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Global behavior of positive solutions for some semipositone fourth-order problems

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Abstract

In this paper, we study the global behavior of positive solutions of fourth-order boundary value problems

$$\begin{cases} u'''' = \lambda f(x,u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function with $f(x,0) < 0$ in $(0,1)$, and $\lambda > 0$. The proof of our main results are based upon bifurcation techniques.

MSC: Positive solutions; Topological degree; Connected set; Bifurcation

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1 Introduction

In this paper, we study the global behavior of positive solutions of fourth-order boundary value problems

$$\begin{cases} u'''' = \lambda f(x,u), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$. If $f(x,0) \geq 0$, then (1.1) is called a *positone problem*.

On the contrary, here we deal with the so-called *semipositone problem* when f is such that

(f_1) $f(x,0) < 0$ $\forall x \in (0,1)$.

The existence of positive solutions of second-order positone problems have been extensively studied via the Leray–Schauder degree theory, fixed point theorem on a cone, and the method of lower and upper solutions; see [1–3] and the references therein.

Ambrosetti [4] studied the existence of positive solutions for semipositone elliptic problems via bifurcation theory. Recently, Hai and Shivaji [5] obtained the existence of positive solutions for second-order semipositone problems

$$\begin{cases} -u'' = \lambda h(t)f(u), & t \in (0,1), \\ u(0) = 0, & u'(1) + c(u(1))u(1) = 0 \end{cases}$$

via a Krasnosel'skii fixed-point-type theorem in a Banach space.

The existence and multiplicity of positive solutions of fourth-order positone problems have been studied by several authors; see [6–11] along this line. However, there are few results for fourth-order semipositone problems; see [12]. Ma [12] used the fixed point theorem in cones to show that the problem

$$\begin{cases} u'''' = \lambda \tilde{f}(x, u(x), u'(x)), & t \in (0,1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0 \end{cases}$$

has a positive solution if $\lambda > 0$ is small enough, where $\tilde{f}(x, u, p) \geq -M$ for some positive constant M, and

$$\lim_{p\to\infty} \frac{f(x,u,p)}{p} = \infty.$$

There is a big difference in the study of fourth- and second-order problems. For example:

1. Spectrum theory for singular second-order linear eigenvalue problems has been established via Prüfer transform in [13]. However, the spectrum structure of singular fourth-order linear eigenvalue problems is not established so far.
2. The uniqueness of solutions of second-order problems

$$\begin{cases} -u'' = \lambda u^q, & x \in (a,b), \\ u > 0, & x \in (a,b), \\ u(a) = u(b) = 0 \end{cases}$$

   has been obtained in [14]. However, the uniqueness of solution of

$$\begin{cases} w'''' = b|w|^\alpha, & x \in (0,1), \\ w(0) = w(1) = w''(0) = w''(1) = 0 \end{cases}$$

   is not obtained so far.
3. It is well known that, for a second-order differential equation with periodic, Neumann, or Dirichlet boundary conditions, the existence of a well-ordered pair of lower and upper solutions $\alpha \leq \beta$ is sufficient to ensure the existence of a solution in the sector enclosed by them. However, this result it is not true for fourth-order differential equations; see Remark 3.1 in [15].

Motivated by Ambrosetti [4], we investigate the global behavior of positive solutions of the fourth-order boundary value problem (1.1). Depending on the behavior of $f = f(x,s)$ as $s \to +\infty$, we handle both asymptotically linear, superlinear, and sublinear problems. All results are obtained by showing that there exists a global branch of solutions of (1.1) "emanating from infinity" and proving that for λ near the bifurcation value, solutions of large norms are indeed positive to which bifurcation theory or topological methods apply in a classical fashion. Since there are a lot of differences between second- and fourth-order cases, we have to overcome several new difficulties in the proof of our main results.

We deal in Sect. 2 with asymptotically linear problems. In Sect. 3, we discuss superlinear problems, and we show that (1.1) possesses positive solutions for $0 < \lambda < \lambda^*$. Similar arguments can be used in the sublinear case, discussed in Sect. 4, to show that (1.1) has positive solutions provided that λ is large enough.

2 Asymptotically linear problems

For Lebesgue spaces, we use standard notation. We work in $X = C[0,1]$. The usual norm in such spaces is denoted by $\|u\| = \max_{t\in[0,1]} |u(t)|$, and we set $B_r = \{u \in X : \|u\| \le r\}$. The first eigenvalue of u'''' with boundary conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ is denoted by λ_1; ϕ_1 is the corresponding eigenfunction such that $\phi_1 > 0$ in $(0,1)$. We also set $\mathbb{R}^+ = [0,\infty)$.

We define $K : X \to X$ by

$$Ku(t) := \int_0^1 \int_0^1 G(t,s)G(s,\tau)f\bigl(\tau,u(\tau)\bigr)\,d\tau\,ds$$

and

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

We write $u = Kv$ if

$$\begin{cases} u'''' = v, & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

With this notation, problem (1.1) is equivalent to

$$u - \lambda Kf(u) = 0, \quad u \in X. \tag{2.1}$$

Hereafter we will use the same symbol to denote both the function and the associated Nemitski operator.

We say that λ_∞ is a bifurcation from infinity for (2.1) if there exist $\mu_n \to \lambda_\infty$ and $u_n \in X$ such that $u_n - \mu_n Kf(u_n) = 0$ and $\|u_n\| \to \infty$.

In some situations, like the specific ones we will discuss later, an appropriate rescaling allows us to find bifurcation from infinity by means of the Leray–Schauder topological degree, denoted by $\deg(\cdot,\cdot,\cdot)$. Recall that $K : X \to X$ is (continuous and) compact, and hence it makes sense to consider the topological degree of $I - \lambda Kf$, where I is the identity map.

We suppose that $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$ satisfies (f_1) and

(f_2) there is $m > 0$ such that

$$\lim_{u\to+\infty} \frac{f(x,u)}{u} = m.$$

Let $\lambda_\infty = \frac{\lambda_1}{m}$ and define

$$a(x) = \liminf_{u\to+\infty}\bigl(f(x,u) - mu\bigr), \qquad A(x) = \limsup_{u\to+\infty}\bigl(f(x,u) - mu\bigr).$$

Theorem 2.1 *Suppose that f satisfies (f_1) and (f_2). Then there exists $\epsilon > 0$ such that (1.1) has positive solutions, provided that either*

(i) $a > 0$ (*possibly* $+\infty$) *in* $[0,1]$, *and* $\lambda \in (\lambda_\infty - \epsilon, \lambda_\infty)$; *or*
(ii) $A < 0$ (*possibly* $-\infty$) *in* $[0,1]$, *and* $\lambda \in (\lambda_\infty, \lambda_\infty + \epsilon)$.

The proof of Theorem 2.1 will be carried out in several steps. First of all, we extend $f(x,\cdot)$ to the whole $\mathbb{R}$ by setting

$$F(x,u) = f\big(x,|u|\big).$$

For $u \in X$,

$$\Phi(\lambda,u) := u - \lambda KF(u).$$

Clearly, any $u > 0$ such that $\Phi(\lambda,u) = 0$ is a positive solution of (1.1).

Lemma 2.1 *For every compact interval $\Lambda \subset \mathbb{R}^+ \backslash \{\lambda_\infty\}$, there exists $r > 0$ such that*

$$\Phi(\lambda,u) \neq 0 \quad \forall \|u\| \geq r.$$

Moreover,
(i) *if $a > 0$, then we can also take $\Lambda = [\lambda_\infty, \lambda]$ for all $\lambda > \lambda_\infty$, and*
(ii) *if $A < 0$, then we can also take $\Lambda = [0, \lambda_\infty]$.*

Proof Suppose on the contrary that there exists a sequence $\{(\mu_n,u_n)\}$ satisfying

$$\mu_n \in \Lambda; \qquad \|u_n\| \geq n \quad \text{for } n \in \mathbb{N}; \qquad u_n = \mu_n KF(u_n).$$

Obviously, $\|u_n\| \geq n$ implies that $u_n(x) \not\equiv 0$. We may assume that $\mu_n \to \mu$ for some $\mu \neq \lambda_\infty$.
Setting $w_n = u_n \|u_n\|^{-1}$, we find

$$w_n = \mu_n \|u_n\|^{-1} KF(u_n).$$

Since w_n is bounded in X, after taking a subsequence if necessary, we have that $w_n \to w$ in X, where w is such that $\|w\| = 1$ and satisfies

$$\begin{cases} w'''' = \mu m|w|, & x \in (0,1), \\ w(0) = w(1) = w''(0) = w''(1). \end{cases}$$

By the maximum principle it follows that $w \geq 0$. Since $\|w\| = 1$, we infer that $\mu m = \lambda_1$, namely $\mu = \lambda_\infty$, a contradiction that proves the first statement.

We will give a short sketch of (i). Taking $\mu_n \downarrow \lambda_\infty$, it follows that $w \geq 0$ satisfies

$$\begin{cases} w'''' = \lambda_1 w, & x \in (0,1), \\ w(0) = w(1) = w''(0) = w''(1) = 0, \end{cases} \tag{2.2}$$

and hence there exists $\beta > 0$ such that $w = \beta\phi_1$. Then we have $u_n = \|u_n\| w_n \to +\infty$ and $F(u_n) = f(u_n)$ for n large.

From $\Phi(\lambda_n, u_n) = 0$ it follows that

$$\lambda_1 \int_0^1 u_n\phi_1\,dx = \mu_n \int_0^1 \bigl(f(u_n) - mu_n\bigr)\phi_1\,dx + \mu_n m \int_0^1 u_n\phi_1\,dx. \tag{2.3}$$

Since $\mu_n > \lambda_\infty$ and $\int_0^1 u_n\phi_1\,dx > 0$ for n large, we infer that $\int_0^1 (f(u_n) - mu_n)\phi_1\,dx < 0$ for n large, and the Fatou lemma yields

$$\begin{aligned} 0 &\geq \liminf \int_0^1 \bigl(f(u_n) - mu_n\bigr)\phi_1\,dx \\ &\geq \int_0^1 a\phi_1\,dx, \end{aligned}$$

a contradiction if $a > 0$.

We prove statement (ii) similarly to (i). Taking $\mu_n \uparrow \lambda_\infty$, it follows that $w \geq 0$ satisfies (2.2), and hence there exists $\beta > 0$ such that $w = \beta\phi_1$. Then we have $u_n = \|u_n\|w_n \to +\infty$ and $F(u_n) = f(u_n)$ for n large.

From $\Phi(\lambda_n, u_n) = 0$ we have (2.3); since $\mu_n < \lambda_\infty$ and $\int_0^1 u_n\phi_1\,dx > 0$ for n large, we infer that $\int_0^1 (f(u_n) - mu_n)\phi_1\,dx > 0$ for n large, and the Fatou lemma yields

$$0 \leq \liminf \int_0^1 \bigl(f(u_n) - mu_n\bigr)\phi_1\,dx \leq \int_0^1 A\phi_1\,dx,$$

a contradiction if $A < 0$. □

Lemma 2.2 *If* $\lambda > \lambda_\infty$, *then there exists* $r > 0$ *such that*

$$\Phi(\lambda, u) \neq t\phi_1 \quad \forall t \geq 0, \|u\| \geq r.$$

Proof Taking into account that $F(x, u) \simeq m|u|$ as $|u| \to \infty$, we can repeat the arguments of Lemma 3.3 of [16] with some minor changes. □

For $u \neq 0$, we set $z = u\|u\|^{-2}$. Letting

$$\Psi(\lambda, z) = \|u\|^2\Phi(\lambda, u) = z - \lambda\|z\|^2 KF\left(\frac{z}{\|z\|^2}\right),$$

we have that λ_∞ is a bifurcation from infinity for (2.1) if and only if it is a bifurcation from the trivial solution $z = 0$ for $\Psi = 0$. From Lemma 2.1 by homotopy it follows that

$$\begin{aligned} \deg\bigl(\Psi(\lambda, \cdot), B_{1/r}, 0\bigr) &= \deg\bigl(\Psi(0, \cdot), B_{1/r}, 0\bigr) \\ &= \deg(I, B_{1/r}, 0) = 1 \quad \forall \lambda < \lambda_\infty. \end{aligned} \tag{2.4}$$

Similarly, by Lemma 2.2 we infer that, for all $\tau \in [0, 1]$ and $\lambda > \lambda_\infty$,

$$\begin{aligned} \deg\bigl(\Psi(\lambda, \cdot), B_{1/r}, 0\bigr) &= \deg\bigl(\Psi(\lambda, \cdot) - \tau\phi_1, B_{1/r}, 0\bigr) \\ &= \deg\bigl(\Psi(\lambda, \cdot) - \phi_1, B_{1/r}, 0\bigr) = 0 \quad \forall \lambda < \lambda_\infty. \end{aligned} \tag{2.5}$$

Let us set

$$\Sigma = \{(\lambda, u) \in \mathbb{R}^+ \times X : u \neq 0, \Phi(\lambda, u) = 0\}.$$

From (2.4) and (2.5) and the preceding discussion we deduce the following:

Lemma 2.3 *λ_∞ is a bifurcation from infinity for (2.1). More precisely, there exists an unbounded closed connected set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity. Moreover, Σ_∞ bifurcates to the left (to the right), provided that $a > 0$ (respectively, $A < 0$).*

Proof of Theorem 2.1 By the previous lemmas it suffices to show that if $\mu_n \to \lambda_\infty$ and $\|u_n\| \to \infty$, then $u_n > 0$ in $[0, 1]$ for n large. Setting

$$w_n = u_n \|u_n\|^{-1}$$

and using the preceding arguments, we find that, up to subsequence, $w_n \to w$ in X and $w = \beta\phi_1$, $\beta > 0$. Then it follows that

$$u_n > 0$$

in $(0, 1)$ for n large. □

Example 2.1 Let us consider the fourth-order semipositone boundary value problem

$$\begin{cases} x''''(t) = \lambda f(t, x), & t \in (0, 1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{2.6}$$

where $\lambda > 0$ and $f(t, x) = 10x + t\ln(1 + x) - t$.

Obviously,

$$f(t, 0) < 0, \quad t \in (0, 1);$$
$$\lim_{x\to\infty} \frac{f(t, x)}{x} = 10 =: m;$$
$$a(t) = \liminf_{x\to+\infty} \bigl(f(t, x) - mx\bigr) = \liminf_{x\to+\infty} \bigl(t\ln(1 + x) - t\bigr) > 0, \quad t \in (0, 1).$$

Notice that $\lambda_1 = \pi^4$ and $\lambda_\infty = \frac{\pi^4}{10}$. Thus by Theorem 2.1 there exists $\epsilon > 0$ such that (2.6) has positive solutions, provided that $\lambda \in (\lambda_\infty - \epsilon, \lambda_\infty)$. Moreover, Lemma 2.3 guarantees that there exists an unbounded closed connected set of positive solutions $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity and bifurcates to the left of λ_∞.

3 Superlinear problems

We study the existence of positive solutions of problem (1.1) when $f(x, \cdot)$ is superlinear. Precisely, we suppose that $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$ satisfies (f_1) and

- (f_3) there is $b \in C([0, 1]), b > 0$, such that $\lim_{u\to\infty} u^{-p} f(x, u) = b$ uniformly in $x \in [0, 1]$ with $p > 1$.

Lemma 3.1 ([6]) *Let X be a Banach space, and let $\Omega \subset X$ be a cone in X. For $p > 0$, define $\Omega_p = \{x \in \Omega \mid |x| < p\}$. Assume that $F : \Omega_p \to \Omega$ is completely continuous such that*

$$Fx \neq x, \quad x \in \partial \Omega_p = \{x \in \Omega \mid |x| = p\}.$$

(1) *If $\|Fx\| \leq \|x\|$ for $x \in \partial\Omega_p$, then $i(F, \Omega_p, \Omega) = 1$.*
(2) *If $\|Fx\| \geq \|x\|$ for $x \in \partial\Omega_p$, then $i(F, \Omega_p, \Omega) = 0$.*

Our main result is the following:

Theorem 3.1 *Let $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$ satisfy (f_1) and (f_3). Then there exists $\lambda_* > 0$ such that (1.1) has positive solutions for all $0 < \lambda \leq \lambda_*$. More precisely, there exists a connected set of positive solutions of (1.1) bifurcating from infinity at $\lambda_\infty = 0$.*

Proof As before, we set

$$F(x,u) = f(x, |u|)$$

and let

$$G(x,u) = F(x,u) - b|u|^p.$$

For the remainder of the proof, we omit the dependence with respect to $x \in [0,1]$.

To prove that $\lambda_\infty = 0$ is a bifurcation from infinity for

$$u - \lambda KF(u) = 0, \tag{3.1}$$

we use the rescaling $w = \gamma u, \lambda = \gamma^{p-1}, \gamma > 0$. A direct calculation shows that (λ, u), $\lambda > 0$, is a solution of (3.1) if and only if

$$w - K\tilde{F}(\gamma, w) = 0, \tag{3.2}$$

where

$$\tilde{F}(\gamma, w) := b|w|^p + \gamma^p G(\gamma^{-1} w). \tag{3.3}$$

We can extend $\tilde{F}$ to $\gamma = 0$ by setting

$$\tilde{F}(0, w) = b|w|^p,$$

and by (f_3) such an extension is continuous. We set

$$S(\gamma, w) = w - K\tilde{F}(\gamma, w), \quad \gamma \in \mathbb{R}^+.$$

Let us point out explicitly that $S(\gamma,\cdot) = I - K$ with compact K. For $\gamma = 0$, solutions of $S_0(w) := S(0,w) = 0$ are nothing but solutions of

$$\begin{cases} w'''' = b|w|^p, & x \in (0,1), \\ w(0) = w(1) = w''(0) = w''(1) = 0. \end{cases} \tag{3.4}$$

We claim that there exist two constants $R > r > 0$ such that

$$S_0(w) \neq 0 \quad \forall \|w\| \geq R, \tag{3.5}$$

$$S_0(w) \neq 0 \quad \forall \|w\| \leq r. \tag{3.6}$$

Assume on the contrary that (3.5) is not true. Then there exists a sequence $\{w_n\}$ of solutions of (3.4) satisfying

$$\|w_n\| \to \infty, n \to \infty. \tag{3.7}$$

In fact, we have from (3.4) that

$$\begin{cases} w_n'''' = (b|w_n|^{p-1})w_n, & x \in (0,1), \\ w_n(0) = w_n(1) = w_n''(0) = w_n''(1) = 0, \end{cases}$$

since

$$\lim_{n\to\infty} \left(b|w_n|^{p-1}\right) = \infty \quad \text{uniformly in } x \in [1/4, 3/4],$$

which means that w_n must change its sign in $[1/4, 3/4]$. However, this is a contradiction. Therefore (3.5) is valid.

Assume on the contrary that (3.6) is not true. Then there exists a sequence w_n of solutions of (3.4) satisfying

$$\|w_n\| > 0 \quad \forall n \in \mathbb{N}; \qquad \|w_n\| \to 0, \quad n \to \infty. \tag{3.8}$$

Let $v_n := w_n/\|w_n\|$. From (3.4) we have

$$\begin{cases} v_n'''' = (b|w_n|^{p-1})v_n, & x \in (0,1), \\ v_n(0) = v_n(1) = v_n''(0) = v_n'' = 0. \end{cases} \tag{3.9}$$

By the standard argument, after taking a subsequence and relabeling if necessary, it follows that

$$\lim_{n\to\infty} \left(b|w_n|^{p-1}\right) = 0 \quad \text{uniformly in } x \in [0,1],$$

and there exists $v_* \in X$ with $\|v_*\| = 1$ such that

$$v_n \to v_*, \quad n \to \infty,$$

and

$$\begin{cases} v_*'''' = 0, \quad x \in (0,1), \\ v_*(0) = v_*(1) = v_*''(0) = v_*''(1) = 0, \end{cases}$$

which implies that $v_* = 0$. However, this is a contradiction, Therefore (3.6) is valid.

Now, from (3.5) and (3.6), we deduce

$$S_0(w) \neq 0 \quad \forall w \in \partial\Omega_R, \qquad S_0(w) \neq 0 \quad \forall w \in \partial\Omega_r.$$

This implies

$$S_0(w) \neq 0 \quad \forall w \in \partial(\bar{\Omega}_R \setminus \Omega_r).$$

Thus the degree $\deg(S_0, \Omega_R \setminus \Omega_r, 0)$ is well defined.

Next, we show that

$$\deg(S_0, \Omega_R \setminus \bar{\Omega}_r, 0) = -1.$$

To this end, let us define

$$\Omega = \{u \in C[0,1] : u(t) \geq 0 \text{ for } t \in [0,1]\}$$

and

$$\Omega_\rho = \{u \in \Omega : \|u(t)\| < \rho\}.$$

Using Lemma 3.1 and an argument similar to that in the proof of [6], Theorem 3, we deduce

$$\mathrm{i}(K\tilde{F}(0,\cdot), \Omega_r, \Omega) = 1, \qquad \mathrm{i}(K\tilde{F}(0,\cdot), \Omega_R, \Omega) = 0. \tag{3.10}$$

By the excision and the additivity properties of the degree it follows that

$$\mathrm{i}(K\tilde{F}(0,\cdot), \Omega_R \setminus \bar{\Omega}_r, \Omega) + \mathrm{i}(K\tilde{F}(0,\cdot), \Omega_r, \Omega) = \mathrm{i}(K\tilde{F}(0,\cdot), \Omega_R, \Omega), \tag{3.11}$$

and accordingly,

$$\mathrm{i}(K\tilde{F}(0,\cdot), \Omega_R \setminus \bar{\Omega}_r, \Omega) = \mathrm{i}(K\tilde{F}(0,\cdot), \Omega_R, \Omega) - \mathrm{i}(K\tilde{F}(0,\cdot), \Omega_r, \Omega) = -1, \tag{3.12}$$

that is,

$$\deg(S_0, \Omega_R \setminus \bar{\Omega}_r, 0) = -1.$$

Lemma 3.2 *There exists $\gamma > 0$ such that*

(i) $\deg(S(\gamma,\cdot), \Omega_R \setminus \bar{\Omega}_r, 0) = -1$ $\forall 0 \leq \gamma \leq \gamma_0$;

(ii) *if $S(\gamma, w) = 0$, $\gamma \in [0, \gamma_0]$, $r \leq \|w\| \leq R$, then $w > 0$ in $(0,1)$.*

Proof Clearly, (i) follows if we show that

$$S(\gamma, w) \neq 0, \quad 0 \le \gamma \le \gamma_0, \|w\| \in \{r, R\}.$$

Otherwise, there exists a sequence (γ_n, w_n) with $\gamma_n \to 0$, $\|w_n\| \in \{r, R\}$, and $w_n = K\tilde{F}(\gamma_n, w_n)$. Since K is compact, then, up to a subsequence, $w_n \to w$, and

$$S_0(w) = 0, \quad \|w\| \in \{r, R\},$$

a contradiction with (3.5) and (3.6).

Thus, by (3.7) and homotopy we get that

$$\deg\bigl(S(\gamma, \cdot), \Omega_R \setminus \Omega_r, 0\bigr) = -1.$$

To prove (ii), we argue again by contradiction. As in the preceding argument, we find a sequence $w_n \in X$ with $\{x \in [0,1] : w_n(x) \le 0\} \neq \varnothing$ such that $w_n \to w$, $\|w\| \in [r, R]$, and $S_0(w) = 0$; namely, w solves (3.4). By the maximum principle, $w > 0$ on (0,1) and X. Moreover, without relabeling, $w_n \to w$ in X. Therefore

$$w_n > 0, \quad x \in (0,1),$$

for n large, a contradiction. □

Proof of Theorem 3.1 completed By Lemma 3.2 problem (3.2) has a positive solution w_γ for all $0 \le \gamma \le \gamma_0$. As remarked before, for $\gamma > 0$, the rescaling $\lambda = \gamma^{p-1}, u = w/\gamma$ gives a solution (λ, u_λ) of (3.1) for all $0 < \lambda < \lambda_* := \gamma_0^{p-1}$. Since $w_\gamma > 0$, (λ, u_λ) is a positive solution of (1.1). Finally, $\|w_\gamma\| \ge r$ for all $\gamma \in [0, \gamma_0]$ implies that

$$\|u_\lambda\| = \|w\|_\gamma / \gamma \to \infty \quad \text{as } \gamma \to 0.$$

This completes the proof. □

4 Sublinear problems

In this final section, we deal with sublinear f, namely with $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$ that satisfy (f_1) and

- (f_4) $\exists b \in C([0,1]), b > 0$, such that $\lim_{u\to\infty} u^{-q} f(x,u) = b$ uniformly in $x \in [0,1]$ with $0 \le q < 1$.

We will show that in this case positive solutions of (1.1) branch off from ∞ for $\lambda_\infty = +\infty$. First, some preliminaries are in order. It is convenient to work on X. Following the same procedure as for the superlinear case, we employ the rescaling $w = \gamma u, \lambda = \gamma^{q-1}$ and use the same notation with q instead of p. As before, (λ, u) solves (3.1) if and only if (γ, w) satisfies (3.2). Note that now, since $0 \le q < 1$, we have that

$$\lambda \to +\infty \quad \Leftrightarrow \quad \gamma \to 0. \tag{4.1}$$

For future reference, note that by Lemma 3.1

$$\begin{cases} u'''' = bw^q, \quad x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{4.2}$$

has a unique positive solution w_0.

We claim that there exist two constants $R > r > 0$ such that

$$S_0(w) \neq 0 \quad \forall \|w\| \leq R; \tag{4.3}$$

$$S_0(w) \neq 0 \quad \forall \|w\| \geq r; \tag{4.4}$$

$$\deg(S_0, O_R \setminus O_r, 0) = 1. \tag{4.5}$$

Assume on the contrary that (4.3) is not true. Then there exists a sequence w_n of solutions of (4.4) satisfying

$$\|w_n\| \to 0, \quad n \to \infty, \tag{4.6}$$

then $w_n \equiv 0$ in $[0,1]$ for n large.

Let $v_n := w_n / \|w_n\|$. From (3.4) we have

$$\begin{cases} v_n'''' = (b|w_n|^{q-1})v_n, \quad x \in (0,1), \\ v_n(0) = v_n(1) = v_n''(0) = v_n''(1) = 0. \end{cases} \tag{4.7}$$

By the standard argument, after taking a subsequence and relabeling if necessary, it follows that

$$\lim_{n\to\infty} \left(b|w_n|^{q-1}\right) = 0 \quad \text{uniformly in } x \in [0,1],$$

and there exists $v_* \in X$ with $\|v_*\| = 1$ such that

$$v_n \to v_*, \quad n \to \infty,$$

and

$$\begin{cases} v_*'''' = 0, \quad x \in (0,1), \\ v_*(0) = v_*(1) = v_*''(0) = v_*''(1) = 0, \end{cases}$$

which implies that $v_* = 0$. However, this is a contradiction, Therefore (4.3) is valid.

Assume on the contrary that (4.4) is not true. Then there exists a sequence $\{w_n\}$ of solutions of (4.4) satisfying

$$\|w_n\| \to \infty, \quad n \to \infty. \tag{4.8}$$

In fact, we have from (3.4) that

$$\begin{cases} w_n'''' = (b|w_n|^{q-1})w_n, \quad x \in (0,1), \\ w_n(0) = w_n(1) = w_n''(0) = w_n''(1) = 0, \end{cases}$$

since

$$\lim_{n\to\infty}\left(b|w_n|^{q-1}\right) = \infty \quad \text{uniformly in } x \in [1/4, 3/4],$$

which shows that w_n must change its sign in $[1/4, 3/4]$. However, this is a contradiction. Therefore (4.4) is valid.

Now, from (4.3) and (4.4) we deduce

$$S_0(w) \neq 0 \quad \forall w \in \partial O_R, \qquad S_0(w) \neq 0 \quad \forall w \in \partial O_r.$$

This implies that

$$S_0(w) \neq 0 \quad \forall w \in \partial(\bar{O}_R \setminus O_r).$$

Thus, the degree $\deg(S_0, O_R \setminus \bar{O}_r, 0)$ is well defined.

Next, we show that

$$\deg(S_0, O_R \setminus \bar{O}_r, 0) = 1.$$

To this end, let us define

$$O = \left\{u \in C[0,1] : u(t) \geq 0 \text{ for } t \in [0,1]\right\}$$

and

$$O_\rho = \left\{u \in O : \|u\| < \rho\right\}.$$

Using Lemma 3.1 and an argument similar to that in the proof of [6], Theorem 3, we deduce

$$\mathrm{i}\left(K\tilde{F}(0,\cdot), O_r, O\right) = 0, \qquad \mathrm{i}\left(K\tilde{F}(0,\cdot), O_R, O\right) = 1.$$

By the excision and the additivity properties of the degree it follows that

$$\mathrm{i}\left(K\tilde{F}(0,\cdot), O_R \setminus \bar{O}_r, O\right) + \mathrm{i}\left(K\tilde{F}(0,\cdot), O_r, O\right) = \mathrm{i}\left(K\tilde{F}(0,\cdot), O_R, O\right),$$

and accordingly,

$$\mathrm{i}\left(K\tilde{F}(0,\cdot), O_R \setminus \bar{O}_r, O\right) = \mathrm{i}\left(K\tilde{F}(0,\cdot), O_R, O\right) - \mathrm{i}\left(K\tilde{F}(0,\cdot), O_r, O\right) = 1,$$

that is,

$$\deg(S_0, O_R \setminus \bar{O}_r, 0) = 1.$$

Lemma 4.1 *There exists $\gamma > 0$ such that*

(i) $\deg(S(\gamma,\cdot), O_R \setminus \bar{O}_r, 0) = 1 \ \forall 0 \leq \gamma \leq \gamma_0$;

(ii) *if $S(\gamma, w) = 0$, $\gamma \in [0, \gamma_0]$, $r \leq \|w\| \leq R$, then $w > 0$ in $(0,1)$.*

Proof Clearly, (i) follows if we show that

$$S(\gamma, w) \neq 0, \quad 0 \leq \gamma \leq \gamma_0, \|w\| \in \{r, R\}.$$

Otherwise, there exists a sequence (γ_n, w_n) with $\gamma_n \to 0$, $\|w_n\| \in \{r, R\}$, and $w_n = K\tilde{F}(\gamma_n, w_n)$. Since K is compact, then, up to a subsequence, $w_n \to w$, and

$$S_0(w) = 0, \quad \|w\| \in \{r, R\},$$

a contradiction with (4.3) and (4.4).

To prove (ii), we argue again by contradiction. As in the preceding argument, we find a sequence $w_n \in X$ with $\{x \in [0,1] : w_n(x) \leq 0\} \neq \varnothing$ such that $w_n \to w$, $\|w\| \in [r, R]$, and $S_0(w) = 0$; namely, w solves (3.2). By the maximum principle, $w > 0$ on (0,1) and X. Moreover, without relabeling, $w_n \to w$ in X. Therefore

$$w_n > 0, \quad x \in (0, 1),$$

for n large, a contradiction. □

Theorem 4.1 *Let $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$ satisfy (f_1) and (f_4). Then there is $\lambda^* > 0$ such that (1.1) has positive solutions for all $\lambda \geq \lambda^*$. More precisely, there exists a connected set of positive solutions of (1.1) bifurcating from infinity for $\lambda_\infty = +\infty$.*

Proof of Theorem 4.1 By Lemma 4.1 problem (3.2) has a positive solution w_γ for all $0 \leq \gamma \leq \gamma_0$. As remarked before, for $\gamma > 0$, the rescaling

$$\lambda = \gamma^{q-1}, \quad u = w/\gamma$$

gives a solution (λ, u_λ) of (3.1) for all $\lambda \geq \lambda^* := \gamma_0^{q-1}$. Since $w_\gamma > 0$, (λ, u_λ) is a positive solution of (1.1). Finally, $\|w_\gamma\| \geq r$ for all $\gamma \in [0, \gamma_0]$ implies that

$$\|u_\lambda\| = \|w\|_\gamma/\gamma \to \infty \quad \text{as } \gamma \to 0.$$

This completes the proof. □

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Abbreviation
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The authors claim that the research was realized in collaboration with the same responsibility. Both authors read and approved the last version of the manuscript.

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