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Ibragimov–Gadjiev operators based on q-integers

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Abstract

In this paper, we define the q-analogue of the generalized linear positive operators introduced by Ibragimov and Gadjiev in 1970. We study some approximation properties of these new operators, and we show that this sequence of operators is a generalization of well-known q-Bernstein, q-Chlodowsky, and q-Szász–Mirakyan operators as a particular case.

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1 Introduction

In [1], Ibragimov and Gadjiev defined a general sequence of positive operators and showed that this sequence of linear positive operators contains as a particular case the well-known operators of Bernstein, Bernstein–Chlodowsky, Szász, and Baskakov. They also studied the uniform convergence of these operators in the class of continuous functions $C[0,A]$, where $A > 0$ is a given number, and the properties of their convexity or concavity.

Now, we recall this construction. Let $A > 0$ be a given number, $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ be the sequences of functions in $C[0,A]$ such that $\varphi_n(0) = 0$, and $\psi_n(t) > 0$ for each $t \in [0,A]$. Let also $\{\alpha_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{\alpha_n}{n} = 1$, $\lim_{n\to\infty} \frac{1}{n^2\psi_n(0)} = 0$.

Assume that a sequence of functions of three variables $\{K_n(x,t,u)\}$ ($x,t \in [0,A]$, $-\infty < u < \infty$) satisfies the following conditions:

1°. Each function of this family is an entire analytic function with respect to u for fixed $x,t \in [0,A]$;

2°. $K_n(x,0,0) = 1$ for any $x \in [0,A]$ and for all $n \in \mathbb{N}$;

3°. $\{(-1)^\nu \frac{\partial^\nu}{\partial u^\nu} K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(t) \\ t=0}}\} \geq 0$ ($\nu, n = 1, 2, \ldots; x \in [0,A]$);

4°. $\frac{\partial^\nu}{\partial u^\nu} K_n(x,t,u)|_{\substack{u=\alpha_n\psi_n(t) \\ t=0}} = -nx[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}} K_{n+m}(x,t,u)|_{\substack{u=\alpha_n\psi_n(t) \\ t=0}}]$ ($\nu, n = 1, 2, \ldots; x \in [0,A]$),

where m is a number such that $m + n$ is zero or a natural number.

Ibragimov–Gadjiev operators have the following form:

$$L_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\left[\frac{\partial^\nu}{\partial u^\nu} K_n(x,t,u)\bigg|_{\substack{u=\alpha_n\psi_n(t) \\ t=0}}\right]\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \tag{1.1}$$

for $x \in \mathbb{R}_+$ and any function f defined on $\mathbb{R}_+$.

By ν-multiple application of property $\mathbf{4}^o$, the operator (1.1) can be reduced to the form

$$L_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{n(n+m)\cdots(n+(\nu-1)m)}{\nu!}$$
$$\times \left(x\alpha_n\psi_n(0)\right)^\nu K_{n+\nu m}\left(x, 0, \alpha_n\psi_n(0)\right).$$

As we mentioned before, these operators contain as a particular case a series of known operators. For instance, by choosing $K_n(x,t,u) = (1 - \frac{ux}{1+t})^n$ and $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, the operators defined by (1.1) are transformed into Bernstein polynomials; for $\alpha_n = n$, $\psi_n(0) = \frac{1}{nb_n}$ ($\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$), we get Bernstein–Chlodowsky polynomials; and for $K_n(x,t,u) = e^{-n(t+ux)}$, $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, we get Szasz–Mirakjan operators. Moreover, if we choose $K_n(x,t,u) = K_n(t+ux)$, $\alpha_n = n$, $\psi_n(0) = \frac{1}{n}$, then we obtain Baskakov operators. Ibragimov–Gadjiev operators were studied widely in different papers, see [2–8].

After the construction of a generalization of Bernstein polynomials involving q-integers by Lupaş [9] and then by Phillips [10], many authors studied a q-based generalization of the linear positive operators, some of them may be found in [11–18]. This is because, the q-generalized operators are more flexible than the classical ones, that is, they coincide with the classical ones, while for $q = 1$ they also possess interesting properties and, depending on the selection of q, can obtain better approximation while $q \neq 1$. The aim of this paper is to introduce a q-analogue of the Ibragimov–Gadjiev operators defined by (1.1).

2 Construction and some properties of q-Ibragimov–Gadjiev operators

Before introducing the q-operators, we mention some basic definitions of q-calculus, see [19–22].

Let $q > 0$. For any natural number $n \in \mathbb{N} \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] = \frac{1-q^n}{1-q}, \qquad [0] = 0,$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! = [1][2]\cdots[n], \qquad [0]! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \qquad [n]_1! = n!, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x}, & x \neq 0, \\ f'(0), & x = 0, \end{cases}$$

for functions which are differentiable at $x = 0$, and the higher q-derivatives are

$$D_q^0 f = f, D_q^n f = D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots.$$

Note that

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

The q-analogue of $(x - a)^n$ is the polynomial

$$(x - a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x - a)(x - qa) \cdots (x - q^{n-1} a) & \text{if } n \geq 1. \end{cases}$$

The q-exponential functions are given as follows:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}, \quad |x| < \frac{1}{1-q}, |q| < 1,$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]!}, \quad x \in \mathbb{R}, |q| < 1.$$

Now, we are ready to introduce a q-analogue of the Ibragimov–Gadjiev operators as follows:

Let $0 < q < 1$ and $A > 0$ be a given number and $\{\psi_n(t)\}$ be a sequence of functions in $C[0, A]$ such that $\psi_n(t) > 0$ for each $t \in [0, A]$, $\lim_{n \to \infty} \frac{1}{[n]^2 \psi_n(0)} = 0$. Also, let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} \frac{\alpha_n}{[n]} = 1$.

Assume that a function $K_{n,\nu}^q(x, t, u)$ $(x, t \in [0, A], -\infty < u < \infty)$ depends on the three parameters n, ν, and q, and satisfies the following conditions:

(i) The function $K_{n,\nu}^q$ is infinitely q-differentiable with respect to u for fixed $x, t \in [0, A]$;

(ii) For any $x \in [0, A]$ and for all $n \in \mathbb{N}$,

$$\sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \left[D_{q,u}^{\nu} K_{n,\nu}^q(x, t, u) \big|_{\substack{u = \alpha_n \psi_n(t) \\ t=0}} \right] \frac{(-q \alpha_n \psi_n(0))^{\nu}}{[\nu]!} = 1;$$

(iii) $\{(-1)^{\nu} D_{q,u}^{\nu} K_{n,\nu}^q(x, t, u) \big|_{\substack{u = \alpha_n \psi_n(t) \\ t=0}} \} \geq 0$ $(\nu, n = 1, 2, \ldots; x \in [0, A])$;

(This notation means that the q-derivative with respect to u is taken ν times, then one sets $u = \alpha_n \psi_n(t)$ and $t = 0$.)

(iv) $D_{q,u}^{\nu} K_{n,\nu}^q(x, t, u) \big|_{\substack{u = \alpha_n \psi_n(t) \\ t=0}} = -[n] x q^{1-\nu} [D_{q,u}^{\nu-1} K_{n+m,\nu-1}^q(x, t, u) \big|_{\substack{u = \alpha_n \psi_n(t) \\ t=0}}]$ $(\nu, n = 1, 2, \ldots;$ $x \in [0, A])$, where m is a number such that $m + n$ is zero or a natural number.

According to these conditions, we define q-Ibragimov–Gadjiev operators as follows:

$$L_n(f; q; x) = \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} f\left(\frac{[\nu]}{[n]^2 \psi_n(0)} \right) \left[D_{q,u}^{\nu} K_{n,\nu}^q(x, t, u) \big|_{\substack{u = \alpha_n \psi_n(t) \\ t=0}} \right] \frac{(-q \alpha_n \psi_n(0))^{\nu}}{[\nu]!} \qquad (2.1)$$

for $x \in \mathbb{R}_+$ and any function f defined on $\mathbb{R}_+$.

By ν-multiple application of property (iv), $L_n(f;q;x)$ can be reduced to the form

$$L_n(f;q;x) = \sum_{\nu=0}^{\infty} f\left(\frac{[\nu]}{[n]^2 \psi_n(0)}\right) \frac{[n][n+m][n+2m]\cdots[n+(\nu-1)m]}{[\nu]!}$$
$$\times \left(xq\alpha_n\psi_n(0)\right)^\nu K_{n+\nu m,0}^q\left(x,0,\alpha_n\psi_n(0)\right). \tag{2.2}$$

Note that for $q = 1$, the operators $L_n(f;q;x)$ are classical Ibragimov–Gadjiev operators. It is easily verified that these operators are linear and positive from (iii).

In this section we give the following moment estimate, which is necessary to prove our theorems.

Lemma 1 *For $L_n(t^s;q;x)$, $s = 0, 1, 2$, one has*

$$L_n(1;q;x) = 1, \tag{2.3}$$

$$L_n(t;q;x) = \frac{q\alpha_n}{[n]}x, \tag{2.4}$$

$$L_n(t^2;q;x) = \left(\frac{q\alpha_n}{[n]}x\right)^2 \left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]}\right) + \left(\frac{q\alpha_n}{[n]}x\right)\frac{1}{[n]^2\psi_n(0)}. \tag{2.5}$$

Proof Using the definition (2.1) of $L_n(f;q;x)$ and from condition (ii), we have

$$L_n(1;q;x) = \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} D_{q,u}^\nu K_{n,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}} \frac{(-q\alpha_n\psi_n(0))^\nu}{[\nu]!}$$
$$= 1.$$

Therefore, we can see that the result can be verified for $s = 0$. Next we consider the case $s = 1$ as follows:

$$L_n(t;q;x) = \sum_{\nu=1}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{[\nu]}{[n]^2\psi_n(0)} \left[D_{q,u}^\nu K_{n,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^\nu}{[\nu]!}$$
$$= -\frac{q\alpha_n}{[n]^2} \sum_{\nu=1}^{\infty} q^{\frac{(\nu-2)(\nu-1)}{2}} q^{\nu-1} \left[D_{q,u}^\nu K_{n,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}.$$

By using conditions (ii) and (iv), we have

$$L_n(t;q;x) = \frac{q\alpha_n x}{[n]} \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \left[D_{q,u}^\nu K_{n+m,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^\nu}{[\nu]!}$$
$$= \frac{q\alpha_n x}{[n]}.$$

Finally, for $s = 2$,

$$L_n(t^2;q;x) = \sum_{\nu=1}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{[\nu]^2}{[n]^4\psi_n^2(0)} \left[D_{q,u}^\nu K_{n,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^\nu}{[\nu]!}$$
$$= -\frac{q\alpha_n}{[n]^4} \sum_{\nu=1}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{[\nu]}{\psi_n(0)} \left[D_{q,u}^\nu K_{n,\nu}^q(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$= -\frac{q\alpha_n}{[n]^4} \sum_{\nu=1}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{[\nu]-1}{\psi_n(0)} \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big] \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$- \frac{q\alpha_n}{[n]^4} \sum_{\nu=1}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{1}{\psi_n(0)} \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big] \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$= -\frac{q\alpha_n}{[n]^4} \sum_{\nu=2}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \frac{q[\nu-1]}{\psi_n(0)} \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big] \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$- \frac{q\alpha_n}{[n]^4\psi_n(0)} \sum_{\nu=1}^{\infty} q^{\frac{(\nu-2)(\nu-1)}{2}} q^{\nu-1} \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$= \frac{q^2\alpha_n^2}{[n]^4} \sum_{\nu=2}^{\infty} q^{\frac{(\nu-3)(\nu-2)}{2}} q^{2\nu-3} q \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-2}}{[\nu-2]!}$$

$$- \frac{q\alpha_n}{[n]^4} \frac{1}{\psi_n(0)} \sum_{\nu=1}^{\infty} q^{\frac{(\nu-2)(\nu-1)}{2}} q^{\nu-1} \big[D_{q,u}^{\nu} K_{n,\nu}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}.$$

Using conditions (ii) and (iv), we have

$$L_n(t^2;q;x) = -\frac{q^2\alpha_n^2 x}{[n]^3} \sum_{\nu=2}^{\infty} q^{\frac{(\nu-3)(\nu-2)}{2}} q^{\nu-1} \big[D_{q,u}^{\nu-1} K_{n+m,\nu-1}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-2}}{[\nu-2]!}$$

$$+ \frac{q\alpha_n x}{[n]^3\psi_n(0)} \sum_{\nu=1}^{\infty} q^{\frac{(\nu-2)(\nu-1)}{2}} \big[D_{q,u}^{\nu-1} K_{n+m,\nu-1}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-1}}{[\nu-1]!}$$

$$= -\frac{q^2\alpha_n^2 x}{[n]^3} \sum_{\nu=2}^{\infty} q^{\frac{(\nu-3)(\nu-2)}{2}} q^{\nu-1} \big[D_{q,u}^{\nu-1} K_{n+m,\nu-1}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-2}}{[\nu-2]!} + \frac{q\alpha_n x}{[n]^3\psi_n(0)}.$$

Using condition (iv) for the second time, we obtain

$$L_n(t^2;q;x) = \frac{q^2\alpha_n^2 [n+m]x^2}{[n]^3} \sum_{\nu=2}^{\infty} q^{\frac{(\nu-3)(\nu-2)}{2}} q \big[D_{q,u}^{\nu-2} K_{n+2m,\nu-2}^{q}(x,t,u)\big|_{\substack{u=\alpha_n\psi_n(t)\\t=0}}\big]$$

$$\times \frac{(-q\alpha_n\psi_n(0))^{\nu-2}}{[\nu-2]!} + \frac{q\alpha_n x}{[n]^3\psi_n(0)}$$

$$= \frac{q^3 \alpha_n^2 [n+m] x^2}{[n]^3} \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \left[D_{q,u}^{\nu} K_{n+2m,\nu}^{q}(x,t,u) \big|_{\substack{u=\alpha_n \psi_n(t) \\ t=0}} \right]$$

$$\times \frac{(-q\alpha_n \psi_n(0))^{\nu}}{[\nu]!} + \frac{q\alpha_n x}{[n]^3 \psi_n(0)}$$

$$= \frac{q^3 \alpha_n^2 [n+m] x^2}{[n]^3} + \frac{q\alpha_n x}{[n]^3 \psi_n(0)}$$

$$= \left(\frac{q\alpha_n x}{[n]} \right)^2 \frac{[n+m]q}{[n]} + \left(\frac{q\alpha_n x}{[n]} \right) \frac{1}{[n]^2 \psi_n(0)}$$

$$= \left(\frac{q\alpha_n x}{[n]} \right)^2 \left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]} \right) + \left(\frac{q\alpha_n x}{[n]} \right) \frac{1}{[n]^2 \psi_n(0)}. \qquad \square$$

If we denote kth order central moment of the operator (2.1) as $\mu_{n,k}(x) = L_n((t-x)^k; q; x)$, for abbreviation, we obtain the following from the equalities in (2.3)–(2.5) by direct computation:

$$\mu_{n,1}(x) = x \left(\frac{q\alpha_n}{[n]} - 1 \right), \tag{2.6}$$

$$\mu_{n,2}(x) = x^2 \left[\left(\frac{q\alpha_n}{[n]} \right)^2 \left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]} \right) - 2\frac{q\alpha_n}{[n]} + 1 \right]$$

$$+ \frac{xq\alpha_n}{[n]} \frac{1}{[n]^2 \psi_n(0)}. \tag{2.7}$$

Examining relations (2.4) and (2.5), it is observed that for $0 < q < 1$, one has $\lim_{n \to \infty} \frac{1}{[n]} = 1 - q$. This implies that the sequence of the operators in (2.1) does not satisfy the conditions of the Bohman–Korovkin-type theorem. For this reason, we consider a sequence (q_n), $0 < q_n < 1$, such that $\lim_{n \to \infty} q_n = 1$. We can now give the following Bohman–Korovkin-type theorem.

Theorem 1 *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n \to \infty} q_n = 1$. Then, for $f \in C[0,A]$, the sequence of $\{L_n(f; q_n; \cdot)\}$ converges uniformly to f on any closed interval $[0,A]$, where A is a fixed positive real number.*

Proof Replacing q by a sequence (q_n) with the given conditions, the result follows from (2.3)–(2.5) and the Bohman–Korovkin theorem. $\qquad \square$

3 Rate of convergence

Using the standard methods, we will estimate the rate of convergence in terms of the modulus of continuity and the Peetre-K functional.

Let $f \in C[0,A]$. The modulus of continuity of f denoted by $\omega(f; \delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and given by the relation

$$\omega(f; \delta) = \sup_{|t-x| \leq \delta} \left| f(t) - f(x) \right|, \quad t, x \in [0,A].$$

For $f \in C[0,A]$ and $\delta > 0$, the Peetre K-functional is defined by

$$K(f; \delta) = \inf_{g \in C^2[0,A]} \left\{ \|f - g\|_{C[0,A]} + \delta \|g\|_{C^2[0,A]} \right\}.$$

Here we note that

$$\|g\|_{C^2[0,A]} = \|g\|_{C[0,A]} + \|g'\|_{C[0,A]} + \|g''\|_{C[0,A]}.$$

Theorem 2 *Let* (q_n) *be a sequence of real numbers such that* $0 < q_n < 1$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f \in C[0,A]$, *then we have*

$$\left| L_n(f; q_n; x) - f(x) \right| \le 2\omega\left(f; \sqrt{\mu_{n,2}(x)}\right), \tag{3.1}$$

where $\mu_{n,2}(x)$ *is given by* (2.7).

Proof By using the well-known technique of Popoviciu, we have

$$\left| L_n(f; q_n; x) - f(x) \right| \le \omega(f; \delta)\left[1 + \frac{1}{\delta}\sqrt{\mu_{n,2}(x)}\right].$$

By choosing $\delta = \sqrt{\mu_{n,2}(x)}$, we obtain the quantitative estimate in (3.1). Since $\sqrt{\mu_{n,2}(x)} \to 0$ for $n \to \infty$, the estimation in (3.1) gives the rate of approximation for the operators (2.1). □

Theorem 3 *Let* (q_n) *be a sequence of real numbers such that* $0 < q_n < 1$ *and* $\lim_{n\to\infty} q_n = 1$. *If* $f \in C[0,A]$, *then*

$$\left\| L_n(f; q_n; \cdot) - f \right\|_{C[0,A]} \le 2K(f; \delta_n),$$

where

$$\delta_n = \frac{A}{2}\left(\frac{q_n\alpha_n}{[n]} - 1\right) + \frac{A^2}{4}\left[\left(\frac{q_n\alpha_n}{[n]}\right)^2\left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]}\right) - 2\frac{q_n\alpha_n}{[n]} + 1\right]$$
$$+ \frac{A}{4}\frac{q_n\alpha_n}{[n]}\frac{1}{[n]^2\psi_n(0)}.$$

Proof Let $g \in C^2[0,A]$, from Taylor's expansion we have

$$g(t) - g(x) = g'(x)(t - x) + \int_x^t g''(s)(t - s)\,ds.$$

Applying operators (2.1) on both sides of the above equation, we get

$$\left| L_n(g; q_n; x) - g(x) \right| = \left| L_n\left(g'(x)(t - x); q_n; x\right) + L_n\left(\int_x^t g''(s)(t - s)\,ds; q_n; x\right) \right|$$
$$\le \|g'\|_{C[0,A]}\left| L_n\left((t - x); q_n; x\right) \right| + \|g''\|_{C[0,A]}$$
$$\times \left| L_n\left(\int_x^t (t - s)\,ds; q_n; x\right) \right|.$$

Since

$$\int_x^t (t - s)\,ds = \frac{(t - x)^2}{2},$$

one obtains from (2.6) and (2.7)

$$\left|L_n(g;q_n;x) - g(x)\right| \le \|g'\|_{C[0,A]}|\mu_{n,1}(x)| + \frac{1}{2}\|g''\|_{C[0,A]}|\mu_{n,2}(x)|.$$

Finally we have

$$\left|L_n(g;q_n;x) - g(x)\right| \le \|g\|_{C^2[0,A]}\left\{x\left(\frac{q_n\alpha_n}{[n]} - 1\right) + \frac{x^2}{2}\left[\left(\frac{q_n\alpha_n}{[n]}\right)^2\left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]}\right)\right.\right.$$
$$\left.\left. - 2\frac{q_n\alpha_n}{[n]} + 1\right] + \frac{xq_n\alpha_n}{2[n]}\frac{1}{[n]^2\psi_n(0)}\right\}. \tag{3.2}$$

On the other hand, by the linearity property of (2.1), we get

$$\left|L_n(f;q_n;x) - f(x)\right| \le \left|L_n(f;q_n;x) - L_n(g;q_n;x)\right| + \left|L_n(g;q_n;x) - g(x)\right|$$
$$+ \left|g(x) - f(x)\right|$$
$$\le \|f - g\|_{C[0,A]}\left|L_n(1;q_n;x)\right| + \|f - g\|_{C[0,A]}$$
$$+ \left|L_n(g;q_n;x) - g(x)\right|.$$

From inequality (3.2), one has

$$\left|L_n(f;q_n;x) - f(x)\right| \le 2\|f - g\|_{C[0,A]} + \|g\|_{C^2[0,A]}\left\{x\left(\frac{q_n\alpha_n}{[n]} - 1\right)\right.$$
$$+ \frac{x^2}{2}\left[\left(\frac{q_n\alpha_n}{[n]}\right)^2\left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]}\right) - 2\frac{q_n\alpha_n}{[n]} + 1\right]$$
$$\left. + \frac{xq_n\alpha_n}{2[n]}\frac{1}{[n]^2\psi_n(0)}\right\}.$$

Taking the infimum on the right-hand side of the above inequality over all $g \in C^2[0,A]$, we obtain

$$\left\|L_n(f;q_n;\cdot) - f\right\|_{C[0,A]} \le 2K(f;\delta_n),$$

where

$$\delta_n = \frac{A}{2}\left(\frac{q_n\alpha_n}{[n]} - 1\right) + \frac{A^2}{4}\left[\left(\frac{q_n\alpha_n}{[n]}\right)^2\left(\frac{[n+m+1]}{[n]} - \frac{1}{[n]}\right) - 2\frac{q_n\alpha_n}{[n]} + 1\right]$$
$$+ \frac{A}{4}\frac{q_n\alpha_n}{[n]}\frac{1}{[n]^2\psi_n(0)}.$$

This completes the proof. □

4 An application of *q*-Ibragimov–Gadjiev operators

It should be emphasized that the operator (2.1), in a special case, consists of some well-known operators related to *q*-integers:

1. In the case

$$K_{n,v}^q(x,t,u) = \left(1 - \frac{q^{1-v}ux}{1+t}\right)_q^n,$$

the operator (2.1) assumes the form

$$L_n(f;q;x) = \sum_{v=0}^n f\left(\frac{[v]}{[n]^2\psi_n(0)}\right)\begin{bmatrix}n\\v\end{bmatrix}\left(xq\alpha_n\psi_n(0)\right)^v\left(1 - qx\alpha_n\psi_n(0)\right)_q^{n-v}. \tag{4.1}$$

Conditions (i)–(iv) are fulfilled and $m = -1$. For $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]}$, we have q-Bernstein polynomials

$$L_n^{\langle 1 \rangle}(f;q;x) = \sum_{v=0}^n f\left(\frac{[v]}{[n]}\right)\begin{bmatrix}n\\v\end{bmatrix}x^v(1-x)_q^v$$

which are introduced by Phillips [10].

2. In (4.1) for $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]b_n}$ ($\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$), the operators become q-Chlodowsky polynomials

$$L_n^{\langle 2 \rangle}(f;q;x) = \sum_{v=0}^n f\left(\frac{[v]}{[n]}b_n\right)\begin{bmatrix}n\\v\end{bmatrix}\left(\frac{x}{b_n}\right)^v\left(1 - \frac{x}{b_n}\right)_q^{n-v}$$

defined by Karslıand Gupta [14].

3. By choosing

$$K_{n,v}^q(x,t,u) = E_q\left(-[n]\left(t + q^{1-v}ux\right)\right),$$

conditions (i)–(iv) are fulfilled and $m = 0$. The operator (2.1) becomes

$$L_n(f;q;x) = \sum_{v=0}^{\infty} f\left(\frac{[v]}{[n]^2\psi_n(0)}\right)\frac{[n][n]\cdots[n]}{[v]!}$$
$$\times \left(qx\alpha_n\psi_n(0)\right)^v E_q\left(-[n]xq\alpha_n\psi_n(0)\right).$$

If we choose $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]b_n}$ ($\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$), we get the q-analogue of the classical Szász–Mirakjan operators

$$L_n^{\langle 3 \rangle}(f;q;x) = \sum_{v=0}^{\infty} f\left(\frac{[v]b_n}{[n]}\right)\frac{([n]x)^v}{[v]!(b_n)^v}E_q\left(-[n]\frac{x}{b_n}\right).$$

These operators were defined by Aral [11].

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