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Notes on oscillation of linear delay differential equations

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Abstract

This paper deals with the oscillation criteria for the linear delay differential equations. We present new sufficient conditions for the oscillation of all solutions of such equations. The results improve and complement some earlier ones in the literature.

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1 Introduction

In this article we consider the linear delay differential equation

$$x'(t) + p(t)x\bigl(\tau(t)\bigr) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0,\infty),(0,\infty))$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Our aim is to establish new sufficient conditions for the oscillation of all solutions of Eq. (1). This problem has been recently investigated by many authors. See, for example, [1–14] and the references cited therein.

A continuously differentiable function defined on $[\tau(T_0),\infty)$ for some $T_0 \geq t_0$ and satisfying Eq. (1) for $t \geq T_0$ is called a solution of Eq. (1). Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

We will denote by k the lower limit of the average

$$\int_{\tau(t)}^{t} p(s)\,ds$$

as $t \to \infty$, i.e.,

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds,$$

and constant L is defined by

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds.$$

We refer the readers to the papers [6, 13] for the historical and chronological review of the results. We mention some results in the literature (cf. [6, 13]) for the purpose of this article. In 1972 Ladas, Lakshmikantham, and Papadakis [11] proved that if the following holds

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 1, \tag{2}$$

then every solution of Eq. (1) oscillates.

In 1979 Ladas [10] and in 1982 Koplatadze and Chanturija [7] showed that the same conclusion holds if

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}. \tag{3}$$

We point out that if the inequality

$$\int_{\tau(t)}^{t} p(s)\,ds \le \frac{1}{e}$$

holds eventually, then, according to a result in [7], Eq. (1) has a nonoscillatory solution.

It is obvious that there is a gap between conditions (2) and (3) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors, e.g., [1–6, 8, 9, 12–15].

In the case $0 < k \le 1/e$, all conditions in the papers [4–6, 8, 9, 12–14] are dependent on $0 < L < 1$. The aim of this article is to establish such conditions for oscillation of solutions of Eq. (1) which are independent of L.

We assume for the analysis of asymptotic behavior of the function

$$w(t) = \frac{x(\tau(t))}{x(t)}$$

that Eq. (1) has a solution $x(t)$ which is positive for all large t.

In the second section we will use the next lemma by Jaroš and Stavroulakis [5].

Lemma 1.1 ([5]) *Suppose that $k > 0$ and Eq.* (1) *has an eventually positive solution $x(t)$. Then $k \le 1/e$ and*

$$\lambda_1 \le \liminf_{t\to\infty} w(t) \le \lambda_2,$$

where λ_1 is the smaller and λ_2 is the greater root of the equation $\lambda = e^{k\lambda}$.

2 Oscillatory properties

In this section we will study the oscillatory properties of Eq. (1).

Lemma 2.1 *Let $x(t)$ be an eventually positive solution of Eq.* (1) *and* $0 < k \leq \frac{1}{e}$. *Suppose that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(u)\, du \right) ds > 1 - \frac{1}{\beta}, \tag{4}$$

where $\beta \in [\lambda_1, \lambda_2]$, *and* λ_1 *is the smaller and* λ_2 *is the greater root of the equation* $\lambda = e^{k\lambda}$. *Then*

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} > \beta.$$

Proof Let $t > t_0$ be large enough so that $\tau(t) > t_0$. Integrating (1) from $\tau(t)$ to t, we obtain

$$x\bigl(\tau(t)\bigr) = x(t) + \int_{\tau(t)}^{t} p(s) x\bigl(\tau(s)\bigr)\, ds. \tag{5}$$

Let $0 < \lambda < \lambda_1$. Then the function

$$\varphi(t) = x(t) \exp\left(\lambda \int_{t_0}^{t} p(s)\, ds \right), \quad t \geq t_1, \tag{6}$$

is decreasing for appropriate $t_1 \geq t_0$ (cf. [6, 13]). Indeed by Lemma 1.1

$$\frac{x(\tau(t))}{x(t)} > \lambda$$

for $t \geq t_2$, where $t_2 \geq t_1$ is sufficiently large, and consequently,

$$0 = x'(t) + p(t) x\bigl(\tau(t)\bigr) > x'(t) + \lambda p(t) x(t),$$

which implies $\varphi'(t) < 0$ for $t \geq t_2$. Substituting (6) into (5), we derive for $t \geq t_2$ that

$$\begin{aligned}
x\bigl(\tau(t)\bigr) &= x(t) + \int_{\tau(t)}^{t} p(s) \varphi\bigl(\tau(s)\bigr) \exp\left(-\lambda \int_{t_0}^{\tau(s)} p(u)\, du \right) ds, \\
x\bigl(\tau(t)\bigr) &\geq x(t) + \varphi\bigl(\tau(t)\bigr) \int_{\tau(t)}^{t} p(s) \exp\left(-\lambda \int_{t_0}^{\tau(s)} p(u)\, du \right) ds, \\
x\bigl(\tau(t)\bigr) &\geq x(t) + x\bigl(\tau(t)\bigr) \exp\left(\lambda \int_{t_0}^{\tau(t)} p(u)\, du \right) \\
&\quad \times \int_{\tau(t)}^{t} p(s) \exp\left(-\lambda \int_{t_0}^{\tau(s)} p(u)\, du \right) ds, \\
x\bigl(\tau(t)\bigr) &\geq x(t) + x\bigl(\tau(t)\bigr) \int_{\tau(t)}^{t} p(s) \exp\left(\lambda \int_{\tau(s)}^{\tau(t)} p(u)\, du \right) ds, \\
0 &\geq x(t) + x\bigl(\tau(t)\bigr) \left(-1 + \int_{\tau(t)}^{t} p(s) \exp\left(\lambda \int_{\tau(s)}^{\tau(t)} p(u)\, du \right) ds \right).
\end{aligned} \tag{7}$$

From (4) it follows that there exists a constant c such that $c > 1 - \frac{1-\varepsilon}{\beta}$, where $0 < \varepsilon < \beta[c - (1 - \frac{1}{\beta})] \leq 1$, and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds \geq c > 1 - \frac{1-\varepsilon}{\beta}. \tag{8}$$

Then, for λ sufficiently close to λ_1, we get

$$\int_{\tau(t)}^{t} p(s) \exp\left(\lambda \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds \geq 1 - \frac{1-\varepsilon}{\beta}, \quad t \geq t_3,$$

where $t_3 \geq t_2$ is sufficiently large. If it is not true, then for all $0 < \lambda < \lambda_1$ we have

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds \leq 1 - \frac{1-\varepsilon}{\beta} < c.$$

By letting $\lambda \to \lambda_1$, the last inequality leads to

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds < c.$$

This inequality contradicts (8). Thus we obtain from (7)

$$\begin{aligned}
&0 \geq x(t) - \frac{1-\varepsilon}{\beta} x\bigl(\tau(t)\bigr), \\
&0 \geq 1 - \frac{1-\varepsilon}{\beta} \frac{x(\tau(t))}{x(t)}, \\
&\frac{x(\tau(t))}{x(t)} \geq \frac{\beta}{1-\varepsilon} > \frac{(1-\varepsilon)\beta}{1-\varepsilon} = \beta, \quad t \geq t_3.
\end{aligned}$$

Therefore, we have

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} > \beta.$$

The proof is complete. □

Theorem 2.1 *Let* $0 < k \leq \frac{1}{e}$. *Suppose that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds > 1 - \frac{1}{\lambda_2}.$$

Then all solutions of Eq. (1) *oscillate.*

Proof Assume that Eq. (1) eventually has a positive solution $x(t)$. It follows from Lemma 2.1 that

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} > \lambda_2.$$

This contradicts the result of Lemma 1.1 and completes the proof of the theorem. □

Theorem 2.2 *Let* $0 < k \leq \frac{1}{e}$. *Suppose that there exists* $\gamma \in (\lambda_1, \lambda_2)$ *such that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds > 1 - \frac{1}{\gamma}, \tag{9}$$

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \exp\left(\gamma \int_{\tau(s)}^{\tau(t)} p(u)\,du\right) ds > 1 - \frac{1}{\lambda_2}. \tag{10}$$

Then all solutions of Eq. (1) *oscillate.*

Proof Assume that Eq. (1) has an eventually positive solution $x(t)$. By Lemma 2.1, condition (9) implies that

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} > \gamma.$$

With regard to condition (10) and repeating the procedure as in the proof of Lemma 2.1, we get

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} > \lambda_2.$$

This contradicts the result of Lemma 1.1. The proof is complete. □

In the next example we observe the case $k = 1/e$. Then $\lambda_1 = \lambda_2 = e$.

Example ([6, 13]) Consider the linear delay differential equation

$$x'(t) + px\left(t - a\sin^2\sqrt{t} - \frac{1}{pe}\right) = 0, \quad t \geq t_0, \tag{11}$$

where $p > 0$, $a > 0$.

Then

$$\begin{aligned} k &= \liminf_{t\to\infty} \int_{\tau(t)}^{t} p\,ds = \liminf_{t\to\infty} p\left(a\sin^2\sqrt{t} + \frac{1}{pe}\right) \\ &= \liminf_{t\to\infty} \left(pa\sin^2\sqrt{t} + \frac{1}{e}\right) = \frac{1}{e}. \end{aligned}$$

We get

$$\begin{aligned} \int_{\tau(t)}^{t} p\exp\left(e\int_{\tau(s)}^{\tau(t)} p\,du\right) ds &= p\int_{\tau(t)}^{t} \exp\bigl(pe\bigl(\tau(t) - \tau(s)\bigr)\bigr)\,ds \\ &= p\exp\bigl(pe\tau(t)\bigr) \int_{\tau(t)}^{t} \exp\bigl(-pe\tau(s)\bigr)\,ds \\ &= p\exp\bigl(pe\tau(t)\bigr) \int_{\tau(t)}^{t} \exp\left(-pe\left(s - a\sin^2\sqrt{s} - \frac{1}{pe}\right)\right) ds \\ &= ep\exp\bigl(pe\tau(t)\bigr) \int_{\tau(t)}^{t} \exp(-pes)\exp\bigl(ape\sin^2\sqrt{s}\bigr)\,ds. \end{aligned}$$

Then we have

$$\begin{aligned}
&\liminf_{t\to\infty} ep\exp\bigl(pe\tau(t)\bigr)\int_{\tau(t)}^{t}\exp(-pes)\exp\bigl(ape\sin^2\sqrt{s}\bigr)\,ds\\
&\quad>\liminf_{t\to\infty} ep\exp\bigl(pe\tau(t)\bigr)\int_{\tau(t)}^{t}\exp(-pes)\,ds\\
&\quad=\liminf_{t\to\infty}\exp\bigl(pe\tau(t)\bigr)\bigl(\exp\bigl(-pe\tau(t)\bigr)-\exp(-pet)\bigr)\\
&\quad=\liminf_{t\to\infty}\bigl(1-\exp\bigl(pe\bigl(\tau(t)-t\bigr)\bigr)\bigr)=\liminf_{t\to\infty}\bigl(1-\exp\bigl(-ape\sin^2\sqrt{t}-1\bigr)\bigr)\\
&\quad=\liminf_{t\to\infty}\left(1-\frac{1}{e}\exp\bigl(-ape\sin^2\sqrt{t}\bigr)\right)=1-\frac{1}{e}.
\end{aligned}$$

We conclude that

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}p\exp\left(e\int_{\tau(s)}^{\tau(t)}p\,du\right)ds>1-\frac{1}{e},$$

and by Theorem 2.1 all solutions of Eq. (11) oscillate.

We point out that the results in [6, 13] are dependent on the constant $0<L<1$, while our results do not depend on the constant L.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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References
1. Baštinec, J., Berezansky, L., Diblík, J., Šmarda, Z.: On the critical case in oscillation for differential equations with a single delay and with several delays. Abstr. Appl. Anal. **2010**, Article ID 417869 (2010)
2. Diblík, J., Růžičková, M., Šutá, Z.: Asymptotical convergence of the solutions of a linear differential equation with delays. Adv. Differ. Equ. **2010**, Article ID 749852 (2010)
3. Diblík, J., Svoboda, Z., Šmarda, Z.: Explicit criteria for the existence of positive solutions for a scalar differential equation with variable delay in the critical case. Comput. Math. Appl. **56**(2), 556–564 (2008)
4. Erbe, L.H., Zhang, B.G.: Oscillation for first order linear differential equations with deviating arguments. Differ. Integral Equ. **1**, 305–314 (1988)
5. Jaroš, J., Stavroulakis, I.P.: Oscillation tests for delay equations. Rocky Mt. J. Math. **29**, 139–145 (1999)
6. Kon, M., Scifas, Y.G., Stavroulakis, I.P.: Oscillation criteria for delay equations. Proc. Am. Math. Soc. **128**, 2989–2997 (2000)
7. Koplatadze, R.G., Chanturija, T.A.: On oscillatory and monotonic solutions of first order differential equations with deviating arguments. Differ. Uravn. **18**, 1463–1465 (1982) (Russian)
8. Koplatadze, R.G., Kvinikadze, G.: On the oscillation of solutions of first order delay differential inequalities and equations. Georgian Math. J. **1**, 675–685 (1994)
9. Kwong, M.K.: Oscillation of first order delay equations. J. Math. Anal. Appl. **156**, 274–286 (1991)
10. Ladas, G.: Sharp conditions for oscillations caused by delays. Appl. Anal. **9**, 93–98 (1979)
11. Ladas, G., Lakshmikantham, V., Papadakis, L.S.: Oscillations of higher-order retarded differential equations generated by the retarded arguments. In: Delay and Functional Differential Equations and Their Applications, pp. 219–231. Academic Press, New York (1972)

12. Philos, C.H.G., Sficas, Y.G.: An oscillation criterion for first order linear delay differential equations. Can. Math. Bull. **41**, 207–213 (1998)
13. Sficas, Y.G., Stavroulakis, I.P.: Oscillation criteria for first-order delay equations. Bull. Lond. Math. Soc. **35**, 239–246 (2003)
14. Yu, J.S., Wang, Z.C., Zhang, B.G., Qian, X.Z.: Oscillations of differential equations with deviating arguments. Panam. Math. J. **2**, 59–78 (1992)
15. Diblík, J., Kúdelčíková, M.: Existence and asymptotic behavior of positive solutions of functional differential equations of delayed type. Abstr. Appl. Anal. **2011**, Article ID 754701 (2011)