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A HODIE finite difference scheme for pricing American options

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Abstract

In this paper, we introduce a new numerical method for pricing American-style options, which has long been considered as a very challenging problem in financial engineering. Based on the HODIE (high order via differential identity expansion) finite difference scheme, we discretize the spatial variable on a piecewise uniform mesh, and meanwhile, use the implicit Euler method to discretize the time variable. Under such a discretization, we show that the resulting matrix is an M-matrix, which ensures the stability of the current scheme in the maximum-norm sense. By applying the discrete maximum principle, an error estimate of the current scheme is theoretically obtained first and then tested numerically. It is shown that our method is first order and second order convergent in the time and spatial directions, respectively. The results of various numerical experiments show that this new approach is quite accurate, and can be easily extended to price other kinds of American-style options.

MSC: 65M06; 65M12; 65M15

Keywords: American options; Linear complementarity problem; HODIE scheme; Implicit Euler scheme; Convergence analysis

1 Introduction

An American-style option is an option that can be exercised at any time during its lifespan. In financial engineering, the pricing of this kind of options has long been acknowledged as a very challenging problem [33]. The challenge mainly comes from the early exercise nature of the American options. Mathematically, this early exercise feature gives the pricing problem its free boundary form; in turn this free boundary problem may be recast as an LCP (linear complementarity problem). Nowadays, it is important to ensure that American-style options can be priced accurately as well as efficiently due to the popularity of such options in today's financial markets.

Zhu [48] has made a breakthrough by successfully deriving a closed-form pricing formula for American options. His formula is not, however, computationally as well as practically appealing, as pointed out in [51]. Till now, approximation methods, especially various numerical approaches, are still popular among market practitioners because most of them are usually faster with acceptable accuracy.

In the literature, of all the approximation methods, there are predominantly two types, analytical approximations and numerical methods, for the valuation of American options. Typical methods in the first category include the compound-option approximation method [20], the randomization approach [4], the integral equation method [9, 52], and the Laplace transform method [49]. However, as pointed out in [43], the generalization of these quasi-analytical approaches to some exotic options may not be easy. Moreover, it seems quite difficult to extend all these methods to high-dimensional problems. On the other hand, the numerical methods for the valuation of American options typically include the FD (finite difference) method [15, 29, 37, 38, 43, 53], the FE (finite element) method [2, 45], the IFE (inverse finite element) method [51], the radius basis function method [25, 36], the binomial tree method [17], the moving boundary approach [10, 35], the Monte Carlo simulation technique [19], and the least square approach [33]. We remark that the FD method is common but elegant in solving either linear or nonlinear heat equations. Examples include [23, 24, 27, 34, 44] and the references therein.

It is worth mentioning that, from a purely numerical point of view, there are two difficulties associated with the pricing of American options. Firstly, the Black–Scholes differential operator at zero underlying asset price is degenerative. A common and widely used approach dealing with such a singularity is to apply the Euler transformation to remove the singularity [16, 39, 40, 47]. As a result of the transformation, the pricing domain becomes $(-\infty, +\infty)$. However, the truncation on the left-hand side of the domain to artificially remove the degeneracy may cause additional computational errors. Furthermore, the uniform mesh on the transformed interval will lead to the originally grid points concentrating around $x = 0$. While the standard finite difference method is used to discretize the Black–Scholes differential operator in the original form, numerical difficulty can be caused. The main reason is that when the volatility or the asset price is small, the Black–Scholes differential operator becomes a convection-dominated operator. Hence, the implicit Euler scheme with central spatial difference method may lead to nonphysical oscillations in the computed solution [5, 6]. The implicit Euler scheme with upwind spatial difference method does not have this disadvantage, but this difference scheme is only first order convergent. Recently, a stable fitted finite volume method [3, 41] has been employed for the discretization of the Black–Scholes differential operator, but it is also first order convergent. The second difficulty of the pricing of American options is that the payoff functions of vanilla options are not smooth at the strike price. This would result in the solution of the LCP governing the price of American options being not smooth enough and, consequently, difficult to be determined accurately [5]. Although the second difficulty will also be faced in the numerical pricing of European options, it is admittedly exacerbated in the American case due to the need to find the free boundary using "smooth pasting" condition.

In this paper we propose a HODIE (high order via differential identity expansion) finite difference scheme (see [12–14] for details) to solve the linear complementarity problem of the pricing of American puts. We remark that in the literature, the HODIE method is restricted to solve PDE (partial differential equation) formulated problems only; the application of this method to solve the linear complementarity problem, as presented in the current paper, is achieved for the first time. In particular, in our work, we concentrate on overcoming the two difficulties mentioned to further improve the accuracy. To avoid the computational error brought in by the truncation around $x = 0$ (corresponding to $-\infty$ in the transformed domain), we deal with the Black–Scholes equation in the original form, but with the HODIE finite difference approximations on a piecewise uniform mesh for the spatial direction. Fortunately, with the HODIE finite difference approximations on a piecewise uniform mesh, we show theoretically that the coefficient matrix after discretization is an M-matrix. This ensures that the scheme is maximum-norm stable for arbitrary parameter settings. Most remarkably, we have successfully obtained a sharp error estimate of the current scheme by applying the maximum principle to the discrete linear complementarity problem in two mesh sets. It is proved that the scheme is first and second order convergent with respect to the time and spatial variables, respectively. Numerical results agree with the theoretical statement and indicate that our method is more accurate than the other existing methods.

On the other hand, to deal with the non-smoothness of the payoff function, we use the singularity-separating method developed by Zhu et al. [54, 55] for solving shock waves in fluid mechanics. The application of this method to the quantitative finance area can be found in [5]. According to the essence of this approach, we first compute the difference between the American option and its corresponding European counterpart, because the terminal value of such a difference is zero, which is smooth. The final option price could then be obtained by adding the corresponding European option price to the difference. Financially, this approach reveals the fact that the price of an American option could be decomposed into the price of its European counterpart and the early exercise premium.

The rest of the paper is organized as follows. In Sect. 2, we describe some theoretical results on the linear complementarity formulation of American put options. In Sect. 3, we introduce the HODIE method in detail including its stability and error analysis. In Sect. 4, some numerical experiments are provided to test the theoretical error estimation obtained in the previous section. Concluding remarks are given in the last section.

2 The continuous problem

To demonstrate the HODIE method in a clear way, we shall consider the pricing of American puts under the classical Black–Scholes model throughout the paper. In fact, the extension of the current method to price other American-style securities is quite straightforward.

It is well known that the pricing of American puts can be formulated as a linear complementarity problem [26, 42]. Let $v(x,t)$ denote the value of an American put option on the underlying price x at any time t before the expiry date T with strike price E. It can be shown that v satisfies the following linear complementarity problem:

$$Lv(x,t) \geq 0, \quad x > 0 \text{ and } t \in [0,T), \tag{2.1}$$

$$v(x,t) - g(x) \geq 0, \quad x > 0 \text{ and } t \in [0,T], \tag{2.2}$$

$$Lv(x,t) \cdot \bigl(v(x,t) - g(x)\bigr) = 0, \quad x > 0 \text{ and } t \in [0,T), \tag{2.3}$$

$$v(x,T) = g(x), \quad x \geq 0, \tag{2.4}$$

$$v(0,t) = E, \quad t \in [0,T], \tag{2.5}$$

$$v(x,t) \to 0, \quad x \to +\infty \text{ and } t \in [0,T], \tag{2.6}$$

where L denotes the Black–Scholes differential operator defined as

$$Lv(x,t) \equiv -\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} - (r-d)x\frac{\partial v}{\partial x} + rv,$$

with σ being the volatility of the underlying, r and d being the risk-free interest rate and dividend yield, respectively.

We further remark that $g(x)$ appearing in (2.2) is the payoff function defined as

$$g(x) = \max(E - x, 0).$$

For simplicity, we assume that $r - d > 0$.

One should notice that the final condition $g(x) = \max(E - x, 0)$ is not smooth, and hence $v(x,t)$ should not be very smooth in the region where x near E and t near T. This would result in large truncation error in the numerical solution near $x = E$ and $t = T$. Fortunately, such a singularity can be ingeniously dealt with by the singularity-separating method introduced in [54, 55], as will be illustrated in the following.

According to the essence of the so-called singularity-separating method, we split the price of an American put option $v(x,t)$ into two parts, i.e.,

$$v(x,t) = v_1(x,t) + v_2(x,t).$$

Here, $v_1(x,t)$ is the price of the European put, which has a closed-form representation as

$$v_1(x,t) = Ee^{-r(T-t)}N(-d_2) - xe^{-d(T-t)}N(-d_1), \tag{2.7}$$

where

$$d_1 = \frac{\ln\frac{x}{E} + (r - d + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{s^2}{2}}\,\mathrm{d}s.$$

Since $v_1(x,t)$ has the same terminal value as $v(x,t)$, it is clear that $v_2(x,t)$ satisfies the following linear complementarity problem:

$$Lv_2(x,t) \geq 0, \quad x > 0 \text{ and } t \in [0,T), \tag{2.8}$$

$$v_2(x,t) - \left[g(x) - v_1(x,t)\right] \geq 0, \quad x > 0 \text{ and } t \in [0,T], \tag{2.9}$$

$$Lv_2(x,t) \cdot \left[v_2(x,t) - \left(g(x) - v_1(x,t)\right)\right] = 0, \quad x > 0 \text{ and } t \in [0,T), \tag{2.10}$$

$$v_2(x,T) = 0, \quad x \geq 0, \tag{2.11}$$

$$v_2(0,t) = E\left(1 - e^{-r(T-t)}\right), \quad t \in [0,T], \tag{2.12}$$

$$v_2(x,t) \to 0, \quad x \to +\infty \text{ and } t \in [0,T]. \tag{2.13}$$

We remark that such a splitting indeed coincides with the financial fact that the price of an American option can be written as the sum of its European counterpart and the early exercise premium. Most importantly, after splitting v into two parts, the terminal value of $v_2(x,t)$ becomes zero, which is a smooth function in the entire solution domain.

Now, to solve the linear complementarity problem (2.8)–(2.13) on a computer, we need to truncate the domain $(0,+\infty)$ into $(0,X)$. Based on Wilmott et al.'s estimate [42] that the upper bound of the asset price is typically three or four times of the strike price, it is reasonable to set $X = 4E$. Remarkably, this truncation of the domain only leads to a negligible error in the computed option price [30]. The boundary condition at $x = X$ is now artificially set to $v_2(X,t) = 0$. Therefore, in the remainder of this paper, we shall concentrate on solving the following linear complementarity problem numerically:

$$Lv_2(x,t) \geq 0, \quad (x,t) \in (0,X) \times (0,T), \tag{2.14}$$

$$v_2(x,t) - \bigl[g(x) - v_1(x,t)\bigr] \geq 0, \quad (x,t) \in (0,X) \times [0,T], \tag{2.15}$$

$$Lv_2(x,t) \cdot \bigl[v_2(x,t) - \bigl(g(x) - v_1(x,t)\bigr)\bigr] = 0, \quad (x,t) \in (0,X) \times [0,T], \tag{2.16}$$

$$v_2(x,T) = 0, \quad x \in [0,X], \tag{2.17}$$

$$v_2(0,t) = E\bigl(1 - e^{-r(T-t)}\bigr), \quad v_2(X,t) = 0, t \in [0,T]. \tag{2.18}$$

3 The HODIE method

In this section, the HODIE method used to price American puts will be illustrated in great detail. Although this method has been used to solve many problems recently, it is the first time that the method is successfully applied to solve problems in a linear complementarity formulation. This section is further organized into two subsections, according to two important issues to be addressed. In the first subsection, the discretization of the HODIE method will be introduced, whereas in the second subsection, the stability and the error estimate of the current method will be analyzed.

3.1 Discretization

Our numerical method is based on a HODIE finite difference scheme for the spatial discretization and the implicit Euler scheme for the temporal discretization.

Similar to [6], to ensure the stability of the discrete scheme, we use a piecewise uniform mesh Ω^N on the spatial interval $[0,X]$:

$$x_i = h\left[1 + \frac{\sigma^2}{r-d}(i-1)\right], \quad i = 1,\dots,N, \tag{3.1}$$

where $h = \frac{X}{1+\frac{\sigma^2}{r-d}(N-1)}$.

For the temporal discretization, we use a uniform mesh Ω^K on $[0,T]$ with K mesh elements. Then the piecewise uniform mesh $\Omega^{N\times K}$ on $\Omega = (0,X) \times (0,T)$ is defined as the tensor product $\Omega^{N\times K} = \Omega^N \times \Omega^K$. We remark that the quasi-uniform mesh is used to ensure that the current scheme is maximum-norm stable for arbitrary parameter settings.

Now, applying the implicit Euler scheme with uniform mesh size $\Delta t = t_j - t_{j-1} = \frac{T}{K}$ to the time direction, (2.14)–(2.18) could be discretized into the following semi-linear complementarity problem:

$$\tilde{L}v_2^j(x) \geq v_2^{j+1}(x), \quad x \in (0,X), \tag{3.2}$$

$$v_2^j(x) - \left[g(x) - v_1^j(x)\right] \geq 0, \quad x \in (0, X), \tag{3.3}$$

$$\left(\tilde{L}v_2^j(x) - v_2^{j+1}(x)\right) \cdot \left[v_2^j(x) - \left(g(x) - v_1^j(x)\right)\right] = 0, \quad x \in (0, X), \tag{3.4}$$

$$v_2^K(x) = 0, \quad x \in [0, X], \tag{3.5}$$

$$v_2^j(0) = E\left(1 - e^{-r(T-t_j)}\right), \quad v_2^j(X) = 0, \tag{3.6}$$

where $v_2^j(x)$ is the solution of (2.14)–(2.18) at the jth time level, and $\tilde{L}$ is the operator defined as

$$\tilde{L}v_2^j(x) = v_2^j(x) + \Delta t\left[-\frac{1}{2}\sigma^2 x^2 \frac{d^2 v_2^j}{dx^2} - (r-d)x\frac{dv_2^j}{dx} + rv_2^j(x)\right].$$

To approximate the solution $v_2^j(x)$ of the semi-discrete problem (3.2)–(3.6), we use the HODIE finite difference scheme, which will be illustrated in detail in the following.

The first step in applying the HODIE technique is to propose a discrete scheme

$$\tilde{L}^N U_i \equiv a_i^- U_{i-1} + a_i^c U_i + a_i^+ U_{i+1} - q_i^1 \tilde{L}u_{i-1} - q_i^2 \tilde{L}u_i \tag{3.7}$$

for the differential operator $\tilde{L}$.

We now assume that the space $P_2[x]$ is in the kernel of the operator $\tilde{L}$; moreover, the normalization conditions

$$q_i^1 + q_i^2 = 1, \qquad q_i^j \geq 0, \quad j = 1, 2, \tag{3.8}$$

are satisfied. Based on the above two assumptions, the following linear system can be obtained after we transform the interval $[x_{i-1}, x_{i+1}]$ into $[-h_i, h_{i+1}]$:

$$\begin{aligned}
&a_i^- + a_i^c + a_i^+ = 1 + r\Delta t, \\
&-h_i a_i^- + h_{i+1}a_i^+ = q_i^1\left\{\left[-(r-d)x_{i-1} - h_i r\right]\Delta t - h_i\right\} - q_i^2(r-d)x_i\Delta t, \\
&h_i^2 a_i^- + h_{i+1}^2 a_i^+ = q_i^1\left\{\left[-\sigma^2 x_{i-1}^2 + 2h_i(r-d)x_{i-1} + h_i^2 r\right]\Delta t + h_i^2\right\} - q_i^2\sigma^2 x_i^2\Delta t, \\
&q_i^2 = 1 - q_i^1.
\end{aligned}$$

Solving the above linear system, we obtain

$$\begin{aligned}
a_i^- = {}& \frac{q_i^1[-\sigma^2 x_{i-1}^2 + (r-d)h_i x_{i-1}]\Delta t - q_i^2[\sigma^2 x_i^2 + (r-d)h_i x_i]\Delta t}{(h_i + h_{i+1})h_i} \\
&+ \frac{q_i^1\{[(r-d)x_{i-1} + h_i r]\Delta t + h_i\} + q_i^2(r-d)x_i\Delta t}{h_i},
\end{aligned} \tag{3.9}$$

$$a_i^+ = \frac{q_i^1[-\sigma^2 x_{i-1}^2 + (r-d)h_i x_{i-1}]\Delta t - q_i^2[\sigma^2 x_i^2 + (r-d)h_i x_i]\Delta t}{(h_i + h_{i+1})h_{i+1}}, \tag{3.10}$$

$$a_i^c = 1 + r\Delta t - a_i^- - a_i^+. \tag{3.11}$$

Now, with all the (a_i)s available, it suffices for us to summarize the finite difference equation written on a grid point (i, j) as

$$L^{N,K} V_{2,i}^j \geq 0, \quad (i, j) \in \tilde{\Omega}_h, \tag{3.12}$$

$$V_{2,i}^j - [g_i - v_{1,i}^j] \geq 0, \quad (i,j) \in \tilde{\Omega}_h, \tag{3.13}$$

$$L^{N,K} V_{2,i}^j \cdot [V_{2,i}^j - (g_i - v_{1,i}^j)] = 0, \quad (i,j) \in \tilde{\Omega}_h, \tag{3.14}$$

$$V_{2,i}^K = 0, \quad i = 0, 1, \ldots, N, \tag{3.15}$$

$$V_{2,0}^j = E(1 - e^{-r(T-t_j)}), \qquad V_{2,N}^j = 0, \quad j = K-1, \ldots, 1, 0, \tag{3.16}$$

where

$$L^{N,K} V_{2,i}^j \equiv a_i^- V_{2,i-1}^j + a_i^c V_{2,i}^j + a_i^+ V_{2,i+1}^j - q_i^1 V_{2,i-1}^{j+1} - q_i^2 V_{2,i}^{j+1}, \tag{3.17}$$

and

$$\tilde{\Omega}_h = \{(i,j) | 1 \leq i \leq N-1, 0 \leq j \leq K-1\}.$$

We remark that in (3.12)–(3.16), the parameter q_i^1 provides a degree of freedom to ensure that the current scheme is stable. In the next subsection, we will show that the discrete operator $L^{N,K}$ satisfies a discrete maximum principle when $q_i^1 = 0$. Hence, there exists a unique solution V_2 satisfying (3.12)–(3.16). For a detailed discussion on the uniqueness of the solution of the discrete linear complementarity problem, we refer to [22].

To solve numerically problem (3.12)–(3.16), the projection scheme introduced in [21, p. 433] can be adopted. Assuming that the function value V_2 at the $(j+1)$th time step is already known, and we set $L^{N,K} V_{2,i}^{j+1/2} = 0$, which is then solved together with the boundary condition (3.16), i.e., we solve for $\{V_2^{j+1/2}\}$ from

$$L^{N,K} V_{2,i}^{j+1/2} = 0, \quad i = 1, 2, \ldots, N-1,$$

$$V_{2,0}^{j+1/2} = E(1 - e^{-r(T-t_j)}), \qquad V_{2,N}^{j+1/2} = 0.$$

With $\{V_2^{j+1/2}\}$ available, we set

$$V_{2,i}^j = \max\{V_{2,i}^{j+1/2}, g_i - v_{1,i}^j\}, \quad i = 0, 1, \ldots, N,$$

which ensures that (3.13) and (3.14) are satisfied.

After V_2^j is solved, the desired option price at (x_i, t_j) can be obtained by

$$v(x_i, t_j) = V_{2,i}^j + v_1(x_i, t_j),$$

where $v_1(x_i, t_j)$ is the corresponding European put price and can be determined directly from (2.7).

3.2 Analysis of the method

As an error analysis forms an indispensable part of any numerical approach, the proposed HODIE approach needs an error estimation as well. Remarkably, in the literature, such an issue has never been formally addressed when the HODIE method is applied to solve linear complementarity problems. In the following, based on the discrete maximum principle, truncation error analysis, and the barrier function technique, we have managed to provide an error estimation for the HODIE method applied to solve the price of American puts.

Lemma 1 (Discrete maximum principle) *The operator $L^{N,K}$ defined in (3.17) on the piecewise uniform mesh $\Omega^{N\times K}$ satisfies a discrete maximum principle when $q_i^1 = 0$, i.e., if u_i^j is a mesh function that satisfies $u_0^j \geq 0, u_N^j \geq 0$ $(0 \leq j < K)$, $u_i^K \geq 0$ $(0 \leq i \leq N)$, and $L^{N,K}u_i^j \geq 0$ for $(i,j) \in \tilde{\Omega}_h$, then $u_i^j \geq 0$ for all i,j.*

Proof Let $q_i^1 = 0$ in (3.9) be zero, we obtain

$$\begin{aligned} a_i^- &= \frac{-[\sigma^2 x_i^2 + (r-d)h_i x_i]\Delta t}{(h_i + h_{i+1})h_i} + \frac{(r-d)x_i\Delta t}{h_i} \\ &= \frac{[-\sigma^2 x_i^2 + (r-d)h_{i+1}x_i]\Delta t}{(h_i + h_{i+1})h_i} \leq \frac{[-\sigma^2 x_1 + (r-d)h_{i+1}]x_i\Delta t}{(h_i + h_{i+1})h_i} \\ &\leq \frac{[-\sigma^2 h + (r-d)\frac{\sigma^2}{r-d}h]x_i\Delta t}{(h_i + h_{i+1})h_i} = 0 \end{aligned}$$

for $(i,j) \in \tilde{\Omega}_h$. It is also not difficult to show that, for $(i,j) \in \tilde{\Omega}_h$,

$$\begin{aligned} a_i^+ &= \frac{-[\sigma^2 x_i^2 + (r-d)h_i x_i]\Delta t}{(h_i + h_{i+1})h_{i+1}} < 0, \\ a_i^c &= 1 + r\Delta t - a_i^- - a_i^+ > 0, \end{aligned}$$

and

$$a_i^- + a_i^c + a_i^+ - 1 = r\Delta t > 0.$$

Therefore, the matrix associated with $L^{N,K}$ is an M-matrix.[a] By applying the same argument as Lemma 3.1 in [31] used, it is straightforward to obtain the result of our lemma. □

It should be remarked that from the above lemma, one can also conclude that our scheme is stable because the associated matrix is a strictly diagonally dominant M-matrix. Next, by using the Taylor expansion, the following truncation error estimate can be obtained.

Lemma 2 *Let $u(x,t)$ be a smooth function defined on $\Omega^{N\times K}$. Then, if $q_i^1 = 0$, the following estimate for the truncation error holds true*:

$$\left|L^{N,K}u(x_i,t_j) - \Delta t(Lu)(x_i,t_j)\right| \leq C\Delta t\left(h^2 + \Delta t\right) \quad \text{for } (i,j) \in \tilde{\Omega}_h,$$

where C is a positive constant independent of the mesh.

Based on Lemmas 1 and 2, our main result for the HODIE finite difference scheme can be achieved as follows.

Theorem 1 *Let $v_2(x,t)$ be the solution of problem (2.14)–(2.18) and $V_{2,i}^j$ be the solution of problem (3.12)–(3.16). Then, if $q_i^1 = 0$, we have the following error estimates*:

$$\left|v_2(x_i,t_j) - V_{2,i}^j\right| \leq C\left(h^2 + \Delta t\right)$$

for $0 \leq i \leq N$ and $0 \leq j \leq K$, where C is a positive constant independent of the mesh.

Proof We apply the maximum principle to the discrete linear complementarity problem (3.12)–(3.16) in two mesh sets to derive the error estimate [8].

Now, let

$$\bar{\Omega}_h = \bigl\{(i,j) | 0 \le i \le N, 0 \le j \le K\bigr\}, \quad \partial\Omega_h = \bar{\Omega}_h \setminus \tilde{\Omega}_h,$$
$$\Omega^{(1)} = \bigl\{(i,j) \in \tilde{\Omega}_h | v_2(x_i,t_j) = g_i - v_{1,i}^j\bigr\}, \quad \Omega^{(2)} = \tilde{\Omega}_h \setminus \Omega^{(1)}.$$

From (2.14)–(2.18), it is clear that

$$Lv_2(x_i,t_j) \ge 0, \quad (i,j) \in \Omega^{(1)},$$
$$Lv_2(x_i,t_j) = 0, \quad (i,j) \in \Omega^{(2)}.$$

Denote

$$\Omega_h^{(1)} = \bigl\{(i,j) \in \tilde{\Omega}_h | V_{2,i}^j = g_i - v_{1,i}^j\bigr\}, \quad \Omega_h^{(2)} = \tilde{\Omega}_h \setminus \Omega_h^{(1)}.$$

It is obvious that

$$L^{N,K} V_{2,i}^j = 0, \quad (i,j) \in \Omega_h^{(2)}. \tag{3.18}$$

Define the function on $\tilde{\Omega}_h$ by

$$W_i^j = C\bigl[(T - t_j + 1)\Delta t + (X - x_i)h^2\bigr] > 0, \tag{3.19}$$

where C is a sufficiently large constant independent of the mesh.

For $(i,j) \in \Omega_h^{(2)}$, using the fact that $Lv_2(x_i,t_j) \ge 0$, together with (3.18), (3.19), and Lemma 2, we obtain

$$\begin{aligned} L^{N,K}\bigl(v_2(x_i,t_j) - V_{2,i}^j + W_i^j\bigr) &= L^{N,K} v_2(x_i,t_j) + L^{N,K} W_i^j \\ &= \bigl[L^{N,K} v_2(x_i,t_j) - \Delta t L v_2(x_i,t_j) + L^{N,K} W_i^j\bigr] \\ &\quad + \Delta t L v_2(x_i,t_j) \ge 0. \end{aligned}$$

For the nodes on the 'boundary' of $\Omega_h^{(2)}$, if $(i,j) \in \Omega_h^{(1)}$, we have

$$v_2(x_i,t_j) - V_{2,i}^j + W_i^j = v_2(x_i,t_j) - \bigl(g_i - v_{1,i}^j\bigr) + W_i^j \ge 0,$$

and if $(i,j) \in \partial\Omega_h$, we obtain

$$v_2(x_i,t_j) - V_{2,i}^j + W_i^j = W_i^j \ge 0.$$

Applying the maximum principle to $\Omega_h^{(2)}$, it is clear at this stage that

$$v_2(x_i,t_j) - V_{2,i}^j + W_i^j \ge 0, \quad (i,j) \in \Omega_h^{(2)},$$

and thus

$$v_2(x_i,t_j) - V_{2,i}^j + W_i^j \geq 0, \quad (i,j) \in \bar{\Omega}_h. \tag{3.20}$$

On the other hand, for $(i,j) \in \Omega^{(2)}$, it is known that $Lv_2(x_i,t_j) = 0$, but $L^{N,K}V_{2,i}^j \geq 0$. Therefore, we have

$$\begin{aligned} L^{N,K}\bigl(v_2(x_i,t_j) - V_{2,i}^j - W_i^j\bigr) &= \bigl[L^{N,K}v_2(x_i,t_j) - \Delta t L v_2(x_i,t_j) - L^{N,K}W_i^j\bigr] - L^{N,K}V_{2,i}^j \\ &\leq 0. \end{aligned}$$

On the 'boundary' of $\Omega^{(2)}$, if the nodes $(i,j) \in \Omega^{(1)}$, we have

$$v_2(x_i,t_j) - V_{2,i}^j - W_i^j = \bigl(g_i - v_{1,i}^j\bigr) - V_{2,i}^j - W_i^j \leq 0,$$

and if $(i,j) \in \partial\Omega_h$, we obtain

$$v_2(x_i,t_j) - V_{2,i}^j - W_i^j = -W_i^j \leq 0.$$

Now, applying the maximum principle to $\Omega^{(2)}$, we obtain

$$v_2(x_i,t_j) - V_{2,i}^j - W_i^j \leq 0, \quad (i,j) \in \Omega^{(2)}.$$

Thus

$$v_2(x_i,t_j) - V_{2,i}^j - W_i^j \leq 0, \quad (i,j) \in \bar{\Omega}_h. \tag{3.21}$$

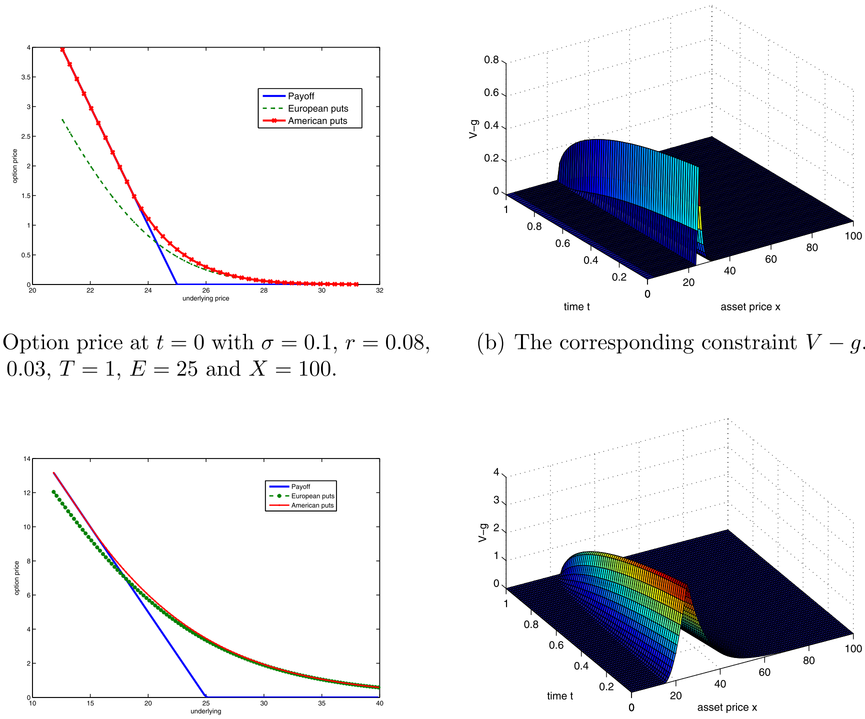
From (3.20) and (3.21), we obtain

$$\max_{(i,j)\in\bar{\Omega}_h} \bigl|v_2(x_i,t_j) - V_{2,i}^j\bigr| \leq \max_{(i,j)\in\bar{\Omega}_h} W_i^j \leq C\bigl(h^2 + \Delta t\bigr).$$

This completes the proof. □

It should be remarked that although the current scheme is second order accurate in space, much more accurate solutions may be obtained by the idea of the HODIE discretization. Based on the adapted mesh used in [14], a third order HODIE scheme could be constructed, based on the current framework, to solve for the price of American-style options accurately. Also, as pointed out in [11], it is quite possible to combine the current method with some other techniques, such as the defect-correction or Richardson extrapolation, to increase the order of convergence in the time direction. These possibilities will be further explored in a forthcoming paper.

It is also worth mentioning that although the HODIE method is currently used to price American puts under the Black–Scholes model, this method may be extended to price other kinds of American-style options, for example, American-style options in multi-dimensions such as options with stochastic volatility [50] or on multi-assets [46], American-style options under a modified Black–Scholes equation with fractional derivatives [7, 28].



(a) Option price at $t = 0$ with $\sigma = 0.1$, $r = 0.08$, $d = 0.03$, $T = 1$, $E = 25$ and $X = 100$.

(b) The corresponding constraint $V - g$.

(c) Option price at $t = 0$ with $\sigma = 0.4$, $r = 0.06$, $d = 0.02$, $T = 1$, $E = 25$ and $X = 100$.

(d) The corresponding constraint $V - g$

Figure 1 Option prices with different parameter settings

4 Numerical experiments

In this section, we shall present some numerical results as well as useful discussions on the performance and convergence of the HODIE finite difference scheme. In particular, we shall concentrate on calculating the errors and the convergence rate of the current scheme through the following examples.

4.1 Computed option price and optimal exercise prices

Depicted in Fig. 1(a) and Fig. 1(c) are two sets of American put prices as a function of the underlying but with different parameter settings. From these figures, it is clear that the option price is a decreasing function of the underlying. Moreover, the "smooth pasting" conditions across the free boundary, which are usually difficult to implement numerically, are also satisfied well. One could observe from these two figures that the American option price dominates its European counterpart. This makes sense, as the right of being able to exercise earlier, in comparison to the European-style options, has added additional values to the option price. Besides the option prices, the differences between the option price, and the payoff, i.e., $V - g$, are further shown in Fig. 1(b) and Fig. 1(d) for the two cases. From these two figures, it is observed that the option prices computed using our method are always greater than or equal to the payoff value at any time to maturity, which is one of the essential properties of American options.

As pointed out in [30], the truncation of the domain only leads to a negligible error in the computed option price. To corroborate this, we have compared the option prices

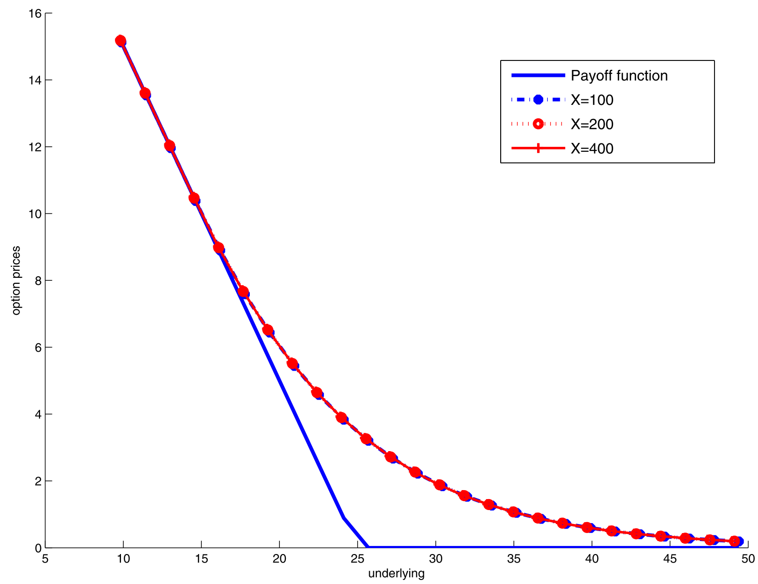


Figure 2 Impacts of the truncation of the domain. Model parameters are $\sigma = 0.3$, $r = 0.1$, $D = 0$, $T - t = 1$(year), $E = \$25$

computed with otherwise identical but different X, as displayed in Fig. 2. One could clearly observe from this figure that the option prices with different choices of X agree perfectly well with each other. Numerical calculation further shows that the maximum pointwise error among them is in the order of 10^{-4}, which is indeed negligible from a financial point of view.

It is known that the optimal exercise price is far more difficult to determine accurately than the option price. From our algorithm described in Sect. 3.1, it is clear that the optimal exercise price can be obtained as the maximum underlying at which $V^j_{2,i} = g_i - v^j_{1,i}$ is satisfied. Based on this fact, we calculate the optimal exercise prices with $N = 2048$ and $K = 4096$, and compare the numerical values with those calculated by the IFE method and Zhu's analytical approach [48], as shown in Fig. 3. From this figure, one can clearly observe that our numerical results agree well with those calculated using other methods. A close examination reveals that the current approach slightly underestimates the S_f values when the time is close to expiry. This is not surprising because of the well-known singularity at expiry [18], which is not possible for most of the numerical schemes to deal with.

Another important issue in finance industry is the speed of calculation, which is equally important, sometimes even more, than the accuracy. To clearly demonstrate the overall numerical performance of the current approach, we display in Fig. 4 the relationship between the efficiency and accuracy of our scheme. Here, the computational efficiency is measured by the total CPU time in seconds consumed for each run, whereas the accuracy is measured by the relative error over the whole computational domain, which is defined as

$$\text{Relative error} = \frac{\| V - V_{\text{exact}} \|_\infty}{\| V_{\text{exact}} \|_\infty},$$

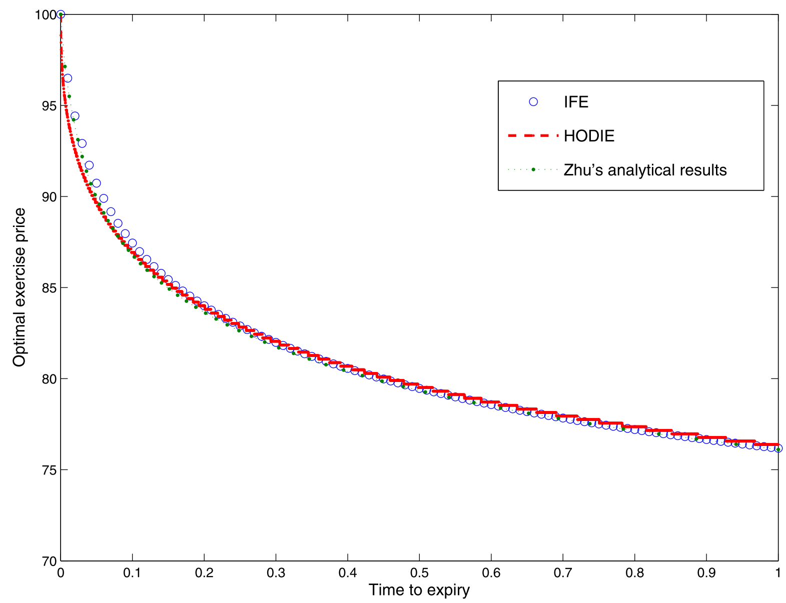


Figure 3 Comparison of the optimal exercise price. Model parameters are $\sigma = 0.3$, $r = 0.1$, $D = 0$, $T - t = 1$(year), $E = \$100$, $X = 400$

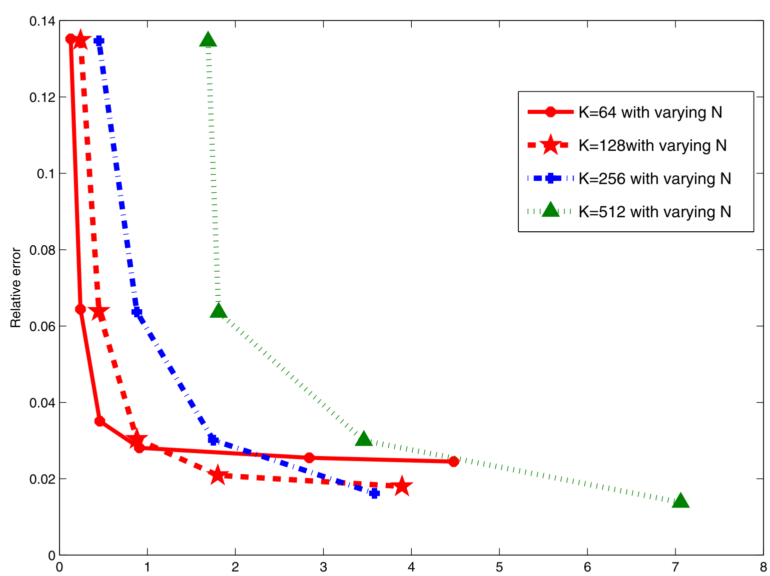


Figure 4 CPU VS Accuracy. Model parameters are $\sigma = 0.4$, $r = 0.06$, $D = 0.02$, $T - t = 1$(year), $E = \$25$, $X = 100$

with V and V_{exact} denoting the computed values and the reference values, respectively, and $\|\cdot\|_\infty$ being the infinite norm. We remark that the approximate solution calculated on very fine grid sizes, i.e., with $N = 4096$ and $K = 8192$, is used as the reference solution here. In this figure, four sets of relative errors with fixed number of time steps are displayed as a function of the varying number of spatial steps and CPU times. All the experiments were performed within Matlab2011b on an Intel(R) Core(TM)i7-2600 CPU, 3.40 GHz machine.

Table 1 Error estimate and convergence rate for the spatial direction with the same parameters as used in Fig. 1(c)

K	N	error	rate
4096	128	4.0555e–3	–
	256	1.0697e–3	1.923
	512	2.8216e–4	1.923
	1024	8.7300e–5	1.692

From Fig. 4, one can clearly observe that the efficiency of the current scheme is inversely varying with the accuracy. In other words, a demand in high accuracy in the computed option price requires more computational time. However, by using our method, it is possible to produce a result with the relative error being less than 5% within 1 second. This level of accuracy and efficiency certainly suits the practical needs of financial market [51].

4.2 Convergence rate

As demonstrated in Sect. 3.2, our method is first and second order convergent in the time and spatial directions, respectively. To further verify this theoretical result, we turn to investigating the error estimate and convergence rate of the current method from a numerical point of view. Since we do not have the exact solution of American options in hand, we shall adopt the approximate solution calculated on very fine grid sizes, i.e., with $N = 2048$ and $K = 4096$, as the exact solution. For comparison purpose, we use linear interpolation to get values at desired points, because "the exact solution" here is only known on mesh points.

To obtain the convergence rate along the spatial direction, we fix the size of the time step to be fairly small, i.e., $K = 4096$, and vary the number of spatial intervals from 128 to 1024. The errors reported in Table 1 are measured by the discrete maximum norm defined as

$$e^{N,K} = \max_{i,j} \left| V_{i,j}^{N,K} - \bar{V}(x_i, t_j) \right|.$$

With the errors associated with different N available, the convergence rate along the spatial direction is then calculated from

$$R^{N,K} = \log_2 \left(\frac{e^{N,K}}{e^{2N,K}} \right).$$

From Table 1, one can clearly observe that $\frac{e^{N,K}}{e^{2N,K}}$ is close to 4 for sufficiently large K, which indicates that our method is second order convergent in the spatial direction. This agrees with our theoretical convergence result stated in Theorem 1.

Similarly, when we fix the spatial step size to $N = 2048$ and increase the grid number in the time direction, we find that the rate approaches 1, as shown in Table 2. This indicates that a first order convergence is achieved in the time direction.

To better investigate the convergence rate of the current scheme, we also calculate the ratio with the time and spatial steps adjusted to each other according to the expected order of error $O(h^2 + \Delta t)$. Specifically, we choose varying grid numbers $N_i = n_i N_1$ and $K_i = n_i^2 K_1$, where N_i and K_i are the grid numbers used in the ith row of Table 3 with $N_1 = 32$ and $K_1 = 16$. It is anticipated that if the theoretic order of convergence is achieved, the rate should approximately equal 2. This has been confirmed by the results shown in Table 3 as well.

Table 2 Error estimate and convergence rate for the time direction with the same parameters as used in Fig. 1(c)

K	N	error	rate
2048	64	2.551e–1	–
	128	1.208e–2	1.078
	216	5.471e–2	1.143
	512	2.5300e–2	1.112

Table 3 Global error estimate and convergence rate with the same parameters as used in Fig. 1(c)

n	error	rate
1	4.24e–2	–
2	1.33e–2	1.67
4	4.20e–3	1.66

Table 4 Comparison of the American put option values at different underlying values with parameters: $\sigma = 0.3$, $t = 0$, $T = 1$, $r = 0.04$, $d = 0.02$, $E = 100$, $X = 400$

Stock price x	Binomial method	Analytic approx. method	Compact method 1	Compact method 2	Compact method 3	Our stable scheme	True values
75.9572	25.33949	25.4509	25.10042	25.32570	25.32739	25.32939	25.32986
83.9457	19.49101	19.6617	19.34597	19.49193	19.49383	19.49647	19.49691
92.7743	14.27957	14.4477	14.16375	14.25707	14.25914	14.26231	14.26265
102.5315	9.87092	10.0278	9.78167	9.83789	9.84000	9.84332	9.84354
113.3148	6.35580	6.53401	6.32881	6.36044	6.36241	6.36555	6.36558
125.2323	3.84473	3.97728	3.81244	3.82898	3.83064	3.83327	3.83337
138.4031	2.14801	2.25467	2.12653	2.13452	2.13578	2.13775	2.13784

We also compare our scheme with binomial method, analytic approximation method proposed in [20], and compact finite difference methods proposed in [47], as shown in Table 4. The values in this table are computed with the following parameter settings. The binomial method is based on time step $\Delta t = 0.01$, the analytical approximation method is based on time step $\Delta t = 0.02$, whereas compact finite difference methods 1, 2, and 3 are based on space step $h = 0.02$ and time step $\Delta t = 0.0005$. In our stable scheme, we use time step $\Delta t = 0.0005$ and $N = 20{,}000$ mesh points for spatial discretization which has almost the same number of mesh points as the compact methods. The true option values are based on the trinomial method proposed in [1] using time step $\Delta t = 0.00005$. From Table 4, one can observe that our solution agrees perfectly well with the true option price, which indicates that our scheme is more accurate than other methods.

5 Conclusion

In this paper, we have proposed and tested a HODIE finite difference method for the pricing of American put options. Based on the HODIE finite difference discretization with a piecewise uniform mesh, we have proved theoretically that the resulting matrix is an M-matrix, and consequently, we obtain the stability of the current method. Furthermore, with the discrete maximum principle, a theoretical error estimate for the current scheme is also obtained, which is further verified numerically through our numerical experiments. With the application of the HODIE method to the option pricing field for the first time, it is promising that higher order methods could be constructed, based on the current method, to price American-style options in multi-dimensions accurately. Moreover, the HODIE method also may be extended to price other kinds of American-style options, for example, American-style options in multi-dimensions, American-style options under a modified Black–Scholes equation with fractional derivatives.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors have equal contributions. While the first author carried out the literature review, designed the numerical algorithm, and conducted numerical experiments, the second author participated in designing the algorithm and doing numerical experiments. The second author wrote the draft. All authors read and approved the final manuscript.

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Endnote
[a] An M-matrix is a Z-matrix with eigenvalues whose real parts are nonnegative. Here, the Z-matrix is a matrix with off-diagonal entries less than or equal to zero.

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