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# $N$-Fold Darboux transformation of the discrete Ragnisco–Tu system

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**Abstract**

In this paper, based on gauge transformation of Lax pairs, we construct an $N$-fold Darboux transformation for the discrete Ragnisco–Tu equation which is a typical member in the Ragnisco–Tu hierarchy. By using the $N$-fold Darboux transformation, new multi-soliton solutions of the discrete Ragnisco–Tu equation can be directly constructed starting from a seed solution.

**Keywords:** Discrete Ragnisco–Tu hierarchy; $N$-Fold Darboux transformation; Soliton solutions

## 1 Introduction

It is well known that constructing explicit solutions for an integrable system, whether continuous or discrete, plays an important role in describing and explaining nonlinear phenomena such as nonlinear optics effect, fusion reaction in plasma physics, superconduction phenomenon, magnetohydrodynamic phenomenology, etc. Furthermore, the investigation of integrable discrete systems and their related properties has always been important and has become a focus of recent research. In recent years, a great deal of progress has been made on the theory of discrete integrable systems. Lots of important nonlinear integrable differential-difference equations have been obtained [1–12]. In particular, constructing exact solutions for a differential-difference equation is one of the most fundamental and significant topics. There are some methods to construct solutions such as the inverse scattering transform method [13], the bilinear transformation method of Hirota [14], the Bäcklund and Darboux transformation techniques [15, 16], the Fokas unified approach [17], the long-time asymptotics approach [18], and so on. Among them, the Darboux transformation is the most effective technique to find explicit solutions of the integrable differential-difference equations [10, 11, 17–26]. This method based on Lax pairs has been proven to be one of the most fruitful algorithmic procedures to get explicit solutions of nonlinear evolution equations. In this paper, we construct an N-Fold Darboux transformation of Lax pair of the discrete Ragnisco–Tu equation, new multi-soliton solutions can be directly constructed starting from a seed solution.

The main purpose of the present paper is to construct the $N$-fold Darboux transformation of the following discrete Ragnisco–Tu equation [27, 28]:

$$\begin{cases} u_{n,t} = u_{n+1} - u_n^2 v_n, \\ v_{n,t} = u_n v_n^2 - v_{n-1}, \end{cases} \tag{1}$$

where $u_n$, $v_n$ are two potentials. System (1) is a member of the Ragnisco–Tu hierarchy and admits the following Lax pair:

$$E\psi_n = U_n \psi_n = \begin{pmatrix} \lambda + u_n v_n & u_n \\ v_n & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \varphi_2 \end{pmatrix}, \tag{2}$$

$$\frac{d\psi_n}{dt} = V_n \psi_n, \qquad V_n = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\lambda & u_n \\ v_{n-1} & -\frac{1}{2}\lambda \end{pmatrix}, \tag{3}$$

where the shift operator $E$ is defined as

$$Ef_n = f_{n+1}, \qquad E^{-1} f_n = f_{n-1} \quad (n \in Z),$$

and $\lambda$ is the spectral parameter and $\lambda_t = 0$. System (1) can be obtained from the following discrete zero-curvature equation:

$$U_{n,t} - (EV_n)U_n + U_n V_n = 0.$$

Therefore, in this paper, we construct the N-fold Darboux transformation for Eq. (1). Outline of this paper is organized as follows. In Sect. 2, based on its Lax pairs, the N-fold Darboux transformation is constructed for Eq. (1). In Sect. 3, as one of the applications of the N-fold Darboux transformation, we give some exact solutions of Eq. (1). Conclusions are made in the last section.

## 2 *N*-Fold DT of Ragnisco–Tu system

At present, much work has been done on a single Darboux transformation (DT) of the Lax integrable nonlinear integrable systems [29–32]. Here we would like to construct an $N$-fold DT for the Ragnisco–Tu system (1). For this purpose, we search for the following gauge transformation:

$$\tilde{\psi}_n = T_n \psi_n, \tag{4}$$

which can change the Lax pair (2) and (3) into

$$\widetilde{\psi}_{n+1} = \widetilde{U}_n \widetilde{\psi}_n, \qquad \widetilde{\psi}_{nt} = \widetilde{V}_n \widetilde{\psi}_n,$$

with

$$\widetilde{U}_n = T_{n+1} U_n T_n^{-1}, \qquad \widetilde{V}_n = (T_{nt} + T_n V_n) T_n^{-1}. \tag{5}$$

To make $\widetilde{U}_n$, $\widetilde{V}_n$ and $U_n$, $V_n$ have the same form respectively, we take $\phi_n = (\phi_{1n}, \phi_{2n})^T$, $\varphi_n = (\varphi_{1n}, \varphi_{2n})^T$ as two basic solutions of the Lax pair (2) and (3) and define a $2 \times 2$ matrix $T_n$ by

$$T_n = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{6}$$

where $T_{11}$, $T_{12}$, $T_{21}$, $T_{22}$ depend on variables $n$ and $t$:

$$\begin{aligned} T_{11} &= \lambda^N + \sum_{i=0}^{N-1} T_i^{11}(n)\lambda^i, \qquad T_{12} = \sum_{i=0}^{N-1} T_i^{12}(n)\lambda^i, \\ T_{21} &= \sum_{i=0}^{N-1} T_i^{21}(n)\lambda^i, \qquad T_{22} = \lambda^N + \sum_{i=0}^{N-1} T_i^{22}(n)\lambda^i, \end{aligned} \tag{7}$$

and $T_i^{11}(n)$, $T_i^{12}$, $T_i^{21}(n)$, $T_i^{22}(n)$ can be determined by the following linear algebraic system:

$$\begin{aligned} &\sum_{i=0}^{N-1} T_i^{11}(n)\lambda_j^i + \alpha_j \sum_{i=0}^{N-1} T_i^{12}(n)\lambda_j^i = -\lambda_j^N, \\ &\sum_{i=0}^{N-1} T_i^{21}(n)\lambda_j^i + \alpha_j \sum_{i=0}^{N-1} T_i^{22}(n)\lambda_j^i = -\alpha_j\lambda_j^N, \end{aligned} \tag{8}$$

with

$$\alpha_j(n) = \frac{\phi_{2n}(\lambda_j) - \gamma_j\varphi_{2n}(\lambda_j)}{\phi_{1n}(\lambda_j) - \gamma_j\varphi_{1n}(\lambda_j)} \quad (j = 1, 2, \ldots, 2N), \tag{9}$$

$\lambda_j$ and $\gamma_j$ ($\lambda_j \neq \lambda_k$, $\gamma_j \neq \gamma_k$, as $k \neq j$) are some parameters suitably chosen such that the determinants of coefficients for system (8) are nonzero.

From (7)–(9), it is easy to see that

$$\det T_n = T_{11}(\lambda_j)T_{22}(\lambda_j) - T_{12}(\lambda_j)T_{21}(\lambda_j), \tag{10}$$

where $\det T_n$ is a $(2N)$*th-degree* polynomial in $\lambda$.

On the other hand, from (8) we have

$$T_{11}(\lambda_j) = -\alpha_j T_{12}(\lambda_j), \qquad T_{21}(\lambda_j) = -\alpha_j T_{22}(\lambda_j).$$

Therefore,

$$\det T_n(\lambda_j) = 0,$$

which implies that $\lambda_j$ ($1 \leq j \leq 2N$) are $2N$ roots of $\det T_n$, that is,

$$\det T_n(\lambda) = \prod_{j=1}^{2N}(\lambda - \lambda_j). \tag{11}$$

By using the above fact, we can prove the following proposition.

**Proposition 1** *The matrix $\widetilde{U}_n$ defined by* (5) *has the same form as $U_n$, that is,*

$$\tilde{U}_n = \begin{pmatrix} \lambda + \tilde{u}_n\tilde{v}_n & \tilde{u}_n \\ \tilde{v}_n & 1 \end{pmatrix},$$

*in which the transformation formulae between old and new potentials are defined by*

$$\tilde{u}_n = u_n - T_{N-1}^{12}(n), \qquad \tilde{v}_n = v_n + T_{N-1}^{21}(n+1), \tag{12}$$

*the transformation $(\psi_n, u_n, v_n) \to (\tilde{\psi}_n, \tilde{u}_n, \tilde{v}_n)$ is usually called a DT of the spectral problem* (2).

*Proof* Let $T_n^{-1} = T_n^* / \det T_n$ and

$$T_{n+1}U_nT_n^* = \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}.$$

It is easy to see that $f_{11}(\lambda, n)$ and $f_{22}(\lambda, n)$ are $(2N+1)$*th-degree* polynomials in $\lambda$, $f_{12}(\lambda, n)$ and $f_{21}(\lambda, n)$ are $2N$*th-degree* polynomials in $\lambda$, respectively. From (2) and (9), we can find that

$$\alpha_j(n+1) = \frac{\mu_j(n)}{\nu_j(n)}, \tag{13}$$

where $\mu_j(n) = v_n + \alpha_j(n)$, $\nu_j(n) = \lambda_j + u_nv_n + u_n\alpha_j(n)$ $(j = 1, 2, \ldots, 2N)$.

By virtue of (11) and (13), it can be verified that $\lambda_j$ $(j = 1, 2, \ldots, 2N)$ are roots of $f_{k,l}(\lambda, n)$ $(k, l = 1, 2)$. Again noticing (8), we can conclude that

$$\det T_n | f_{k,l}(\lambda, n) \quad (k, l = 1, 2).$$

Therefore, we have

$$T_{n+1}U_nT_n^* = (\det T_n)P_n = (\det T_n)\begin{pmatrix} p_{11}{}^{(1)}\lambda + p_{11}{}^{(0)} & p_{12}{}^{(0)} \\ p_{21}{}^{(0)} & p_{22}{}^{(1)}\lambda + p_{22}{}^{(0)} \end{pmatrix},$$

where $p_{ij}^{(l)}$ $(i, j = 1, 2; l = 0, 1)$ are independent of $\lambda$. At the same time, the above equation can be rewritten as

$$T_{n+1}U_n = P_nT_n. \tag{14}$$

By comparing the coefficients of $\lambda^{N+1}$, $\lambda^N$ in (14), we obtain

$$\begin{aligned}
&p_{11}^{(1)} = p_{22}^{(0)} = 1, \qquad p_{22}^{(1)} = 0, \\
&p_{11}^{(0)} = u_nv_n + T_{N-1}^{11}(n+1) - T_{N-1}^{11}(n) = \tilde{u}_n\tilde{v}_n, \\
&p_{12}^{(0)} = u_n - T_{N-1}^{12}(n) = \tilde{u}_n, \\
&p_{21}^{(0)} = v_n + T_{N-1}^{21}(n+1) = \tilde{v}_n.
\end{aligned}$$

Thus we complete the proof. □

**Proposition 2** *Under transformation* (4), *the matrix* $\tilde{V}_n$ *defined by* (5) *has the same form as* $V_n$, *that is*,

$$\tilde{V}_n = \begin{pmatrix} \frac{1}{2}\lambda & \tilde{u}_n \\ \tilde{v}_{n-1} & -\frac{1}{2}\lambda \end{pmatrix},$$

*the old potentials* $u_n$ *and* $v_n$ *are mapped into new* $\tilde{u}_n$ *and* $\tilde{v}_n$ *according to the same DT Eq*. (4).

*Proof* Let $T_n^{-1} = T_n^*/\det T_n$ and

$$(T_{nt} + T_n V_n)T_n^* = \begin{pmatrix} g_{11}(\lambda, n) & g_{12}(\lambda, n) \\ g_{21}(\lambda, n) & g_{22}(\lambda, n) \end{pmatrix}.$$

With a direct calculation we know that $g_{11}(\lambda, n), g_{22}(\lambda, n)$ or $g_{12}(\lambda, n), g_{21}(\lambda, n)$ are $(2N + 1)$th or $2N$th polynomials in $\lambda$, respectively. It can be checked that $\lambda_j$ $(j = 1, 2, \dots, 2N)$ are roots of $g_{k,l}(\lambda, n)$ $(k, l = 1, 2)$. And the matrix $(T_{nt} + T_n V_n)T_n^*$ is written as

$$(T_{nt} + T_n V_n)T_n^* = (\det T_n)R_n = (\det T_n)\begin{pmatrix} r_{11}^{(1)}\lambda + r_{11}^{(0)} & r_{12}^{(0)} \\ r_{21}^{(0)} & r_{22}^{(1)}\lambda + r_{22}^{(0)} \end{pmatrix},$$

namely

$$T_{nt} + T_n V_n = R_n T_n, \tag{15}$$

where $r_{ij}^l$ $(i, j = 1, 2;\ l = 0, 1)$ are independent of $\lambda$. Comparing the coefficients of $\lambda^i$ $(i = N + 1, N, N - 1)$ in Eq. (15), we gain the following formulas:

$$\begin{aligned} &r_{11}^{(1)} = -r_{22}^{(1)} = \frac{1}{2}, \qquad r_{11}^{(0)} = r_{22}^{(0)} = 0, \\ &r_{12}^{(0)} = u_n - T_{N-1}^{12}(n) = \tilde{u}_n, \\ &r_{21}^{(0)} = v_{n-1} + T_{N-1}^{21}(n) = \tilde{v}_{n-1}. \end{aligned}$$

The proof is thus completed. □

From the fact of equivalence between differential-difference Eq. (5) and the discrete zero-curvature equation $\tilde{U}_{nt} - \tilde{V}_{n+1}\tilde{U}_n + \tilde{U}_n\tilde{V}_n = 0$, with the help of Proposition 1 and Proposition 2, we obtain the following proposition.

**Theorem 1** *The solutions* $(u_n, v_n)$ *of the Ragnisco–Tu system* (1) *are mapped into new solutions* $(\tilde{u}_n, \tilde{v}_n)$ *under the N-fold DT* (4) *and* (12): $(\psi_n, u_n, v_n) \to (\tilde{\psi}_n, \tilde{u}_n, \tilde{v}_n)$.

## 3 Applications of DT and exact solutions

In this section, we will give some exact solutions of the Ragnisco–Tu system (1) via transformations (4) and (12). Substituting the trivial solution $u_n = 1$, $v_n = 1$ into the Lax pair (2) and (3), we can give two real basic solutions as follows:

$$\phi_n = \left(\frac{\lambda_i + 2 + \sqrt{\lambda_i^2 + 4}}{2}\right)^n \exp\left(\frac{\sqrt{\lambda_i^2 + 4}}{2} t\right) \begin{pmatrix} 1 \\ \frac{\sqrt{\lambda_i^2+4}-\lambda_i}{2} \end{pmatrix}, \tag{16}$$

$$\varphi_n = \left(\frac{\lambda_i + 2 - \sqrt{\lambda_i^2 + 4}}{2}\right)^n \exp\left(\frac{-\sqrt{\lambda_i^2 + 4}}{2} t\right) \begin{pmatrix} 1 \\ -\frac{\lambda_i+\sqrt{\lambda_i^2+4}}{2} \end{pmatrix}. \tag{17}$$

According to (9), we have

$$\alpha_i(n) = \frac{\nabla_i \frac{\sqrt{\lambda_i^2+4}-\lambda_i}{2} + \gamma_i \frac{\sqrt{\lambda_i^2+4}+\lambda_i}{2}}{\nabla_i - \gamma_i}, \tag{18}$$

$$\alpha_i(n+1) = \frac{\nabla_i \frac{\sqrt{\lambda_i^2+4}-\lambda_i+2}{2} + \gamma_i \frac{\sqrt{\lambda_i^2+4}+\lambda_i-2}{2}}{\nabla_i \frac{\sqrt{\lambda_i^2+4}+\lambda_i+2}{2} + \gamma_i \frac{\sqrt{\lambda_i^2+4}-\lambda_i-2}{2}}, \tag{19}$$

where $\nabla_i = \frac{(\lambda_i+2+\sqrt{\lambda_i^2+4})^{2n}}{(4\lambda_i)^n} \exp(\sqrt{\lambda_i^2 + 4}t)$, $i = 1, 2, \ldots, 2N$.

Solving the linear algebraic system (8) by using Cramer's rule, we have

$$T_{N-i}^{11} = \frac{\Delta T_{N-i}^{11}}{\Delta_1}, \qquad T_{N-i}^{12} = \frac{\Delta T_{N-i}^{12}}{\Delta_1} \tag{20}$$

with

$$\Delta_1 = \begin{vmatrix} \lambda_1^{N-1} & \lambda_1^{N-2} & \ldots & \lambda_1 & 1 & \alpha_1\lambda_1^{N-1} & \alpha_1\lambda_1^{N-2} & \cdots & \alpha_1\lambda_1 & \alpha_1 \\ \lambda_2^{N-1} & \lambda_2^{N-2} & \ldots & \lambda_2 & 1 & \alpha_2\lambda_2^{N-1} & \alpha_2\lambda_2^{N-2} & \cdots & \alpha_2\lambda_2 & \alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & \ldots & \ldots \\ \lambda_{2N}^{N-1} & \lambda_{2N}^{N-2} & \ldots & \lambda_{2N} & 1 & \alpha_{2N}\lambda_{2N}^{N-1} & \alpha_{2N}\lambda_{2N}^{N-2} & \cdots & \alpha_{2N}\lambda_{2N} & \alpha_{2N} \end{vmatrix},$$

and $\Delta T_{N-i}^{11}(n)$ is produced from $\Delta_1$ by replacing its $i$th column with $(-\lambda_1^N, -\lambda_2^N, \ldots, -\lambda_{2N}^N)$, $\Delta T_{N-i}^{12}(n)$ is produced from $\Delta_1$ by replacing its $(N+i)$th column with $(-\lambda_1^N, -\lambda_2^N, \ldots, -\lambda_{2N}^N)$ where $i = 1, 2, \ldots, N$. Similarly, we have

$$T_{N-i}^{21} = \frac{\Delta T_{N-i}^{21}}{\Delta_2}, \qquad T_{N-i}^{22} = \frac{\Delta T_{N-i}^{22}}{\Delta_2} \tag{21}$$

with

$$\Delta_2 = \begin{vmatrix} \lambda_1^{N-1} & \lambda_1^{N-2} & \ldots & \lambda_1 & 1 & \alpha_1\lambda_1^{N-1} & \alpha_1\lambda_1^{N-2} & \cdots & \alpha_1\lambda_1 & \alpha_1 \\ \lambda_2^{N-1} & \lambda_2^{N-2} & \ldots & \lambda_2 & 1 & \alpha_2\lambda_2^{N-1} & \alpha_2\lambda_2^{N-2} & \cdots & \alpha_2\lambda_2 & \alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \cdots & \ldots & \ldots \\ \lambda_{2N}^{N-1} & \lambda_{2N}^{N-2} & \ldots & \lambda_{2N} & 1 & \alpha_{2N}\lambda_{2N}^{N-1} & \alpha_{2N}\lambda_{2N}^{N-2} & \cdots & \alpha_{2N}\lambda_{2N} & \alpha_{2N} \end{vmatrix},$$

and $\Delta T_{N-i}^{21}(n)$ is produced from $\Delta_2$ by replacing its $i$th column with $(-\alpha_1\lambda_1^N, -\alpha_2\lambda_2^N, \ldots, -\alpha_{2N}\lambda_{2N}^N)$, $\Delta T_{N-i}^{22}(n)$ is produced from $\Delta_2$ by replacing its $(N+i)$th column with $(-\alpha_1\lambda_1^N, -\alpha_2\lambda_2^N, \ldots, -\alpha_{2N}\lambda_{2N}^N)$, where $i = 1, 2, \ldots, N$.

On the basis of (19) and (20), we can derive a new solution to system (1), here we discus three cases when $N = 1, 2, 3$.

*(I)* When $N = 1$, let $\lambda = \lambda_i$ $(i = 1, 2)$. Solving the linear system (8) leads to

$$T_0^{12}(n) = \frac{\Delta T_0^{12}(n)}{\Delta_1}, \qquad T_0^{21}(n) = \frac{\Delta T_0^{21}(n)}{\Delta_2} \tag{22}$$

with

$$\Delta_1 = \Delta_2 = \begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix}, \qquad T_0^{12}(n) = \begin{vmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{vmatrix}, \qquad T_0^{21}(n) = \begin{vmatrix} -\alpha_1\lambda_1 & \alpha_1 \\ -\alpha_2\lambda_2 & \alpha_2 \end{vmatrix}.$$

Hence, an explicit solution of Eq. (1) is obtained as follows:

$$\tilde{u}_n = 1 - \frac{2(\lambda_1 - \lambda_2)(\triangledown_1 - \gamma_2)(\triangledown_2 - \gamma_2)}{\blacktriangledown_1}, \tag{23}$$

where

$$\begin{aligned} \blacktriangledown_1 = {} & \triangledown_1\triangledown_2\Big(\sqrt{4+\lambda_2^2} - \sqrt{4+\lambda_1^2} + \lambda_1 - \lambda_2\Big) + \triangledown_1\gamma_2\Big(\sqrt{4+\lambda_2^2} + \sqrt{4+\lambda_1^2} + \lambda_2 - \lambda_1\Big) \\ & + \triangledown_2\gamma_1\Big(-\sqrt{4+\lambda_2^2} - \sqrt{4+\lambda_1^2} + \lambda_2 - \lambda_1\Big) + \gamma_1\gamma_2\Big(\sqrt{4+\lambda_1^2} - \sqrt{4+\lambda_2^2} + \lambda_1 - \lambda_2\Big) \end{aligned}$$

and

$$\tilde{v}_n = 1 + \frac{(\lambda_2 - \lambda_1)\blacktriangle_1\blacktriangle_2}{\blacktriangledown_2}, \tag{24}$$

where

$$\begin{aligned} \blacktriangle_1 &= \triangledown_2\Big(\sqrt{\lambda_2^2+4} - \lambda_2 + 2\Big) + \gamma_2\Big(\sqrt{\lambda_2^2+4} + \lambda_2 - 2\Big); \\ \blacktriangle_2 &= \triangledown_1\Big(\sqrt{\lambda_1^2+4} - \lambda_1 + 2\Big) + \gamma_1\Big(\sqrt{\lambda_1^2+4} + \lambda_1 - 2\Big); \\ \blacktriangledown_2 &= 2\lambda_1\sqrt{\lambda_2^2+4}(\triangledown_1\triangledown_2 - \triangledown_2\gamma_1 + \triangledown_1\gamma_2 - \gamma_1\gamma_2) \\ &\quad + 2\lambda_2\sqrt{\lambda_1^2+4}(-\triangledown_1\triangledown_2 - \triangledown_2\gamma_1 + \triangledown_1\gamma_2 + \gamma_1\gamma_2) \\ &\quad + 4(\lambda_1 - \lambda_2)(\triangledown_1\triangledown_2 - \triangledown_2\gamma_1 - \triangledown_1\gamma_2 + \gamma_1\gamma_2). \end{aligned}$$

*(II)* When $N = 2$, let $\lambda = \lambda_i$ $(i = 1, 2, 3, 4)$. Solving the linear system (8) leads to

$$T_1^{12}(n) = \frac{\Delta T_1^{12}(n)}{\Delta_1}, \qquad T_1^{21}(n) = \frac{\Delta T_1^{21}(n)}{\Delta_2} \tag{25}$$

with

$$\Delta_1 = \Delta_2 = \begin{vmatrix} \lambda_1 & 1 & \alpha_1 & \alpha_1\lambda_1 \\ \lambda_2 & 1 & \alpha_2 & \alpha_2\lambda_2 \\ \lambda_3 & 1 & \alpha_3 & \alpha_3\lambda_3 \\ \lambda_4 & 1 & \alpha_4 & \alpha_4\lambda_4 \end{vmatrix},$$

$$T_1^{12}(n) = \begin{vmatrix} \lambda_1 & 1 & \alpha_1 & -\lambda_1^2 \\ \lambda_2 & 1 & \alpha_2 & -\lambda_2^2 \\ \lambda_3 & 1 & \alpha_3 & -\lambda_3^2 \\ \lambda_4 & 1 & \alpha_4 & -\lambda_4^2 \end{vmatrix}, \qquad T_1^{21}(n) = \begin{vmatrix} -\alpha_1\lambda_1^2 & 1 & \alpha_1 & \alpha_1\lambda_1 \\ -\alpha_2\lambda_2^2 & 1 & \alpha_2 & \alpha_2\lambda_2 \\ -\alpha_3\lambda_3^2 & 1 & \alpha_3 & \alpha_3\lambda_3 \\ -\alpha_4\lambda_4^2 & 1 & \alpha_4 & \alpha_4\lambda_4 \end{vmatrix}.$$

Therefore, an explicit solution of Eq. (1) is obtained as follows:

$$\tilde{u}_n = 1 - \frac{\Delta_1 T_1^{12}(n)}{\Delta_1(n)}, \tag{26}$$

and

$$\tilde{v}_n = 1 + \frac{\Delta_2 T_1^{21}(n+1)}{\Delta_2(n+1)}. \tag{27}$$

*(III)* When $N = 3$, let $\lambda = \lambda_i$ $(i = 1, 2, 3, 4, 5, 6)$. Solving the linear system (8) leads to

$$T_1^{12}(n) = \frac{\Delta T_1^{12}(n)}{\Delta_1}, \qquad T_1^{21}(n) = \frac{\Delta T_1^{21}(n)}{\Delta_2} \tag{28}$$

with

$$\Delta_1 = \Delta_2 = \begin{vmatrix} \lambda_1^2 & \lambda_1 & 1 & \alpha_1 & \alpha_1\lambda_1 & \alpha_1\lambda_1^2 \\ \lambda_2^2 & \lambda_2 & 1 & \alpha_2 & \alpha_2\lambda_2 & \alpha_2\lambda_2^2 \\ \lambda_3^2 & \lambda_3 & 1 & \alpha_3 & \alpha_3\lambda_3 & \alpha_3\lambda_3^2 \\ \lambda_4^2 & \lambda_4 & 1 & \alpha_4 & \alpha_4\lambda_4 & \alpha_4\lambda_4^2 \\ \lambda_5^2 & \lambda_5 & 1 & \alpha_5 & \alpha_5\lambda_5 & \alpha_5\lambda_5^2 \\ \lambda_6^2 & \lambda_6 & 1 & \alpha_6 & \alpha_6\lambda_6 & \alpha_6\lambda_6^2 \end{vmatrix},$$

$$T_1^{12}(n) = \begin{vmatrix} \lambda_1^2 & \lambda_1 & 1 & \alpha_1 & \alpha_1\lambda_1 & -\lambda_1^2 \\ \lambda_2^2 & \lambda_2 & 1 & \alpha_2 & \alpha_2\lambda_2 & -\lambda_2^2 \\ \lambda_3^2 & \lambda_3 & 1 & \alpha_3 & \alpha_3\lambda_3 & -\lambda_3^2 \\ \lambda_4^2 & \lambda_4 & 1 & \alpha_4 & \alpha_4\lambda_4 & -\lambda_4^2 \\ \lambda_5^2 & \lambda_5 & 1 & \alpha_5 & \alpha_5\lambda_5 & -\lambda_5^2 \\ \lambda_6^2 & \lambda_6 & 1 & \alpha_6 & \alpha_6\lambda_6 & -\lambda_6^2 \end{vmatrix},$$

$$T_1^{21}(n) = \begin{vmatrix} -\alpha_1\lambda_1^2 & \lambda_1 & 1 & \alpha_1 & \alpha_1\lambda_1 & \alpha_1\lambda_1^2 \\ -\alpha_2\lambda_2^2 & \lambda_2 & 1 & \alpha_2 & \alpha_2\lambda_2 & \alpha_2\lambda_2^2 \\ -\alpha_3\lambda_3^2 & \lambda_3 & 1 & \alpha_3 & \alpha_3\lambda_3 & \alpha_3\lambda_3^2 \\ -\alpha_4\lambda_4^2 & \lambda_4 & 1 & \alpha_4 & \alpha_4\lambda_4 & \alpha_4\lambda_4^2 \\ -\alpha_5\lambda_5^2 & \lambda_5 & 1 & \alpha_5 & \alpha_5\lambda_5 & \alpha_5\lambda_5^2 \\ -\alpha_6\lambda_6^2 & \lambda_6 & 1 & \alpha_6 & \alpha_6\lambda_6 & \alpha_6\lambda_6^2 \end{vmatrix}.$$

Therefore, an explicit solution of Eq. (1) is obtained as follows:

$$\tilde{u}_n = 1 - \frac{\Delta_1 T_1^{12}(n)}{\Delta_1(n)}, \tag{29}$$

and

$$\tilde{v}_n = 1 + \frac{\Delta_2 T_1^{21}(n+1)}{\Delta_2(n+1)}. \tag{30}$$

## 4 Conclusions and remarks

In this paper, we have constructed the $N$-fold Darboux transformation for the discrete Ragnisco–Tu system. As an application of the $N$-fold Darboux transformation, we provide three explicit cases of the forms of the exact solutions when $N = 1$, $N = 2$, and $N = 3$. Moreover, if these resulting solutions are taken as new starting points, we may make a Darboux transformation once again and obtain another set of new explicit solutions. This process can be done continuously, and multi-soliton solutions will usually be obtained.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

## Publisher's Note



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