Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On the approximation of time-fractional telegraph equations using localized kernel-based method

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Abstract

In the present work, a hybrid transform-based localized meshless method is constructed for the solution of time-fractional telegraph equations. In the first step the Laplace transform is applied to the time-fractional telegraph equation, which reduces the problem to a finite set of elliptic equations which are solved with the help of local radial basis functions method in parallel. Finally, the solution is represented as an integral along a smooth curve in the complex plane. The integral is then evaluated by quadrature rule. The advantage of this method is that it does not suffer from time instability that may occur in a time stepping procedure. A clear improvement is observed in terms of stability, accuracy and ill-conditioning.

Keywords: Laplace transform; Local kernel based method; Time-fractional telegraph equation

1 Introduction

Fractional calculus is the generalization of differentiation and integration to non-integer orders. Fractional calculus has gained special importance in the last two or three decades. Many phenomena in engineering and other sciences can be successfully modeled by fractional calculus [1–7]. The telegraph equations have many applications in physics and engineering. The applications arise, for example, in signal analysis [8], random walk theory [9], wave propagation [10].

The telegraph equations of fractional order have been investigated by many researchers. The solution of space–time-fractional telegraph equation in a bounded domain is obtained in terms of Mittage-Leffler functions by the method of generalized differential transform [11]. Das et al. [12] used a homotopy analysis method in approximating an analytical solution for the time-fractional telegraph equation and different particular cases have been derived. In [13] Jiang and Lin obtained a series solution for the time-fractional telegraph equation with Robin boundary value conditions using the reproducing kernel theorem. Saadatmandi and Mohabbati [14] have used the Tau method for the approximation of fractional telegraph equation. Liu et al. [15] derived the analytical solution of the nonhomogeneous time-fractional telegraph equation by considering three types of nonhomogeneous boundary conditions using the method of separation of variables. In [16] the authors approximated the solution of fractional telegraph equation using radial basis

functions. More work on fractional telegraph equations can be found in [17–19], and the references therein.

In the present work, the Laplace transform is coupled with localized kernel-based method, and the resulting hybrid method is investigated for solving telegraph equations of fractional order. Following the work [20], the Bromwich integral associated with the inverse Laplace transform is approximated numerically with standard quadrature of M steps. By increasing M round off errors will occur which will make it difficult to find the true solution. The authors in [21] present a method that can safeguard against this. The combination of Laplace transform with some other methods have been successfully achieved earlier and is available in the literature, but only a small amount of work is available. For example, the Laplace transform coupled with the boundary-particle method [22] and the Kansa method [23]. Similarly the authors of [24] studied the combination of Laplace transform with the RBF method on a unit sphere for solving the heat equation. The combination of Laplace transform with the finite element, the finite difference and the spectral methods can be found in Refs. [25–29]. We consider a time-fractional telegraph equation of fractional order $\frac{1}{2} < \alpha \leq 1$ of the form

$$ {}_{C}D_{t}^{2\alpha}u(\mathbf{x},t) + \lambda\, {}_{C}D_{t}^{\alpha}u(\mathbf{x},t) = \mu\mathcal{L}u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega \subset \mathcal{R}^{d}, d \geq 1, \tag{1.1} $$

subject to initial and boundary conditions

$$ u(\mathbf{x},0) = \varphi_1(\mathbf{x}), \qquad u_t(\mathbf{x},0) = \varphi_2(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.2} $$

and

$$ \mathcal{B}u(\mathbf{x},t) = g_1(t), \quad \mathbf{x} \in \partial\Omega, \tag{1.3} $$

respectively, where $\mathcal{L}$ is a linear spatial differential operator and $\mathcal{B}$ is a boundary differential operator and ${}_{C}D_{t}^{\alpha}$ is the Caputo fractional partial derivative of order α.

2 Preliminaries

In this section, we give some important definitions about fractional calculus.

Definition 2.1 Let the Laplace transform of $u(t)$ be defined by

$$ \mathcal{L}\{u(t)\} = U(z) = \int_{0}^{\infty} e^{-zt}u(t)\,dt. \tag{2.1} $$

Definition 2.2 The Riemann–Liouville derivative of fractional order α of a function $u(t)$ is defined as (see [30])

$$ {}_{RL}D_{t}^{\alpha}u(t) = \frac{1}{\Gamma(p-\alpha)}\frac{d^{p}}{dt^{p}}\int_{0}^{t}(t-s)^{p-\alpha-1}u(s)\,ds, \tag{2.2} $$

where $p-1 \leq \alpha \leq p$, $p \in \mathbf{N}$.

Definition 2.3 The Caputo fractional partial derivative of order α of a function $u(t)$ is defined by (see [30]).

$$ {}_{C}D_{t}^{\alpha}u(t) = \frac{1}{\Gamma(p-\alpha)}\int_{0}^{t}(t-s)^{p-\alpha-1}\frac{d^{p}}{ds^{p}}u(s)\,ds, \tag{2.3} $$

$p-1 \leq \alpha \leq p, p \in \mathbf{N}$.

Definition 2.4 If $u(t) \in C^{p}[0,\infty)$ and $p-1 < \alpha < p, p \in \mathbf{N}$, then the Laplace transform of the Caputo fractional derivative is given by

$$ \mathcal{L}\left\{{}_{C}D_{t}^{\alpha}u(t)\right\} = z^{\alpha}U(z) - \sum_{i=0}^{p-1} z^{\alpha-i-1}u^{(i)}(0). \tag{2.4} $$

3 Analysis of the method

In this section, we propose a meshless method based on Laplace transform for time-fractional telegraph equation. In the proposed method we eliminate the time variable by a Laplace transform and for the time independent PDE, the localized meshless numerical scheme will be constructed.

Applying the Laplace transform to Eqs. (1.1)–(1.3), we get

$$ \begin{aligned} &z^{2\alpha}U(\mathbf{x},z) - z^{2\alpha-1}\varphi_{1}(\mathbf{x}) - z^{2\alpha-2}\varphi_{2}(\mathbf{x}) + \lambda\left(z^{\alpha}U(\mathbf{x},z) - z^{\alpha-1}\varphi_{1}(\mathbf{x})\right) \\ &\quad = \mu\mathcal{L}U(\mathbf{x},z) + F(\mathbf{x},z), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{3.1} $$

$$ \mathcal{B}\left\{U(\mathbf{x},z)\right\} = G_{1}(z), \quad \mathbf{x} \in \partial\Omega. \tag{3.2} $$

Thus we have the following system of linear differential equations:

$$ \left[z^{2\alpha}I + \lambda z^{\alpha}I - \mu\mathcal{L}\right]\left\{U(\mathbf{x},z)\right\} = G(\mathbf{x},z), \quad \mathbf{x} \in \Omega, \tag{3.3} $$

$$ \mathcal{B}\left\{U(\mathbf{x},z)\right\} = G_{1}(z), \quad \mathbf{x} \in \partial\Omega, \tag{3.4} $$

where

$$ G(\mathbf{x},z) = z^{2\alpha-1}\varphi_{1}(\mathbf{x}) + z^{2\alpha-2}\varphi_{2}(\mathbf{x}) + \lambda z^{\alpha-1}\varphi_{1}(\mathbf{x}) + F(\mathbf{x},z). $$

In the next section the kernel-based method in local setting is employed to approximate the governing differential operators $\mathcal{L}$ and the boundary differential operator $\mathcal{B}$ and to solve the time independent problem (3.3)–(3.4) in Laplace space.

3.1 Spatial discretization via local kernel based method

We take a given sample data points $\{U(\mathbf{x}_{i}), i = 1,2,\ldots,N\}$ of an unknown smooth function $U(\mathbf{x})$, where $\{\mathbf{x}_{1},\ldots,\mathbf{x}_{N}\} \subset \Omega \subset \mathcal{R}^{d}$, $d \geq 1$. An approximation of the function $U(\mathbf{x})$, at each $\mathbf{x}_{i} \in \Omega$, is defined by

$$ U(\mathbf{x}_{i}) = \sum_{\mathbf{x}_{h} \in \Omega_{i}} \alpha_{h}\psi\left(\|\mathbf{x}_{i} - \mathbf{x}_{h}\|\right), \tag{3.5} $$

where $\boldsymbol{\alpha}^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i]$ is the expansion coefficients vector, and $r = \|\mathbf{x}_i - \mathbf{x}_h\|$ is the distance between centers $\mathbf{x}_i$ and $\mathbf{x}_h$, $\psi(r)$, $r \geq 0$ is a radial kernel and $\Omega_i \subset \Omega$ is a local domain for each center $\mathbf{x}_i$, containing n neighboring centers around $\mathbf{x}_i$. Thus we have N small size linear systems of order $n \times n$ given by

$$\mathbf{U}^i = \Psi^i \boldsymbol{\alpha}^i, \quad i = 1, 2, \ldots, N, \tag{3.6}$$

the entries of Ψ^i are $b_{lh}^i = \psi(\|\mathbf{x}_l - \mathbf{x}_h\|), \mathbf{x}_l, \mathbf{x}_h \in \Omega_i$, the matrix Ψ^i is known as the interpolation matrix, we need to solve each small size $n \times n$ system for the unknowns $\boldsymbol{\alpha}^i = [\alpha_1^i, \alpha_2^i, \ldots, \alpha_n^i]$. Next the $\mathcal{L}U(\mathbf{x})$, is approximated by

$$\mathcal{L}U(\mathbf{x}_i) = \sum_{\mathbf{x}_h \in \Omega_i} \boldsymbol{\alpha}_h^i \mathcal{L}\psi\big(\|\mathbf{x}_i - \mathbf{x}_h\|\big), \tag{3.7}$$

Equation (3.7) can be written as a product of two vectors, given by

$$\mathcal{L}U(\mathbf{x}_i) = \mathbf{v}^i \cdot \boldsymbol{\alpha}^i, \tag{3.8}$$

where $\boldsymbol{\alpha}^i$ of order $n \times 1$ is a vector of unknown coefficients, and $\mathbf{v}^i$ is a vector of order $1 \times n$ with entries given by

$$\mathbf{v}^i = \mathcal{L}\psi\big(\|\mathbf{x}_i - \mathbf{x}_h\|\big), \quad \mathbf{x}_h \in \Omega_i, \tag{3.9}$$

using Eq. (3.6), we eliminate the unknown coefficients,

$$\boldsymbol{\alpha}^i = \big(\Psi^i\big)^{-1} \mathbf{U}^i, \tag{3.10}$$

and by inserting the values of $\boldsymbol{\alpha}^i$ from (3.10) in (3.8) we get

$$\mathcal{L}U(\mathbf{x}_i) = \mathbf{v}^i \big(\Psi^i\big)^{-1} \mathbf{U}^i = \mathbf{w}^i \mathbf{U}^i, \tag{3.11}$$

where

$$\mathbf{w}^i = \mathbf{v}^i \big(\Psi^i\big)^{-1}. \tag{3.12}$$

Hence for each center, the localized approximation of the linear differential operator $\mathcal{L}$ using radial basis functions is given by

$$\mathcal{L}U \equiv \mathbf{D}\mathbf{U}. \tag{3.13}$$

So the spatial operator $\mathcal{L}$ is approximated by the $N \times N$ sparse differentiation matrix $\mathbf{D}$ having $N - n$ zero entries and n non-zero entries, where n is the number of centers in the domain Ω_i. Similarly the boundary operator $\mathcal{B}$ can be approximated using the localized kernel-based method as discussed above.

3.2 Choosing optimal shape parameter

In the literature we can find a variety of kernel functions. In this work the multiquadrics, $\psi(r) = \sqrt{1 + (\varepsilon r)^2}$ are selected. These kernels contain a scale factor ε and accuracy of the solution relies upon this scale factor. For an optimal value of this scale factor ε a large amount of work is available in the literature [31–35] and the references therein. In this paper we utilize the uncertainty principle [36] (e.g., a better accuracy can be achieved comparatively at larger condition numbers of these type of kernel based system matrices) for a decent estimation of the scale factor ε.

Algorithm

- The condition number is kept approximately in the range $10^{12} < \kappa < 10^{16}$ for our problem system matrices.
- Decompose the interpolation matrix as $\mathbf{Q}, \mathbf{S}, \mathbf{V} = \text{svd}(\Psi^i)$ using a singular value decomposition. The interpolation matrix Ψ^i is of order $n \times n$ for each local subdomain Ω_i, and $\mathbf{S}$ is diagonal matrix containing n singular values of Ψ^i, and $\kappa = \|\Psi^i\|\|(\Psi^i)^{-1}\| = \max(\mathbf{S})/\min(\mathbf{S})$ denotes the condition number of the matrix Ψ^i.
- Search for ε until κ satisfy the condition $10^{12} < \kappa < 10^{16}$, using the algorithm

  $\kappa = 1$
  $10^{12} < \kappa < 10^{16}$
  while $\kappa < \kappa_{\min}$ and $\kappa > \kappa_{\max}$
  $\mathbf{Q}, \mathbf{S}, \mathbf{V} = \text{svd}(\Psi^i)$
  $\kappa = \max(\mathbf{S})/\min(\mathbf{S})$
  if $\kappa < \kappa_{\min}$, $\varepsilon = \varepsilon - \delta\varepsilon$
  if $\kappa > \kappa_{\max}$, $\varepsilon = \varepsilon + \delta\varepsilon$
  $\varepsilon(\text{optimal}) = \varepsilon$.
  When the above condition is satisfied a good value of ε is obtained, the inverse is computed using $(\Psi^i)^{-1} = (\mathbf{QSV^T})^{-1} = \mathbf{VS}^{-1}\mathbf{Q}^T$ [37]. Thus we can compute $\mathbf{w}^i$ in (3.12).

After discretization of the operators $\mathcal{L}$ and $\mathcal{B}$ by a localized meshless method the system (3.3)–(3.4) is solved for each point along the contour of integration z. Then the solution $u(\mathbf{x}, t)$ of problem (1.1)–(1.3) can be obtained by the inverse Laplace transform

$$u(\mathbf{x}, t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{zt} U(\mathbf{x}, z)\, dz = \frac{1}{2\pi i}\int_{\Gamma} e^{zt} U(\mathbf{x}, z)\, dz, \quad \sigma > \sigma_0, \tag{3.14}$$

where Γ is suitable path joining $\sigma - i\infty$ to $\sigma + i\infty$ and

$$z = z(\eta), \tag{3.15}$$

are the points chosen along the path Γ. Using (3.15) in (3.14), we find the following expression:

$$u(\mathbf{x}, t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{z(\eta)t} U\big(\mathbf{x}, z(\eta)\big)\acute{z}(\eta)\, d\eta. \tag{3.16}$$

The approximation of (3.16) can be obtained by the trapezoidal rule with uniform step size k, as

$$u_k(\mathbf{x},t) = \frac{k}{2\pi i}\sum_{j=-M}^{M} e^{z_j t}U(\mathbf{x},z_j)\acute{z}_j, \quad z_j = z(\eta_j), \eta_j = jk. \tag{3.17}$$

4 Error analysis of the method

The accuracy of the approximate solution defined by (3.17) is based on the choice of contour Γ. In the literature various such contours are available, for example parabolic [20] and hyperbolic [27]. We used the hyperbolic contour in our computation due to [27]:

$$z(\eta) = \omega + \lambda\big(1 - \sin(\delta - \iota\eta)\big), \quad \text{for } \eta \in \mathcal{R}, \qquad (\Gamma) \tag{4.1}$$

with $\lambda > 0$, $\omega \geq 0$, $0 < \delta < \beta - \frac{1}{2}\pi$, and $\frac{1}{2}\pi < \beta < \pi$. In fact, when $\operatorname{Im}\eta = \gamma$, (4.1) reduces to the left branch of the hyperbola

$$\left(\frac{x-\lambda-\omega}{\lambda\sin(\delta+\gamma)}\right)^2 - \left(\frac{y}{\lambda\cos(\delta+\gamma)}\right)^2 = 1, \tag{4.2}$$

where the strip $Z_r = \{\eta : \operatorname{Im}\eta \leq r\}$ with $r > 0$ is transformed into the hyperbola $\Omega_r = \{z : \eta \in Z_r\} \supset \Gamma$. Let $\Sigma_\phi = \{z \neq 0 : |\arg z| \leq \phi\} \cup 0$, with $0 < \phi < \frac{(1-\alpha)\phi}{2}$, and let $\Sigma_\beta^\omega = \omega + \Sigma_\beta$, $\Gamma \subset \Omega_r \subset \Sigma_\beta^\omega$. The error bound of the proposed method for the hyperbolic path Γ is based on the following theorem.

Theorem 4.1 *Let $u(\mathbf{x},t)$ be the solution of (1.1), with $F(\mathbf{x},t)$ analytic in Σ_β^ω. Let $0 < \theta < 1$, $0 < t_0 < T$, and let $b > 0$ be defined by $b = \cosh^{-1}(1/(\theta\tau\sin(\delta)))$, where $\tau = t_0/T$, and let $\Gamma \subset \Omega_r \subset \Sigma_\beta^\omega$, and scaling factor be $\lambda = \theta\bar{r}M/(bT)$. Then we have for the approximate solution defined by (3.17), with $k = b/M \leq \frac{\bar{r}}{\log 2}$, $|u_k(\mathbf{x},t) - u(\mathbf{x},t)| \leq CQe^{\omega\tau}l(\rho_r M)e^{-\mu M}(\|u_0\| + \|\hat{f}(\mathbf{x},t)\|_{\Sigma_\beta^\omega})$, for $l(x) = \max(1, \log(1/x))$, $r > 0$, $\mu = \bar{r}(1-\theta)/b$, $\bar{r} = 2\pi r$, $\rho_r = \theta\bar{r}\tau\sin(\delta - r)/b$, $t_0 \leq t \leq T$, and $C = C_{\delta,r,\beta}$.*

Proof See [27], Theorem 2.1. □

5 Stability of the method

To discuss the stability of system (3.3)–(3.4), in discrete form this system may be represented as

$$A\mathbf{U} = \mathbf{b}, \tag{5.1}$$

where A is $N \times N$ sparse differentiation matrix which can be obtained by localized kernel-based method discussed in Sect. 3. the stability constant corresponding to system (5.1) is given by

$$C = \sup_{U \neq 0} \frac{\|U\|}{\|AU\|}, \tag{5.2}$$

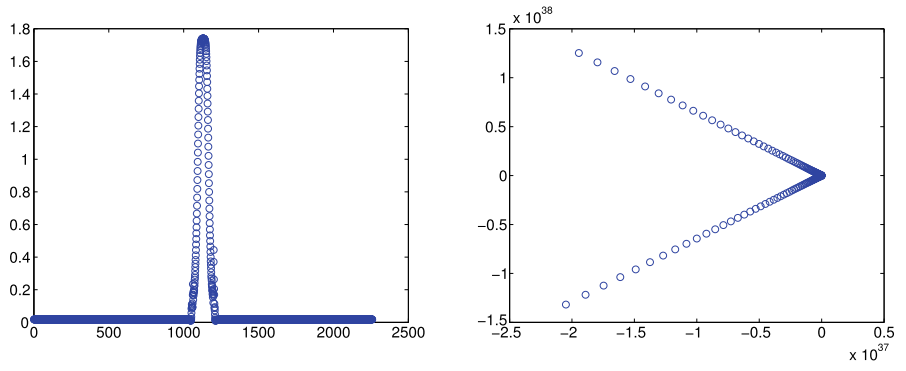

Figure 1 First plot show the stability constant C of our differentiation matrix A for various points along the hyperbolic path. The second plot shows the hyperbolic path, corresponding to Problem 1

where C is finite using any type of discrete norms $\|\cdot\|$ on R^N. The above equation can be expressed as

$$\|A\|^{-1} \leq \frac{\|U\|}{\|AU\|} \leq C. \tag{5.3}$$

Again in terms of the pseudoinverse $A^{\dagger}$ of A, we have

$$\left\|A^{\dagger}\right\| = \sup_{v \neq 0} \frac{\|A^{\dagger} v\|}{\|v\|}. \tag{5.4}$$

Now we write

$$\left\|A^{\dagger}\right\| \geq \sup_{v = AU \neq 0} \frac{\|A^{\dagger} AU\|}{\|AU\|} = \sup_{U \neq 0} \frac{\|U\|}{\|AU\|} = C. \tag{5.5}$$

Hence Eqs. (5.3) and (5.5) ensure the boundedness of the stability constant C. For a numerical approximation of the system (5.1) the calculation of the pseudoinverse may be computationally expansive, but it ensures numerical stability. In the case of square systems, the MATLAB's function condest estimates $\|A^{-1}\|_{\infty}$, thus we have

$$C = \frac{\text{condest}(A')}{\|A\|_{\infty}}. \tag{5.6}$$

This works well for our sparse matrix A with a small amount of computations. The bounds of stability constant C of our system (3.3)–(3.4) corresponding to Problem 1 are shown in Fig. 1. Choosing $M = 90$, $N = 50$ and $n = 7$ at time $t = 1$, we can see that $0.0088 \leq C \leq 1.7401$, which shows the stability constant is bounded by numbers that are not very large, and this implies the numerical stability of localized kernel-based numerical scheme.

6 Numerical results

In this section the proposed method is tested for one dimensional time-fractional telegraph equations. In our computations the multiquadrics $\psi(r) = (1 + (\varepsilon r)^2)^{1/2}$ are used.

The accuracy of the solution depends on the shape parameter ε. A number of criteria are available in the literature for choosing optimal values of the shape parameters. We use the uncertainty principle due to [36] to select the optimal shape parameter. The accuracy of the proposed method is measured by the maximum absolute error (L_∞) defined by

$$L_\infty = \left\| u(\mathbf{x},t) - u_k(\mathbf{x},t) \right\|_\infty = \max_{1\le j\le N} \left(\left| u(\mathbf{x},t) - u_k(\mathbf{x},t) \right| \right).$$

Here u and u_k denotes the exact and approximate solutions, respectively. The error norms are calculated at fixed value of t in time interval $[t_0, T]$, where t_0 and T are given in each numerical experiment.

6.1 Problem 1

Here we apply our proposed numerical method to the one dimensional time-fractional telegraph equation [13],

$${}_CD_t^{2\alpha} u(x,t) + {}_CD_t^{\alpha} u(x,t) = \frac{1}{2} D_x^2 u(x,t) + f(x,t), \quad 0 < x < 1, 0 < t \le 1,$$

where

$$f(x,t) = \frac{2e^x}{\Gamma(3-2\alpha)} t^{2-2\alpha} + \frac{2e^x}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{2} t^2 e^x,$$

subject to the initial condition

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) + u_x(0,t) = 2t^2, \quad 0 < t \le 1,$$
$$u(1,t) - \frac{1}{2} u_x(1,t) = \frac{et^2}{2}, \quad 0 < t \le 1,$$

with exact solution $u(x,t) = e^x t^2$. The problem is solved over the domain $0 \le x \le 1$ at $t = 1$. Different quadrature points are used along the hyperbolic Γ contour. These points are generated by MATLAB statement $\eta = -M : k : M$ for hyperbolic contour Γ. The parameters used are $\theta = 0.1, \delta = 0.1541, r = 0.1387, \omega = 2, t_0 = 0.5$ and $T = 5$. The other optimal parameters are given in (4.1). The L_∞ error and error estimate (E) using fractional orders $\alpha = 0.8, 0.96$ are shown in Table 1. Various numbers of points N in the global domain Ω and n in the local domain Ω_i are used. The shape parameter is optimized using the uncertainty principle [36]. The condition number κ, the shape parameter ε and the CPU time(s) are given in the table. It is observed that the proposed method is less sensitive with respect to the shape parameter. The accuracy is achieved for small shape parameter and large condition number. The results are compared with other methods [13]. It is observed that the proposed method is accurate and computationally efficient. This method gives an almost exact solution in time, an error occurs only in spatial discretization. So

Table 1 The maximum absolute error in our method and in [13] corresponding to Problem 1

	M	L_∞ error	E	ε	κ	CPU time (s)
$\alpha = 0.8, N = 30, n = 5$	5	4.7135	7.2328	0.8	1.0508e+012	0.136899
	7	0.2062	5.9685	0.8	1.0508e+012	0.140582
	10	0.0025	4.4187	0.8	1.0508e+012	0.139019
	25	0.0019	0.9164	0.8	1.0508e+012	0.165614
	30	3.9156e–004	0.5373	0.8	1.0508e+012	0.188401
	50	1.7714e–004	0.0625	0.8	1.0508e+012	0.352443
	70	1.6868e–004	0.0072	0.8	1.0508e+012	0.789112
	90	1.6856e–004	8.1825e–004	0.8	1.0508e+012	1.967992
[13] 5.60e–005						
$\alpha = 0.96, N = 50, n = 7$	5	6.0719	7.2328	3	1.0041e+012	0.136641
	7	0.2656	5.9685	3	1.0041e+012	0.142100
	10	0.0034	4.4187	3	1.0041e+012	0.146196
	25	0.0023	0.9164	3	1.0041e+012	0.207502
	30	6.2477e–004	0.5373	3	1.0041e+012	0.249881
	50	1.1877e–004	0.0625	3	1.0041e+012	0.648773
	70	1.1406e–004	0.0072	3	1.0041e+012	1.786508
	90	1.1400e–004	8.1825e–004	3	1.0041e+012	5.400089
[13] 2.10e–004						

we can approximate the telegraph equation very accurately in time without any time instability issue. The local nature of the method makes it more attractive for such a type of problems.

6.2 Problem 2

Next we consider the one dimensional time-fractional telegraph equation with $\alpha = \frac{2}{3}$,

$$ {}_CD_t^{2\alpha}u(x,t) + {}_CD_t^{\alpha}u(x,t) = D_x^2u(x,t) + f(x,t), \quad 0 < x < 1, 0 < t \leq 1, $$

where

$$ f(x,t) = 6\sin(x+1)\left(\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) + \sin(x+1)\left(t^3+1\right), $$

subject to the initial condition

$$ u(x,0) = \sin(x+1), \qquad u_t(x,0) = 0, \quad 0 < x < 1, $$

and the boundary conditions

$$ u(0,t) = \sin(1)\left(t^3+1\right), \quad 0 < t \leq 1, $$

$$ u(1,t) + 3u_x(1,t) = \left(t^3+1\right)\left(\sin(2) + 3\cos(2)\right), \quad 0 < t \leq 1. $$

The exact solution of the problem is $u(x,t) = (t^3+1)\sin(x+1)$. Here the problem is solved over the domain $[0,1]$ at time $t = 1$. Various quadrature points along the hyperbolic path Γ are used. These points are generated by the MATLAB statement $\eta = -M : k : M$. The parameters used are $\theta = 0.1$, $\delta = 0.1541$, $r = 0.1387$, $\omega = 2$, $t_0 = 0.5$ and $T = 5$. The other optimal parameters are given in Eq. (4.1). The L_∞ error and error estimate (E) using fractional order $\alpha = \frac{2}{3}$ are shown in Table 2. Various numbers of points N in the global domain

Table 2 The maximum absolute error, shape parameter, condition number and computational time corresponding to Problem 2 at $t = 1$

$\alpha = \frac{2}{3}, N = 45, n = 5$	M	L_∞ error	E	ε	κ	CPU time(s)
	10	1.8374	4.4187	1.2	1.1509e+012	0.145084
	20	0.3122	1.5582	1.2	1.1509e+012	0.159080
	35	0.0373	0.3144	1.2	1.1509e+012	0.255237
	50	0.0075	0.0625	1.2	1.1509e+012	0.539619
	70	4.1395e–004	0.0072	1.2	1.1509e+012	1.361986
	90	6.0201e–005	8.1825e–004	1.2	1.1509e+012	4.342044

Table 3 The maximum absolute error for different M, N, n and α corresponding to Problem 3 at $t = 1$

$N = 50, n = 6$	M	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.95$	E
	5	1.0734	1.0734	1.0734	1.0734	7.2328
	7	0.0242	0.0242	0.0242	0.0242	5.9685
	10	0.0086	0.0086	0.0086	0.0086	4.4187
	20	9.4927e–005	9.4845e–005	9.4752e–005	9.4650e–005	1.5582
	30	1.8743e–005	1.8332e–005	1.7849e–005	1.7310e–005	0.5373
	50	1.1274e–005	1.0852e–005	1.0357e–005	9.8078e–006	0.0625

Ω and n in the local domain Ω_i are used. The shape parameter is optimized using the uncertainty principle [36]. The condition number κ, the shape parameter ε and the CPU time(s) are given in Table 2. A similar performance is observed to the one we observed in Problem 1.

6.3 Problem 3

As a third example we consider the one dimensional time-fractional telegraph equation with $\alpha \in (1, 2]$ [16]

$$ {}_CD_t^{\alpha} u(x,t) + {}_CD_t^{\alpha-1} u(x,t) + u(x,t) = \pi D_x^2 u(x,t) + f(x,t), \quad 0 < x < 1, 0 < t \leq 1, $$

where

$$ f(x,t) = 6\big(\sin(x)\big)^2 \left(\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{t^3}{6} \right) - 2\pi t^3 \cos(2x), $$

subject to the initial condition $u(x,0) = u_t(x,0) = 0$, $0 < x < 1$, and the boundary conditions are chosen according to the exact solution $u(x,t) = t^3(\sin(x))^2$. Here we tested our method for the one dimensional telegraph equation in domain $[0,1]$ at time $t = 1$. We used the same hyperbolic contour and the same optimal parameters as discussed in Problem 1 and Problem 2. The absolute errors and error estimate for the contour Γ are shown in Table 3 using fractional order $\alpha = 1.25, 1.5, 1.75, 1.95$. The error functions are shown in Fig. 2, which are calculated for $N = 11$ and $n = 4$. These results can be improved by increasing N. It is observed that as we increase the value of α the absolute error decreases. From Table 3 and Fig. 2 a clear improvement is observed as compared to [16]. So the proposed method is an excellent alternative for solving the fractional order telegraph equations.

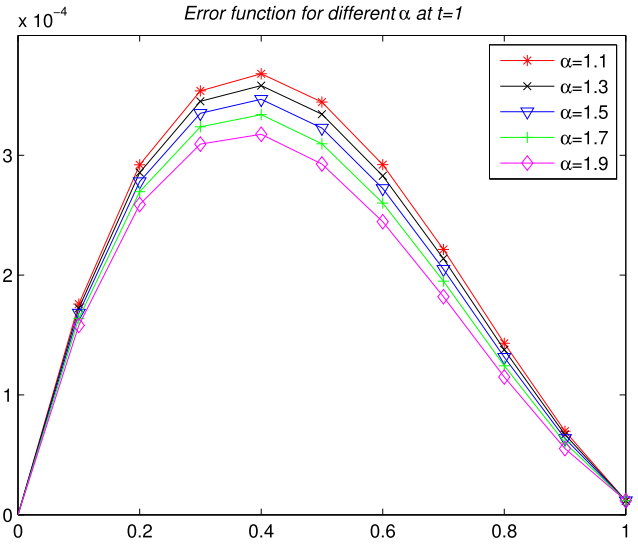

Figure 2 Error function for $N = 11, n = 4, M = 70$ and different α at $t = 1$ corresponding to Problem 3

6.4 Problem 4

In the last example we consider the one dimensional time-fractional telegraph equation with $\alpha \in (1, 2]$ [38]

$$ {}_CD_t^{\alpha} u(x,t) + {}_CD_t^{\alpha-1} u(x,t) + u(x,t) = D_x^2 u(x,t) + f(x,t), \quad 0 < x < 1, 0 < t \leq 1, $$

where

$$ f(x,t) = \left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + t \right) (x^2 - x) - 2t $$

subject to the initial condition

$$ u(x,0) = u_t(x,0) = x^2 - x, \quad 0 < x < 1, $$

and the boundary conditions are

$$ u(0,t) = u(1,t) = 0. $$

Exact solution of the problem is $u(x,t) = (x^2 - x)t$. The problem is solved over the domain $[0, 1]$ at time $t = 1$. The same hyperbolic contour Γ and the same optimal parameters are used in this problem. The absolute errors and error estimate for the contour Γ are shown in Table 4 using fractional order $\alpha = 1.95$. The result given in Table 4 shows that the proposed method is accurate and efficient as compared to [38]. So the proposed method is an excellent alternative for solving the fractional order telegraph equations.

7 Conclusion

In the present work, we propose a local meshless method coupled with the Laplace transform for a time-fractional telegraph equation. The method is almost exact in time without

Table 4 The maximum absolute error of the proposed method for different values of M and $N = 11$, $n = 4$, $\alpha = 1.95$ corresponding to Problem 4 at $t = 1$ and in [38]

Our method	x	$M = 15$	$M = 30$	$M = 40$	$M = 50$
	0	0	0	0	0
	0.1	9.7426e–004	2.4335e–005	1.5203e–005	4.0419e–006
	0.2	1.7345e–003	4.0736e–005	2.9554e–005	9.7154e–006
	0.3	2.2774e–003	5.2554e–005	3.9710e–005	1.3668e–005
	0.4	2.6032e–003	5.9667e–005	4.5773e–005	1.6012e–005
	0.5	2.7118e–003	6.2035e–005	4.7793e–005	1.6794e–005
	0.6	2.6032e–003	5.9666e–005	4.5773e–005	1.6012e–005
	0.7	2.2774e–003	5.2553e–005	3.9711e–005	1.3669e–005
	0.8	1.7345e–003	4.0737e–005	2.9553e–005	9.7145e–006
	0.9	9.7426e–004	2.4337e–005	1.5201e–005	4.0403e–006
	1	0	0	0	0
In [38]	x	$m = 5$	$m = 7$	$m = 10$	$m = 15$
	0	0	0	0	0
	0.1	1.6276e–003	1.6932e–004	2.9057e–004	6.8187e–005
	0.2	2.4790e–003	1.0916e–003	3.7898e–004	8.7378e–005
	0.3	2.3211e–003	1.0749e–003	3.8165e–004	8.8787e–005
	0.4	2.1772e–003	1.0102e–003	3.6413e–004	8.4142e–005
	0.5	2.1507e–003	9.9270e–004	3.5473e–004	8.2186e–005
	0.6	2.1772e–003	1.0102e–003	3.6413e–004	8.4142e–005
	0.7	2.3211e–003	1.0749e–003	3.8165e–004	8.8787e–005
	0.8	2.4790e–003	1.0916e–003	3.7898e–004	8.7378e–005
	0.9	1.6276e–003	1.6932e–004	2.9057e–004	6.8187e–005
	1	0	0	0	0

any time instability, which is commonly encountered in time stepping mesh-free methods. These time stepping methods require a very small time step for greater accuracy on the expense of large computations. We tested our procedure for 1D telegraph equations with time-fractional orders. The accuracy and performance of the methods is excellent for solving time-fractional telegraph equations. The proposed hybrid mesh-free method is an excellent alternative for solving time-fractional partial differential equations.

Acknowledgements
The authors would like to thank the reviewers for their valuable comments.

Funding
This work is self-supported by the authors in respect of funding and technically supported by Islamia College Peshawar and University of Engineering and Technology, Khyber Pakhtun Khwa, Peshawar, Pakistan.

Competing interests
The authors declare that they have no conflict of interests.

Authors' contributions
All the authors contributed in theoretical and computational results and all authors read and approved the final manuscript.

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Received: 17 January 2018 Accepted: 21 August 2018 Published online: 03 September 2018

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