$, it is easy to see that the right-hand side of inequality (20) for $j = 1$ is

$$\frac{Q_{c_1}^{\delta_1}[H_1(\cdot,c_1)]}{H_1(\delta_1,c_1)} + \frac{Q_{\delta_1}^{d_1}[H_2(d_1,\cdot)]}{H_2(d_1,\delta_1)} = \frac{b_{n,1}-a_{n,1}}{4a_{n,1}} + \frac{b_{n,2}-a_{n,2}}{4a_{n,2}}.$$

Thus (20) is satisfied with $j = 1$ if

$$3\sqrt[3]{\mu_1\mu_2}\bigg(5.337+\frac{0.255}{b_{n,1}}+\frac{0.078}{b_{n,2}}\bigg) > 2 + \frac{b_{n,1}-a_{n,1}}{4a_{n,1}} + \frac{b_{n,2}-a_{n,2}}{4a_{n,2}}.$$

When $j = 2$, with the same argument as above we see that the left-hand side of inequality (20) is

$$\begin{aligned} &\frac{\varPi_{c_2}^{\delta_2}[H_1(t,c_2)]}{H_1(\delta_2,c_2)} + \frac{\varPi_{\delta_2}^{d_2}[H_2(d_2,t)]}{H_2(d_2,\delta_2)} \\ &= 3\sqrt[3]{\mu_1\mu_2}\bigg(\int_1^{\frac{3}{2}} \frac{t(4-t)(t-1)^2(t+\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t+\frac{3}{2}}\,\mathrm{d}t \\ &\quad + \int_{\frac{3}{2}}^{1.709} \frac{t(4-t)(t-1)^2(t-\frac{3}{2})}{b_{n,3}(t-\frac{3}{2}+\frac{1}{3}\sin^2(\pi t))}\,\mathrm{d}t \\ &\quad + \int_{1.709}^{2} \frac{t(4-t)(t-1)^2(t-\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{3}{2}}\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\int_{2}^{\frac{5}{2}} \frac{t(4-t)(3-t)^2(t-\frac{3}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{3}{2}}\,\mathrm{d}t \\
&+\int_{\frac{5}{2}}^{2.71} \frac{t(4-t)(3-t)^2(t-\frac{5}{2})}{b_{n,4}(t-\frac{5}{2}+\frac{1}{3}\sin^2(\pi t))}\,\mathrm{d}t \\
&+\int_{2.71}^{3} \frac{t(4-t)(3-t)^2(t-\frac{5}{2}-\frac{1}{3}\sin^2(\pi t))}{t-\frac{5}{2}}\,\mathrm{d}t\biggr)-2 \\
\approx{}& 3\sqrt[3]{\mu_1\mu_2}\biggl(5.337+\frac{0.255}{b_{n,3}}+\frac{0.078}{b_{n,4}}\biggr)-2,
\end{aligned}
$$

and the right-hand side of inequality (20) is

$$
\frac{Q_{c_2}^{\delta_2}[H_1(\cdot,c_2)]}{H_2(\delta_2,c_2)}+\frac{Q_{\delta_2}^{d_2}[H_2(d_2,\cdot)]}{H_2(d_2,\delta_2)}=\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}.
$$

Therefore, (20) is satisfied for $j = 2$, if

$$
3\sqrt[3]{\mu_1\mu_2}\biggl(5.337+\frac{0.255}{b_{n,1}}+\frac{0.078}{b_{n,2}}\biggr)>2+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}.
$$

Hence, by Theorem 2.1, Eq. (26) is oscillatory, if

$$
\begin{cases}
3\sqrt[3]{\mu_1\mu_2}(5.337+\frac{0.255}{b_{n,1}}+\frac{0.078}{b_{n,2}})>2+\frac{b_{n,1}-a_{n,1}}{4a_{n,1}}+\frac{b_{n,2}-a_{n,2}}{4a_{n,2}}, \\
3\sqrt[3]{\mu_1\mu_2}(5.337+\frac{0.255}{b_{n,1}}+\frac{0.078}{b_{n,2}})>2+\frac{b_{n,3}-a_{n,3}}{4a_{n,3}}+\frac{b_{n,4}-a_{n,4}}{4a_{n,4}}.
\end{cases}
\tag{27}
$$

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Authors' contributions
Each of the authors, XZ, CL and RC contributed to each part of this study equally and read and approved the final version of the manuscript.

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