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On the $\alpha\beta$-statistical convergence of the modified discrete operators

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Abstract

The concept of $\alpha\beta$-statistical convergence was introduced by Aktuğlu (J. Comput. Appl. Math. 259:174–181, 2014). In this paper, we apply $\alpha\beta$-statistical convergence to investigate modified discrete operator approximation properties.

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1 Introduction and preliminaries

The notion of statistical convergence was introduced by Fast [2] and Steinhaus [3] independently in the same year 1951 as follows.

Let $K \subset \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n \frac{|K_n|}{n}$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$ or $\lim_n \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n} = 0$. We write st-$\lim x_k = L$.

Statistical convergence is a generalization of concept of ordinary convergence. So, every convergent sequence is statistically convergent, but not conversely. For example, let

$$x_k = \begin{cases} 1, & k = m^2, \\ 0, & k \neq m^2, \end{cases} \qquad m = 1, 2, 3, \ldots.$$

Then, st-$\lim x_k = 0$, but (x_k) is not convergent.

Approximation theory has important applications in the theory of polynomial approximation, various areas of functional analysis, and numerical solutions of differential and integral equations. In the recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved.

Gadjiev and Orhan [4] studied a Korovkin-type approximation theorem by using the notion of statistical convergence for the first time in 2002. Later, generalizations and applications of this concept have been investigated by various authors [5–11].

Aktuğlu [1] introduced $\alpha\beta$-statistical convergence as follows. Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

P_1: α and β are both nondecreasing,

P_2: $\beta(n) \geq \alpha(n)$,

P_3: $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$.

Let Λ denote the set of pairs (α, β) satisfying P_1, P_2, P_3.

For a pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$, and $K \subset \mathbb{N}$, we define

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{|K \cap P_n^{\alpha,\beta}|}{(\beta(n) - \alpha(n) + 1)^{\gamma}},$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$, and $|S|$ represents the cardinality of S.

Definition 1.1 ([1]) A sequence x is said to be $\alpha\beta$-statistically convergent of order γ to L, denoted by $\mathrm{st}^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L$ if for every $\varepsilon > 0$,

$$\delta^{\alpha,\beta}\bigl(\{k : |x_k - L| \geq \varepsilon\}, \gamma\bigr) = \lim_{n\to\infty} \frac{|\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\}|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

For $\gamma = 1$, we say that x is $\alpha\beta$-statistically convergent to L, and this is denoted by $\mathrm{st}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L$.

Let X be a compact subset of $\mathbb{R}$, and let $0 < \gamma \leq 1$; then we can consider the following definition for a sequence of functions $f_r : X \to \mathbb{R}$.

Definition 1.2 A sequence of functions f_r is said to be $\alpha\beta$-statistically uniformly convergent to f on X of order γ and denoted by $f_k \rightrightarrows f(\alpha\beta^{\gamma}\text{-stat})$ if for every $\varepsilon > 0$,

$$\delta^{\alpha,\beta}\bigl(\{k : \|f_k(x) - f(x)\|_{C(X)} \geq \varepsilon\}, \gamma\bigr) = \lim_{n\to\infty} \frac{|\{k \in P_n^{\alpha,\beta} : \|f_k(x) - f(x)\|_{C(X)} \geq \varepsilon\}|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

Theorem 1.3 ([1]) *Let $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$, and let $L_n : C(X) \to C(X)$ be a sequence of positive linear operators satisfying*

$$L_n(e_v, x) \rightrightarrows f\bigl(\alpha\beta^{\gamma}\text{-}stat\bigr), \quad v = 0, 1, 2.$$

Then for all $f \in C(X)$,

$$L_n(f, x) \rightrightarrows f\bigl(\alpha\beta^{\gamma}\text{-}stat\bigr).$$

Throughout this paper, K represents a compact subinterval of $\mathbb{R}^+$, and e_j stands for $e_j(t) = t^j$, $j \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Agratini [12] investigated a general class of positive approximation processes of discrete type expressed by series and modified them into finite sums. Agratini [12] defined the operator

$$L_n(f; x) = \sum_{k=0}^{\infty} \phi_{n,k}(x) f(x_{n,k}), \quad x \geq 0, f \in F, \tag{1}$$

where F stands for the domain of L_n containing the set of all continuous functions on $\mathbb{R}^+$ for which the series in (1) is convergent, by using the following three requirements:

For each $n \in \mathbb{N}$:

(i) For every $k \in \mathbb{N}_0$, there exists a sequence of γ_k such that $x_{n,k} = O(n^{-\gamma_k})$ $(n \to \infty)$ a net on $\mathbb{R}^+$, $\Delta_n = (x_{n,k})_{k\geq 0}$ is fixed.

(ii) A sequence $(\phi_{n,k})_{k\geq 0}$ is given, where $\phi_{n,k} \in C'(\mathbb{R}^+)$ and $C'(\mathbb{R}^+)$ is the space of all real-valued functions continuously differentiable in $\mathbb{R}^+$. This sequence satisfies the following conditions:

$$\phi_{n,k} \geq 0, k \in \mathbb{N}_0, \qquad \sum_{k=0}^{\infty} \phi_{n,k}(x) = e_0, \qquad \sum_{k=0}^{\infty} \phi_{n,k}(x) x_{n,k} = e_1. \tag{2}$$

(iii) There exists a positive function $\psi \in \mathbb{R}^{\mathbb{N}\times\mathbb{R}^+}$, $\psi \in C(\mathbb{R}^+)$, with the property

$$\psi(n,x)\phi'_{n,k}(x) = (x_{n,k} - x)\phi_{n,k}(x), \quad k \in \mathbb{N}_{\nvdash}, x \geq 0. \tag{3}$$

Agratini [12] indicated the following technical result.

Lemma 1.4 *Let $L_n(f;x) = \sum_{k=0}^{\infty} \phi_{n,k}(x) f(x_{n,k})$, $x \geq 0, f \in F$, and let $\zeta_{n,r}$ be the rth central moment of L_n. For every $x \in \mathbb{R}^+$, we have the following identities*:

$$\zeta_{n,0}(x) = 1, \qquad \zeta_{n,1}(x) = 0, \tag{4}$$

$$\zeta_{n,r+1}(x) = \psi(n,x)\big(\zeta'_{n,r}(x) + r\zeta_{n,r+1}(x)\big), \quad r \in \mathbb{N}, \tag{5}$$

$$\zeta_{n,2}(x) = \psi(n,x). \tag{6}$$

In this paper, we present $\alpha\beta$-statistical convergence approximation properties of the operator investigated by Agratini [12].

2 Main results

Theorem 2.1 *Let $L_n(f;x) = \sum_{k=0}^{\infty} \phi_{n,k} f(x_{n,k})$. If $st_{\alpha\beta}^{\gamma}$-$\lim_{n\to\infty} \psi(n,x) = 0$ uniformly on K, then for every $f \in F$, we have $st_{\alpha\beta}^{\gamma}$-$\lim_{n\to\infty} \|L_n(f;x) - f(x)\| = 0$.*

Proof Due to

$$L_n(e_0;x) = \sum_{k=0}^{\infty} \phi_{n,k} = e_0,$$

$$L_n(e_1;x) = \sum_{k=0}^{\infty} \phi_{n,k} x_{n,k} = e_1,$$

we can obtain that

$$\mathrm{st}_{\alpha\beta}^{\gamma}\text{-}\lim_{n\to\infty} \big\| L_n(e_0;x) - e_0 \big\| = 0,$$

$$\mathrm{st}_{\alpha\beta}^{\gamma}\text{-}\lim_{n\to\infty} \big\| L_n(e_1;x) - e_1 \big\| = 0.$$

We know from [12] that $\zeta_{n,r}(x) = L_n((e_1 - xe_0)^r; x)$, $r \in \mathbb{N}_0$. If we choose $r = 2$, then we can write $\psi(n,x) = \zeta_{n,2}(x) = L_n((e_1 - xe_0)^2; x)$. Since L_n is a linear operator, we can easily see that $\psi(n,x) = L_n(e_2;x) - x^2$. So, $\|\psi(n,x)\|_{C(K)} = \|L_n(e_2;x) - x^2\|_{C(K)}$. Since $\mathrm{st}_{\alpha\beta}^{\gamma}$-$\lim_{n\to\infty} \psi(n,x) =$

0 uniformly on K, we have $\|\psi(n,x)\|_{C(K)} = \|L_n(e_2;x) - x^2\|_{C(K)}$. Then from Theorem 1.3 we obtain that $\mathrm{st}^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} \|L_n(f;x) - f(x)\|_{C(K)} = 0$. □

We give some information to investigate $\alpha\beta$-statistical approximation properties of modified discrete operators defined by Agratini [12]. If we specialize the net Δ_n and the function ψ, then we consider that a positive sequence $(a_n)_{n\geq 1}$ and the function $\psi_i \in C(\mathbb{R}^+)$, $i = 1, 2, \ldots, l$, exist such that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} x_{n,k} &= \frac{k}{a_n} \leq k, \quad k \in \mathbb{N}, \text{with } \lim_{n\to\infty} \frac{1}{a_n} = 0, \\ \psi(n,x) &= \sum_{i=1}^{l} \frac{\psi_i(x)}{a_n^i}, \quad x \geq 0. \end{aligned} \tag{7}$$

Under these assumptions, the requirement of Theorem 2.1 is fulfilled. Starting from (1), under the additional assumptions (7), Agratini defined

$$L_{n,\delta}(f;x) = \sum_{k=0}^{[a_n(x+\delta(n))]} \phi_{n,k} f\left(\frac{k}{a_n}\right), \quad x \geq 0, f \in F, \tag{8}$$

where $\delta = (\delta(n))_{n\geq 1}$ is a sequence of positive numbers. The study of these operators was developed in polynomial weighted spaces connected to the weights ω_m, $m \in \mathbb{N}_0$, $\omega_m(x) = \frac{1}{1+x^{2m}}$, $x \geq 0$. For every $m \in \mathbb{N}_0$, the spaces $E_m := \{f \in \mathbb{C}(R^+) : \|f\|_m := \sup_{x\geq 0} \omega_m(x)|f(x)| < \infty\}$ are endowed with the norm $\|\cdot\|_m$.

Lemma 2.2 ([12]) *Let L_n, $n \in \mathbb{N}$, be defined by (1), and let assumptions (7) be fufilled. If $\psi_i \in C^{2m-2}(\mathbb{R}^+)$, $i = 1, 2, 3, \ldots, l$, then the central moment of $(2m)$th order satisfies*

$$\zeta_{n,2m}(x) \leq \frac{C(m,K)}{a_n}, \quad x \in K, \tag{9}$$

where $C(m,K)$ is a constant depending only on m and the compact set $K \subset \mathbb{R}^+$.

Theorem 2.3 *Let $L_{n,\delta}(f;x) = \sum_{k=0}^{[a_n(x+\delta(n))]} \phi_{n,k} f(\frac{k}{a_n})$ be defined by [13] If $\psi_i \in C^{2m-2}(\mathbb{R}^+)$, $i = 1, 2, \ldots, l$, and $st^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} \sqrt{a_n}\delta(n) = 0$, then $st^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} \|L_{n,\delta}(f;x) - f(x)\|_{C(K)} = 0$, for every $f \in E_m \cap F$.*

Proof To prove Theorem 2.3, we need the elementary inequality

$$t^{2m} \leq 2^{2m-1}\left(x^{2m} + (t-x)^{2m}\right), \quad t \geq 0, x \geq 0, m \in \mathbb{N}. \tag{10}$$

On the other hand, for $f \in E_m$, there exist constants $A, B \in \mathbb{R}^+$ and $m \in \mathbb{N}$ such that $|f| \leq A + Bt^{2m}$. Thus, using (1), we get $|f(t)| \leq A + B(2^{2m-1}(x^{2m} + (t-x)^{2m})) = A + B2^{2m-1}x^{2m} + B2^{2m-1}(t-x)^{2m} = g_m(x) + 2^{2m-1}B(t-x)^{2m}$, where $g_m := A + B2^{2m-1}e^{2m}$. Then

$$\left|f\left(\frac{k}{a_n}\right)\right| \leq g_m(x) + 2^{2m-1}B\left(\frac{k}{a_n} - x\right)^{2m}, \quad k \in N_0, x \geq 0.$$

Since $x, \delta(n)$, and a_n are positive, if $k \geq [a_n(x+\delta(n))]+1$, then

$$\frac{k}{a_n} \geq x.$$

So we can write

$$\{k \in \mathbb{N}_0 : k \geq [a_n(x+\delta(n))]+1\} \subset \left\{k \in \mathbb{N}_0 : \left|\frac{k}{a_n} - x\right| > \delta(n)\right\} := I_{n,x,\delta}.$$

Let $R_n := L_n - L_{n,\delta}$. Thus it follows that

$$\begin{aligned}
|R_n(f;x)| &= \left|\sum_{k=[a_n(x+\delta(n))]+1}^{\infty} \phi_{n,k} f\left(\frac{k}{a_n}\right)\right| \\
&\leq \sum_{k=[a_n(x+\delta(n))]+1}^{\infty} \phi_{n,k}\left[g_m(x) + 2^{2m-1}B\left(\frac{k}{a_n} - x\right)^{2m}\right] \\
&\leq \sum_{k \in I_{n,x,\delta}}^{\infty} \phi_{n,k}(x) g_m(x) + 2^{2m-1}B \sum_{k \in I_{n,x,\delta}}^{\infty} \phi_{n,k}(x)\left(\frac{k}{a_n} - x\right)^{2m} \\
&\leq g_m(x)\frac{1}{\delta^{2m}(n)}\sum_{k=0}^{\infty} \phi_{n,k}(x)\left(\frac{k}{a_n} - x\right)^{2m} + 2^{2m-1}B\sum_{k=0}^{\infty} \phi_{n,k}(x)\left(\frac{k}{a_n} - x\right)^{2m} \\
&= g_m(x)\frac{1}{\delta^{2m}(n)}\zeta_{n,2m}(x) + 2^{2m-1}B\zeta_{n,2m}(x).
\end{aligned}$$

Using $\zeta_{n,2m}(x) \leq \frac{C(m,K)}{a_n^m}$, we get

$$|R_n(f;x)| \leq \left(g_m(x)\frac{1}{\delta^{2m}(n)} + 2^{2m-1}B\right)\frac{C(m,K)}{a_n^m}. \tag{11}$$

Taking the norm on K, we have

$$\|R_n(f;x)\|_{C(K)} \leq \|g_m\| C(m,K)\left[\frac{1}{\sqrt{a_n}\delta(n)}\right]^{2m} + 2^{2m-1}B\frac{C(m,K)}{a_n^m}.$$

For a given $\varepsilon > 0$, define the sets

$$\begin{aligned}
A &:= \{k \leq P_n^{\alpha,\beta} : \|R_k(f,x)\| \geq \varepsilon\}, \\
A_1 &:= \left\{k \leq P_n^{\alpha,\beta} : (\sqrt{a_k}\delta(k))^{-2m} \geq \frac{\varepsilon}{2\|g_m\| C(m,K)}\right\},
\end{aligned}$$

and

$$A_2 := \left\{k \leq P_n^{\alpha,\beta} : {a_k}^{-m} \geq \frac{\varepsilon}{2^{2m}BC(m,K)}\right\}.$$

Then from (8) we clearly have $A \subset A_1 \cup A_2$ and $\delta^{\alpha,\beta}(A;\gamma) \leq \delta^{\alpha,\beta}(A_1;\gamma) + \delta^{\alpha,\beta}(A_2;\gamma)$. Since $\mathrm{st}_{\alpha\beta}^{\gamma}\text{-}\lim_{n\to\infty} \sqrt{a_n}\delta(n) = \infty$ and $\mathrm{st}_{\alpha\beta}^{\gamma}\text{-}\lim_{n\to\infty} {a_n}^{-1} = 0$, the proof is complete. □

If we take $\alpha(n) = 1$, $\beta(n) = n$, and $\gamma = 1$, then

$$\delta^{\alpha,\beta}\left(\left\{k : |x_k - L| \geq \varepsilon\right\}, \gamma\right) = \lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n}.$$

Therefore, if we take $\alpha(n) = 1$, $\beta(n) = n$, and $\gamma = 1$, then $\alpha\beta$-statistical convergence reduces to statistical convergence. Thus, Theorems 2.1 and 2.3 reduce to Theorems 1 and 2 of [14], respectively.

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References

1. Aktuğlu, H.: Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence. J. Comput. Appl. Math. **259**, 174–181 (2014)
2. Fast, H.: Sur la convergence statistique. Colloq. Math. **2**, 241–244 (1951)
3. Steinhaus, H.: Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math. **2**, 73–74 (1951)
4. Gadjiev, A.D., Orhan, C.: Some approximation theorems via statistical convergence. Rocky Mt. J. Math. **32**, 129–138 (2002)
5. Mohiuddine, S.A., Alotaibi, A., Hazarika, B.: Weighted *A*-statistical convergence for sequences of positive linear operator. Sci. World J. (2014). https://doi.org/10.1155/2014/437863
6. Karakaya, V., Karaisa, A.: Korovkin type approximation theorems for weighted $\alpha\beta$-statistical convergence. Bull. Math. Sci. **5**, 159–169 (2015)
7. Mohiuddine, S.A.: An application of almost convergence in approximation theorems. Appl. Math. Lett. **23**, 1382–1387 (2010)
8. Duman, O., Orhan, C.: Statistical approximation by positive linear operators. Stud. Math. **161**, 187–197 (2004)
9. Dirik, F., Demirci, K.: Korovkin type approximation theorem for functions of two variables in statistical sense. Turk. J. Math. **34**, 73–83 (2010)
10. Kirişci, M., Karaisa, A.: Fibonacci statistical convergence and Korovkin type approximation theorems. J. Inequal. Appl. **2017**, 229 (2017)
11. Kadak, U.: Weighted statistical convergence based on generalized difference operator involving (p,q)-gamma function and its applications to approximation theorems. J. Math. Anal. Appl. **448**, 1633–1650 (2017)
12. Agratini, O.: On the convergence of truncated class of operators. Bull. Inst. Math. Acad. Sin. **3**, 213–223 (2003)
13. Çakar, Ö., Gadjiev, A.D.: On uniform approximation by Bleimann, Butzer and Hahn on all positive semiaxis. Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **19**, 21–26 (1999)
14. Canatan, R.: A note on the statistical approximation properties of the modified discrete operators. Open J. Discrete Math. **2**, 114–117 (2012)