Mao et al. *Advances in Difference Equations* (2018) 2018:37
https://doi.org/10.1186/s13662-018-1490-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

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Approximate solutions for a class of doubly perturbed stochastic differential equations

Wei Mao[1,2*], Liangjian Hu[3] and Xuerong Mao[4]

[*]Correspondence:
jsjysxx365@126.com
[1]College of Information Sciences and Technology, Donghua University, Shanghai, China
[2]School of Mathematics and Information Technology, Jiangsu Second Normal University, Nanjing, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the Carathéodory approximate solution for a class of doubly perturbed stochastic differential equations (DPSDEs). Based on the Carathéodory approximation procedure, we prove that DPSDEs have a unique solution and show that the Carathéodory approximate solution converges to the solution of DPSDEs under the global Lipschitz condition. Moreover, we extend the above results to the case of DPSDEs with non-Lipschitz coefficients.

Keywords: Carathéodory approximate solution; doubly perturbed stochastic differential equations; global Lipschitz condition; non-Lipschitz condition

1 Introduction

As the limit process from a weak polymers model, the following doubly perturbed Brownian motion

$$x_t = B_t + \alpha \sup_{0\le s\le t} x_s + \beta \inf_{0\le s\le t} x_s, \tag{1.1}$$

was discussed by Norris et al. [1], and it also arises as the scaling limit of some self-interacting random walks (see [2]). During the past few decades, equation (1.1) has attracted much interest from many scholars, for example, [3–9]. Following them, Doney et al. [10] studied the singly perturbed Skorohod equations

$$x_t = x_0 + \int_0^t \sigma(x_s)\,dB_s + \int_0^t b(x_s)\,ds + \alpha \sup_{0\le s\le t} x_s. \tag{1.2}$$

Using the Picard iterative procedure, they showed the existence and uniqueness of the solution to equation (1.2). Hu et al. [11] discussed the existence and uniqueness of the solution to doubly perturbed neutral stochastic functional equations, while Luo [12] obtained the existence and uniqueness of the solution to doubly perturbed jump-diffusion processes.

In fact, the Picard iterative method is a well-known procedure for approximating the solution of stochastic differential equations (SDEs). However, to obtain the Picard iterative sequence $x^n(t)$, one needs to compute $x^i(t)$, $0 \le i \le n-1$. And this brings us a lot of calculations on stepwise iterated Ito's integrals. In the early twentieth century, Carathéodory [13] put forward the Carathéodory approximation scheme for ordinary differential equations. In this scheme, Carathéodory defined the approximate solution via a delay equation, and the delay equation can be solved explicitly by successive integrations over intervals of length $\frac{1}{n}$. In other words, the Carathéodory approximation scheme avoids calculating $x^i(t)$, $0 \le i \le n-1$.

Because of its advantage, this approximation procedure has received great attention, and many people have been devoted to the study of the Carathéodory scheme for SDEs. For example, Bell and Mohammed [14] extended the Carathéodory approximation scheme to the case of SDEs and showed the convergence of the Carathéodory approximate solution. Mao [15, 16] considered a class of SDEs with variable delays and studied the Carathéodory approximate solution of delay SDEs. Turo [17] discussed the Carathéodory approximate solution of stochastic functional differential equations (SFDEs) and established the existence theorem for SFDEs. Liu [18] investigated a class of semilinear stochastic evolution equations with time delays and proved that the Carathéodory approximate solution converges to the solution of stochastic delay evolution equations.

Motivated by the above mentioned papers, we will study the Carathéodory approximate scheme of doubly perturbed stochastic differential equations (DPSDEs)

$$x(t) = x(0) + \int_0^t f\bigl(s, x(s)\bigr)\,ds + \int_0^t g\bigl(s, x(s)\bigr)\,dw(s) + \alpha \sup_{0\le s\le t} x(s) + \beta \inf_{0\le s\le t} x(s). \tag{1.3}$$

To the best of our knowledge, so far little is known about the Carathéodory approximations for equation (1.3), and the aim of this paper is to close this gap. In this paper, we will prove that the Carathéodory approximate solution converges to the solution under the global Lipschitz condition. Moreover, we will replace the global Lipschitz condition by a more general condition proposed by [19, 20] and show that equation (1.3) has a unique solution under the non-Lipschitz condition.

This paper is organized as follows. In Section 2, we establish the existence theorem of equation (1.3) and show that the Carathéodory approximate solution converges to the solution of equation (1.3) under the global Lipschitz condition. While in Section 3, we extend the existence and convergence results of Section 2 to the case of equation (1.3) with non-Lipschitz coefficients.

2 Carathéodory approximation and global Lipschitz DPSDEs

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all P-null sets). Let $\{w(t)\}_{t\ge0}$ be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{L}^2([a,b];R)$ denote the family of $\mathcal{F}_t$-measurable, R-valued processes $f(t) = \{f(t,\omega)\}$, $t \in [a,b]$ such that $\int_a^b |f(t)|^2\,dt < \infty$ a.s.

Consider the following doubly perturbed stochastic differential equations:

$$x(t) = x(0) + \int_0^t f\bigl(s, x(s)\bigr)\,ds + \int_0^t g\bigl(s, x(s)\bigr)\,dw(s) + \alpha \sup_{0\le s\le t} x(s) + \beta \inf_{0\le s\le t} x(s), \tag{2.1}$$

where $\alpha, \beta \in (0,1)$, the initial value $x(0) = x_0 \in R$ and $f : [0,T] \times R \to R$, $g : [0,T] \times R \to R$ are both Borel-measurable functions. In this paper, we assume that the initial value x_0 is independent of w and satisfies $E|x_0|^2 < \infty$.

Now, we define the sequence of the Carathéodory approximate solutions $x^n : [-1,T] \to R$. For all $n \geq 1$, we define

$$\begin{aligned} &x^n(t) = x_0, \quad -1 \leq t \leq 0, \\ &x^n(t) = x_0 + \int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds + \int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s) \\ &\qquad + \alpha \sup_{0 \leq s \leq t} x^n\left(s - \frac{1}{n}\right) + \beta \inf_{0 \leq s \leq t} x^n\left(s - \frac{1}{n}\right), \quad t \in (0,T]. \end{aligned} \tag{2.2}$$

Note that $x^n(t)$ can be calculated step by step on the intervals $[0, \frac{1}{n})$, $[\frac{1}{n}, \frac{2}{n}), \ldots$, etc.

To obtain the main results, we give the following conditions.

Assumption 2.1 For any $x, y \in R$ and $t \in [0,T]$, there exists a positive constant k such that

$$\left|f(t,x) - f(t,y)\right|^2 \vee \left|g(t,x) - g(t,y)\right|^2 \leq k|x - y|^2. \tag{2.3}$$

Assumption 2.2 For any $t \in [0,T]$, there exists a positive constant $\bar{k}$ such that

$$\left|f(t,0)\right|^2 \vee \left|g(t,0)\right|^2 \leq \bar{k}. \tag{2.4}$$

Assumption 2.3 The coefficients satisfy $|\alpha| + |\beta| < 1$.

Remark 2.1 Clearly, Assumptions 2.1 and 2.2 imply the linear growth condition. That is, for any $x \in R$ and $t \in [0,T]$,

$$\left|f(t,x)\right|^2 \leq 2\left|f(t,x) - f(t,0)\right|^2 + 2\left|f(t,0)\right|^2 \leq 2k|x|^2 + 2\bar{k} \leq L\left(1 + |x|^2\right), \tag{2.5}$$

where $L = 2\max(k, \bar{k})$. Similarly, we have $|g(t,x)|^2 \leq L(1 + |x|^2)$.

Now, we state our main results.

Theorem 2.1 *Let Assumptions* 2.1-2.3 *hold. Then there exists a unique $\mathcal{F}_t$-adapted solution* $\{x(t)\}_{t \geq 0}$ *to equation* (2.1). *Moreover, for any* $T > 0$,

$$E \sup_{0 \leq t \leq T} \left|x^n(t) - x(t)\right|^2 \leq C\frac{1}{n}, \tag{2.6}$$

where C is a constant independent of n.

In the sequel, to prove our main results, we need some useful lemmas.

Lemma 2.1 (Gronwall's inequality [21]) *Let $u_0 \geq 0$ and $v(t) \geq 0$, and let $u(\cdot)$ be a real continuous function on* $[0,T]$. *If*

$$u(t) \leq u_0 + \int_0^t v(s)u(s)\,ds, \quad \textit{for all } t \in [0,T],$$

then we have

$$u(t) \le u_0 e^{\int_0^t v(s)\,ds}$$

for all $t \in [0, T]$.

Lemma 2.2 *Under Assumptions* 2.1-2.3, *for all* $n \ge 1$,

$$E \sup_{0\le t\le T} \left|x^n(t)\right|^2 \le C_1, \tag{2.7}$$

where C_1 *is a positive constant.*

Proof For any $t \in [0, T]$, it follows from (2.2) that

$$\begin{aligned}
\left|x^n(t)\right| &\le |x_0| + \left|\int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds\right| + \left|\int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s)\right| \\
&\quad + |\alpha| \left|\sup_{0\le s\le t} x^n\left(s - \frac{1}{n}\right)\right| + |\beta| \left|\inf_{0\le s\le t} x^n\left(s - \frac{1}{n}\right)\right| \\
&\le |x_0| + \left|\int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds\right| + \left|\int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s)\right| \\
&\quad + \left(|\alpha| + |\beta|\right) \sup_{0\le s\le t} \left|x^n\left(s - \frac{1}{n}\right)\right| \\
&\le |x_0| + \left|\int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds\right| + \left|\int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s)\right| \\
&\quad + \left(|\alpha| + |\beta|\right) \sup_{-\frac{1}{n}\le s\le 0} \left|x^n(s)\right| + \left(|\alpha| + |\beta|\right) \sup_{0\le s\le t} \left|x^n(s)\right|.
\end{aligned}$$

By the basic inequality $|a + b + c|^2 \le 3(|a|^2 + |b|^2 + |c|^2)$, one has

$$\begin{aligned}
&\left(1 - |\alpha| - |\beta|\right)^2 E \sup_{0\le t\le t_1} \left|x^n(t)\right|^2 \\
&\quad \le 3\left(\left(1 + |\alpha| + |\beta|\right)^2 E|x_0|^2 + E \sup_{0\le t\le t_1} \left|\int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds\right|^2 \right. \\
&\qquad \left. + E \sup_{0\le t\le t_1} \left|\int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s)\right|^2\right)
\end{aligned} \tag{2.8}$$

for any $t_1 \in [0, T]$. Using the Hölder inequality and the Burkholder-Davis-Gundy inequality, we can easily see that

$$E \sup_{0\le t\le t_1} \left|\int_0^t f\left(s, x^n\left(s - \frac{1}{n}\right)\right) ds\right|^2 \le TE \int_0^{t_1} \left|f\left(s, x^n\left(s - \frac{1}{n}\right)\right)\right|^2 ds \tag{2.9}$$

and

$$E \sup_{0\le t\le t_1} \left|\int_0^t g\left(s, x^n\left(s - \frac{1}{n}\right)\right) dw(s)\right|^2 \le 4E \int_0^{t_1} \left|g\left(s, x^n\left(s - \frac{1}{n}\right)\right)\right|^2 ds. \tag{2.10}$$

Then, by Assumptions 2.1, 2.2 and 2.3, we have

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left|x^n(t)\right|^2\\
&\quad\le\frac{3(1+|\alpha|+|\beta|)^2}{(1-|\alpha|-|\beta|)^2}E|x_0|^2+\frac{3(T+4)L}{(1-|\alpha|-|\beta|)^2}E\int_0^{t_1}\left(1+\left|x^n\left(s-\frac{1}{n}\right)\right|^2\right)ds\\
&\quad\le\frac{3(1+|\alpha|+|\beta|)^2+3(T+4)L}{(1-|\alpha|-|\beta|)^2}E|x_0|^2\\
&\qquad+\frac{3(T+4)L}{(1-|\alpha|-|\beta|)^2}\int_0^{t_1}\left(1+E\sup_{0\le s\le t}\left|x^n(s)\right|^2\right)dt.
\end{aligned}$$

Finally, the Gronwall inequality implies that

$$1+E\sup_{0\le t\le t_1}\left|x^n(t)\right|^2\le\frac{3(1+|\alpha|+|\beta|)^2+3(T+4)L}{(1-|\alpha|-|\beta|)^2}E|x_0|^2e^{\frac{3(T+4)L}{(1-|\alpha|-|\beta|)^2}T}.$$

The proof is therefore complete. □

Lemma 2.3 *For all $n\ge 1$ and $0\le s<t\le T$,*

$$E\left|x^n(t)-x^n(s)\right|^2\le C_2(t-s),\tag{2.11}$$

where C_2 is a positive constant.

Proof For all $n\ge 1$ and $0\le s<t\le T$, it follows from (2.2) that

$$\begin{aligned}
x^n(t)-x^n(s)&=\int_s^t f\left(\sigma,x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\\
&\quad+\int_s^t g\left(\sigma,x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\\
&\quad+\alpha\sup_{0\le\sigma\le t}x^n\left(\sigma-\frac{1}{n}\right)+\beta\inf_{0\le\sigma\le t}x^n\left(\sigma-\frac{1}{n}\right)\\
&\quad-\alpha\sup_{0\le\sigma\le s}x^n\left(\sigma-\frac{1}{n}\right)-\beta\inf_{0\le\sigma\le s}x^n\left(\sigma-\frac{1}{n}\right).
\end{aligned}\tag{2.12}$$

Note that $\inf_{0\le\sigma\le t}x^n(\sigma-\frac{1}{n})\le\inf_{0\le\sigma\le s}x^n(\sigma-\frac{1}{n})$, we have

$$\begin{aligned}
\left|x^n(t)-x^n(s)\right|&\le\left|\int_s^t f\left(\sigma,x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right|\\
&\quad+\left|\int_s^t g\left(\sigma,x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right|\\
&\quad+\alpha\left|\sup_{0\le\sigma\le t}x^n\left(\sigma-\frac{1}{n}\right)-\sup_{0\le\sigma\le s}x^n\left(\sigma-\frac{1}{n}\right)\right|.
\end{aligned}\tag{2.13}$$

Next, let us consider the following two cases.

Case I. If $\sup_{0\le\sigma\le t} x^n(\sigma-\frac{1}{n}) = \sup_{0\le\sigma\le s} x^n(\sigma-\frac{1}{n})$, then we get from (2.13) that

$$\begin{aligned}\left|x^n(t)-x^n(s)\right| &\le \left|\int_s^t f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right| \\ &\quad+\left|\int_s^t g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right|. \end{aligned}\tag{2.14}$$

Case II. If $\sup_{0\le\sigma\le t} x^n(\sigma-\frac{1}{n}) > \sup_{0\le\sigma\le s} x^n(\sigma-\frac{1}{n})$, then there exists $r\in(s,t]$ such that $x^n(r) = \sup_{0\le\sigma\le t} x^n(\sigma-\frac{1}{n})$. So we get from (2.13) that

$$\begin{aligned}\left|x^n(t)-x^n(s)\right| &\le \left|\int_s^t f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right| + \left|\int_s^t g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right| \\ &\quad+\alpha\left|x^n(r)-\sup_{0\le\sigma\le s} x^n\left(\sigma-\frac{1}{n}\right)\right| \\ &\le \left|\int_s^t f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right| + \left|\int_s^t g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right| \\ &\quad+\alpha\left|x^n(r)-x^n(s)\right| \\ &\le \left|\int_s^t f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right| + \left|\int_s^t g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right| \\ &\quad+\alpha\sup_{s\le s'<t'\le t}\left|x^n(t')-x^n(s')\right|.\end{aligned}$$

We therefore have

$$\begin{aligned}\sup_{s\le s'<t'\le t}\left|x^n(t')-x^n(s')\right| &\le \frac{1}{1-\alpha}\left(\sup_{s\le s'<t'\le t}\left|\int_{s'}^{t'} f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right|\right. \\ &\quad\left.+\sup_{s\le s'<t'\le t}\left|\int_{s'}^{t'} g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right|\right).\end{aligned}\tag{2.15}$$

Hence,

$$\begin{aligned}&E\sup_{s\le s'<t'\le t}\left|x^n(t')-x^n(s')\right|^2 \\ &\quad\le \frac{2}{(1-\alpha)^2}\left(E\left|\int_s^t f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)d\sigma\right|^2\right. \\ &\qquad\left.+E\left|\int_s^t g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)dw(\sigma)\right|^2\right).\end{aligned}$$

Then Lemma 2.2 yields

$$\begin{aligned}&E\sup_{s\le s'<t'\le t}\left|x^n(t')-x^n(s')\right|^2 \\ &\quad\le \frac{2}{(1-\alpha)^2}\left((t-s)E\int_s^t \left|f\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)\right|^2 d\sigma\right. \\ &\qquad\left.+4E\int_s^t \left|g\left(\sigma, x^n\left(\sigma-\frac{1}{n}\right)\right)\right|^2 d\sigma\right)\end{aligned}$$

$$\leq \frac{2(T+4)L}{(1-\alpha)^2}\int_s^t \left(1 + E\left|x^n\left(\sigma - \frac{1}{n}\right)\right|^2\right) d\sigma$$
$$\leq C_2(t-s),$$

where $C_2 = \frac{2(T+4)L}{(1-\alpha)^2}(1 + C_1)$. The proof is therefore complete. □

Proof of Theorem 2.1 Firstly, we will show that the sequence $\{x^n(t)\}$ is a Cauchy sequence in $\mathcal{L}^2([0,T];R)$. For any $n > m \geq 1$, it follows that

$$\begin{aligned}\left|x^n(t) - x^m(t)\right| \leq{} & \left|\int_0^t \left[f\left(s, x^n\left(s - \frac{1}{n}\right)\right) - f\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] ds\right| \\ & + \left|\int_0^t \left[g\left(s, x^n\left(s - \frac{1}{n}\right)\right) - g\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] dw(s)\right| \\ & + |\alpha|\left|\sup_{0\leq s\leq t} x^n\left(s - \frac{1}{n}\right) - \sup_{0\leq s\leq t} x^m\left(s - \frac{1}{m}\right)\right| \\ & + |\beta|\left|\inf_{0\leq s\leq t} x^n\left(s - \frac{1}{n}\right) - \inf_{0\leq s\leq t} x^m\left(s - \frac{1}{m}\right)\right|.\end{aligned}$$

Noting that

$$\left|\sup_{0\leq s\leq t} x^n\left(s - \frac{1}{n}\right) - \sup_{0\leq s\leq t} x^m\left(s - \frac{1}{m}\right)\right| \leq \sup_{0\leq s\leq t}\left|x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{m}\right)\right|$$

and

$$\left|\inf_{0\leq s\leq t} x^n\left(s - \frac{1}{n}\right) - \inf_{0\leq s\leq t} x^m\left(s - \frac{1}{m}\right)\right| \leq \sup_{0\leq s\leq t}\left|x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{m}\right)\right|,$$

one can have

$$\begin{aligned}\left|x^n(t) - x^m(t)\right| \leq{} & \left|\int_0^t \left[f\left(s, x^n\left(s - \frac{1}{n}\right)\right) - f\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] ds\right| \\ & + \left|\int_0^t \left[g\left(s, x^n\left(s - \frac{1}{n}\right)\right) - g\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] dw(s)\right| \\ & + \left(|\alpha| + |\beta|\right)\sup_{0\leq s\leq t}\left|x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{m}\right)\right| \\ \leq{} & \left|\int_0^t \left[f\left(s, x^n\left(s - \frac{1}{n}\right)\right) - f\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] ds\right| \\ & + \left|\int_0^t \left[g\left(s, x^n\left(s - \frac{1}{n}\right)\right) - g\left(s, x^m\left(s - \frac{1}{m}\right)\right)\right] dw(s)\right| \\ & + \left(|\alpha| + |\beta|\right)\sup_{0\leq s\leq t}\left|x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{n}\right)\right| \\ & + \left(|\alpha| + |\beta|\right)\sup_{0\leq s\leq t}\left|x^m\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{m}\right)\right|.\end{aligned}$$

By the basic inequality and Assumption 2.3, we obtain that

$$\begin{aligned}
&\left(1-|\alpha|-|\beta|\right)^2 E\sup_{0\le t\le t_1}\left|x^n(t)-x^m(t)\right|^2\\
&\quad\le 3\left(E\sup_{0\le t\le t_1}\left|\int_0^s\left[f\left(s,x^n\left(s-\frac{1}{n}\right)\right)-f\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right]ds\right|^2\right.\\
&\qquad+E\sup_{0\le t\le t_1}\left|\int_0^s\left[g\left(s,x^n\left(s-\frac{1}{n}\right)\right)-g\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right]dw(s)\right|^2\\
&\qquad\left.+\left(|\alpha|+|\beta|\right)^2 E\sup_{0\le t\le t_1}\left|x^m\left(t-\frac{1}{n}\right)-x^m\left(t-\frac{1}{m}\right)\right|^2\right). \qquad (2.16)
\end{aligned}$$

Then, using the Hölder inequality and the Burkholder-Davis-Gundy inequality again, we have

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left|\int_0^t\left[f\left(s,x^n\left(s-\frac{1}{n}\right)\right)-f\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right]d\sigma\right|^2\\
&\quad\le TE\int_0^{t_1}\left|f\left(s,x^n\left(s-\frac{1}{n}\right)\right)-f\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right|^2 ds\\
&\quad\le TkE\int_0^{t_1}\left|x^n\left(s-\frac{1}{n}\right)-x^m\left(s-\frac{1}{m}\right)\right|^2 ds \qquad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left|\int_0^t\left[g\left(s,x^n\left(s-\frac{1}{n}\right)\right)-g\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right]dw(s)\right|^2\\
&\quad\le 4E\int_0^{t_1}\left|g\left(s,x^n\left(s-\frac{1}{n}\right)\right)-g\left(s,x^m\left(s-\frac{1}{m}\right)\right)\right|^2 ds\\
&\quad\le 4kE\int_0^{t_1}\left|x^n\left(s-\frac{1}{n}\right)-x^m\left(s-\frac{1}{m}\right)\right|^2 ds. \qquad (2.18)
\end{aligned}$$

Substituting (2.17) and (2.18) into (2.16), one has

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left|x^n(t)-x^m(t)\right|^2\\
&\quad\le\frac{3}{(1-|\alpha|-|\beta|)^2}\left((4+T)kE\int_0^{t_1}\left|x^n\left(s-\frac{1}{n}\right)-x^m\left(s-\frac{1}{m}\right)\right|^2 ds\right.\\
&\qquad\left.+\left(|\alpha|+|\beta|\right)^2 E\sup_{0\le t\le t_1}\left|x^m\left(t-\frac{1}{n}\right)-x^m\left(t-\frac{1}{m}\right)\right|^2\right)\\
&\quad\le\frac{3}{(1-|\alpha|-|\beta|)^2}\left(2(4+T)kE\int_0^{t_1}\left|x^n\left(s-\frac{1}{n}\right)-x^m\left(s-\frac{1}{n}\right)\right|^2 ds\right.\\
&\qquad+2(4+T)kE\int_0^{t_1}\left|x^m\left(s-\frac{1}{n}\right)-x^m\left(s-\frac{1}{m}\right)\right|^2 ds\\
&\qquad\left.+\left(|\alpha|+|\beta|\right)^2 E\sup_{0\le t\le t_1}\left|x^m\left(t-\frac{1}{n}\right)-x^m\left(t-\frac{1}{m}\right)\right|^2\right).
\end{aligned}$$

Then Lemma 2.3 yields

$$
\begin{aligned}
&E \sup_{0\le t\le t_1} \left|x^n(t)-x^m(t)\right|^2 \\
&\quad\le \frac{3}{(1-|\alpha|-|\beta|)^2}\biggl(2(4+T)kE\int_0^{t_1-\frac{1}{n}} \left|x^n(s)-x^m(s)\right|^2\,ds \\
&\qquad+\left[2(4+T)kT+\left(|\alpha|+|\beta|\right)^2\right]C_2\left(\frac{1}{m}-\frac{1}{n}\right)\biggr) \\
&\quad\le \frac{3}{(1-|\alpha|-|\beta|)^2}\biggl(2(4+T)k\int_0^{t_1} E\sup_{0\le s\le t} \left|x^n(s)-x^m(s)\right|^2\,dt \\
&\qquad+\left[2(4+T)kT+\left(|\alpha|+|\beta|\right)^2\right]C_2\left(\frac{1}{m}-\frac{1}{n}\right)\biggr).
\end{aligned}
$$

Hence,

$$
E \sup_{0\le t\le T} \left|x^n(t)-x^m(t)\right|^2 \le C_3\int_0^T \left[E\sup_{0\le s\le t} \left|x^n(s)-x^m(s)\right|^2\right]dt + C_4\left(\frac{1}{m}-\frac{1}{n}\right), \tag{2.19}
$$

where $C_3 = \frac{6(4+T)k}{(1-|\alpha|-|\beta|)^2}$, $C_4 = \frac{3[2(4+T)kT+(|\alpha|+|\beta|)^2]C_2}{(1-|\alpha|-|\beta|)^2}$. By the Gronwall inequality, we have

$$
E \sup_{0\le t\le T} \left|x^n(t)-x^m(t)\right|^2 \le C_4 e^{C_3 T}\left(\frac{1}{m}-\frac{1}{n}\right), \tag{2.20}
$$

which implies that

$$
E \sup_{0\le t\le T} \left|x^n(t)-x^m(t)\right|^2 \to 0 \quad \text{as } n,m\to\infty.
$$

This shows that the sequence $\{x^n(t)\}$ is a Cauchy sequence in $\mathcal{L}^2([0,T];R)$. Denote the limit by $x(t)$. Letting $m\to\infty$ in (2.20) yields

$$
E \sup_{0\le t\le T} \left|x^n(t)-x(t)\right|^2 \le C_4 e^{C_3 T}\frac{1}{n}. \tag{2.21}
$$

Then the Borel-Cantelli lemma can be used to show that $x^n(t)$ converges to $x(t)$ almost surely uniformly on [0,T] as $n\to\infty$. Taking limits on both sides of (2.2) and letting $n\to\infty$, we can obtain that $x(t)$ is a solution of equation (2.1).

Now we show the uniqueness of the solution. Let $x(t)$ and $y(t)$ be any two solutions of equation (2.1). We can prove using the same procedure as (2.19) that

$$
E \sup_{0\le t\le T} \left|x(t)-y(t)\right|^2 \le C\int_0^T E\Bigl(\sup_{0\le s\le t} \left|x(s)-y(s)\right|^2\Bigr)\,ds
$$

for all $t\in[0,T]$. The Gronwall inequality gives that

$$
E \sup_{0\le t\le T} \left|x(t)-y(t)\right|^2 = 0,
$$

i.e., for any $t\in[0,T]$, $x(t)\equiv y(t)$ a.s. This completes the proof. □

Remark 2.2 By (2.6), we conclude that the Carathéodory approximate solution converges to the true solution of equation (2.1) in the mean square sense, i.e., for any $T > 0$,

$$E \sup_{0\le t\le T} \left|x^n(t) - x(t)\right|^2 \to 0 \quad \text{as } n \to \infty.$$

In fact, the proof of the convergence of the Carathéodory approximation represents an alternative to the procedure for establishing the existence and uniqueness of the solution to delay DPSDEs. In other words, the Carathéodory approximation scheme is applicable to a class of DPSDEs.

3 Non-Lipschitz DPSDEs

In this section, we will replace the global Lipschitz condition (2.3) with a more general condition and show that the Carathéodory approximate solution still converges to the true solution of equation (2.1).

Assumption 3.1 For any $x, y \in R$ and $t \in [0, T]$, there exists a function $k(\cdot)$ such that

$$\left|f(t,x) - f(t,y)\right| \vee \left|g(t,x) - g(t,y)\right| \le k\big(|x - y|\big), \tag{3.1}$$

where $k(u)$ is a concave non-decreasing continuous function such that $k(0) = 0$ and $\int_{0^+} \frac{u}{k^2(u)}\,du = \infty$.

Remark 3.1 Since $k(\cdot)$ is concave and $k(0) = 0$, one can find a pair of positive constants a and b such that

$$k(u) \le a + bu \quad \text{for } u \ge 0.$$

Theorem 3.1 *Let Assumptions* 3.1, 2.2 *and* 2.3 *hold. Then there exists a unique* $\mathcal{F}_t$-*adapted solution* $\{x(t)\}_{t\ge 0}$ *to equation* (2.1). *Moreover, for any* $T > 0$,

$$\lim_{n\to\infty} E \sup_{0\le t\le T} \left|x^n(t) - x(t)\right|^2 = 0. \tag{3.2}$$

To prove Theorem 3.1, we will need the following Bihari inequality.

Lemma 3.1 (Bihari's inequality [22]) *Let* $k : R_+ \to R_+$ *be a continuous, non-decreasing function satisfying* $k(0) = 0$ *and* $\int_{0^+} \frac{ds}{k(s)} = +\infty$. *Let* $u(\cdot)$ *be a Borel measurable bounded non-negative function defined on* $[0, T]$ *satisfying*

$$u(t) \le u_0 + \int_0^t v(s)k\big(u(s)\big)\,ds, \quad t \in [0, T],$$

where $u_0 > 0$ *and* $v(\cdot)$ *is a non-negative integrable function on* $[0, T]$. *Then we have*

$$u(t) \le G^{-1}\left(G(u_0) + \int_0^t v(s)\,ds\right),$$

where $G(t) = \int_{t_0}^t \frac{du}{k(u)}$ *is well defined for some* $t_0 > 0$, *and* G^{-1} *is the inverse function of* G.

In particular, if $u_0 = 0$, then $u(t) = 0$ for all $t \in [0, T]$.

Lemma 3.2 *Under Assumptions* 3.1, 2.2 *and* 2.3, *for all* $n \geq 1$,

$$E \sup_{0\leq t\leq T} \left|x^n(t)\right|^2 \leq \bar{C}_1, \tag{3.3}$$

where $\bar{C}_1$ *is a positive constant.*

Proof By the Hölder inequality and the Burkholder-Davis-Gundy inequality, it follows from (2.8) that

$$\begin{aligned}
&\bigl(1-|\alpha|-|\beta|\bigr)^2 E \sup_{0\leq t\leq t_1} \left|x^n(t)\right|^2 \\
&\quad\leq 3\Biggl(\bigl(1+|\alpha|+|\beta|\bigr)^2 E|x_0|^2 + 2TE\int_0^{t_1}\left|f\left(s, x^n\left(s-\frac{1}{n}\right)\right) - f(s,0)\right|^2 ds \\
&\qquad + 8E\int_0^{t_1}\left|g\left(s, x^n\left(s-\frac{1}{n}\right)\right) - g(s,0)\right|^2 ds \\
&\qquad + 2TE\int_0^{t_1}\left|f(s,0)\right|^2 ds + 8E\int_0^{t_1}\left|g(s,0)\right|^2 ds\Biggr).
\end{aligned} \tag{3.4}$$

By Assumptions 2.2 and 3.1, we have

$$\begin{aligned}
&\bigl(1-|\alpha|-|\beta|\bigr)^2 E \sup_{0\leq t\leq t_1} \left|x^n(t)\right|^2 \\
&\quad\leq 3\Biggl(\bigl(1+|\alpha|+|\beta|\bigr)^2 E|x_0|^2 \\
&\qquad + 2(T+4)E\int_0^{t_1} k^2\left(\left|x^n\left(s-\frac{1}{n}\right)\right|\right) ds + 2(T+4)T\bar{k}\Biggr).
\end{aligned} \tag{3.5}$$

Then the Jensen inequality implies that

$$\begin{aligned}
&\bigl(1-|\alpha|-|\beta|\bigr)^2 E \sup_{0\leq t\leq t_1} \left|x^n(t)\right|^2 \\
&\quad\leq 3\Biggl(\bigl(1+|\alpha|+|\beta|\bigr)^2 E|x_0|^2 \\
&\qquad + 2(T+4)\int_0^{t_1} k^2\left(\left(E\left|x^n\left(s-\frac{1}{n}\right)\right|^2\right)^{\frac{1}{2}}\right) ds + 2(T+4)T\bar{k}\Biggr).
\end{aligned}$$

Let $\rho(x) = k^2(x^{\frac{1}{2}})$, it follows that

$$\begin{aligned}
E \sup_{0\leq t\leq t_1} \left|x^n(t)\right|^2 &\leq \frac{3(1+|\alpha|+|\beta|)^2 E|x_0|^2 + 6(T+4)T\bar{k}}{(1-|\alpha|-|\beta|)^2} \\
&\quad + \frac{6(T+4)}{(1-|\alpha|-|\beta|)^2}\int_0^{t_1} \rho\left(E\left|x^n\left(s-\frac{1}{n}\right)\right|^2\right) ds.
\end{aligned} \tag{3.6}$$

Since $\frac{k(x)}{x}$ and $k'_+(x)$ are non-negative, non-increasing functions, we have that

$$\rho'_+(x) = x^{-\frac{1}{2}} k\bigl(x^{\frac{1}{2}}\bigr) k'_+(x)$$

is a non-negative, non-increasing function which implies that ρ is a non-negative, non-decreasing concave function. Note that $k(0) = 0$, then $\rho(0) = 0$, and there exists a pair of positive constants a and b such that

$$\rho(u) \le a + bu \quad \text{for } u \ge 0.$$

We therefore have

$$\begin{aligned} E \sup_{0\le t\le t_1} \bigl|x^n(t)\bigr|^2 &\le \frac{3(1+|\alpha|+|\beta|)^2 E|x_0|^2 + 6(T+4)T(a+\bar{k})}{(1-|\alpha|-|\beta|)^2} \\ &\quad + \frac{6(T+4)b}{(1-|\alpha|-|\beta|)^2} \int_0^{t_1} E\biggl|x^n\biggl(s-\frac{1}{n}\biggr)\biggr|^2 ds \\ &\le \frac{[3(1+|\alpha|+|\beta|)^2 + \frac{6(T+4)b}{n}]E|x_0|^2 + 6(T+4)T(a+\bar{k})}{(1-|\alpha|-|\beta|)^2} \\ &\quad + \frac{6(T+4)b}{(1-|\alpha|-|\beta|)^2} \int_0^{t_1} E \sup_{0\le s\le t} \bigl|x^n(s)\bigr|^2 dt. \end{aligned} \tag{3.7}$$

Set

$$r(t) = \frac{[3(1+|\alpha|+|\beta|)^2 + \frac{6(T+4)b}{n}]E|x_0|^2 + 6(T+4)T(a+\bar{k})}{(1-|\alpha|-|\beta|)^2} e^{\frac{6(T+4)b}{(1-|\alpha|-|\beta|)^2}t},$$

then $r(\cdot)$ is the solution to the following ordinary differential equation:

$$\begin{aligned} r(t_1) = {}& \frac{[3(1+|\alpha|+|\beta|)^2 + \frac{6(T+4)b}{n}]E|x_0|^2 + 6(T+4)T(a+\bar{k})}{(1-|\alpha|-|\beta|)^2} \\ &+ \frac{6(T+4)b}{(1-|\alpha|-|\beta|)^2} \int_0^{t_1} r(t)\,dt. \end{aligned}$$

By recurrence, it is easy to verify that, for each $n \ge 0$,

$$E \sup_{0\le t\le t_1} \bigl|x^n(t)\bigr|^2 \le r(t_1).$$

Note that $r(t_1)$ is continuous and bounded on $[0, T]$, one can have

$$E \sup_{0\le t\le T} \bigl|x^n(t)\bigr|^2 \le \bar{C}_1 < +\infty$$

for any $n \ge 1$. This completes the proof of Lemma 3.2. □

Lemma 3.3 *For all $n \ge 1$ and $0 \le s < t \le T$,*

$$E\bigl|x^n(t) - x^n(s)\bigr|^2 \le \bar{C}_2(t-s), \tag{3.8}$$

where $\bar{C}_2$ is a positive constant.

The proof is similar to Lemma 2.3, we omit its proof.

Now, let us apply the above lemmas to prove Theorem 3.1.

Proof of Theorem 3.1 By the Hölder inequality and the Burkholder-Davis-Gundy inequality, it follows from (2.16) that

$$\begin{aligned}&\bigl(1-|\alpha|-|\beta|\bigr)^2 E\sup_{0\le t\le t_1}\bigl|x^n(t)-x^m(t)\bigr|^2\\&\le 3\biggl(TE\int_0^{t_1}\biggl|f\biggl(s,x^n\biggl(s-\frac{1}{n}\biggr)\biggr)-f\biggl(s,x^m\biggl(s-\frac{1}{m}\biggr)\biggr)\biggr|^2\,ds\\&\quad+4E\int_0^{t_1}\biggl|g\biggl(s,x^n\biggl(s-\frac{1}{n}\biggr)\biggr)-g\biggl(s,x^m\biggl(s-\frac{1}{m}\biggr)\biggr)\biggr|^2\,ds\\&\quad+\bigl(|\alpha|+|\beta|\bigr)^2E\sup_{0\le t\le t_1}\biggl|x^m\biggl(t-\frac{1}{n}\biggr)-x^m\biggl(t-\frac{1}{m}\biggr)\biggr|^2\biggr).\end{aligned}$$

By Assumption 3.1 and the Jensen inequality, we have

$$\begin{aligned}&\bigl(1-|\alpha|-|\beta|\bigr)^2 E\sup_{0\le t\le t_1}\bigl|x^n(t)-x^m(t)\bigr|^2\\&\le 3\biggl((T+4)E\int_0^{t_1}k^2\biggl(\biggl|x^n\biggl(s-\frac{1}{n}\biggr)-x^m\biggl(s-\frac{1}{m}\biggr)\biggr|\biggr)\,ds\\&\quad+\bigl(|\alpha|+|\beta|\bigr)^2E\sup_{0\le t\le t_1}\biggl|x^m\biggl(t-\frac{1}{n}\biggr)-x^m\biggl(t-\frac{1}{m}\biggr)\biggr|^2\biggr)\\&\le 3\biggl((T+4)\int_0^{t_1}k^2\biggl(\biggl(E\biggl|x^n\biggl(s-\frac{1}{n}\biggr)-x^m\biggl(s-\frac{1}{m}\biggr)\biggr|^2\biggr)^{\frac{1}{2}}\biggr)\,ds\\&\quad+\bigl(|\alpha|+|\beta|\bigr)^2E\sup_{0\le t\le t_1}\biggl|x^m\biggl(t-\frac{1}{n}\biggr)-x^m\biggl(t-\frac{1}{m}\biggr)\biggr|^2\biggr).\end{aligned}\tag{3.9}$$

Similar to (3.6), one obtains

$$\begin{aligned}&E\sup_{0\le t\le t_1}\bigl|x^n(t)-x^m(t)\bigr|^2\\&\le\frac{3}{(1-|\alpha|-|\beta|)^2}\biggl((T+4)\int_0^{t_1}\rho\biggl(E\biggl|x^n\biggl(s-\frac{1}{n}\biggr)-x^m\biggl(s-\frac{1}{m}\biggr)\biggr|^2\biggr)\,ds\\&\quad+\bigl(|\alpha|+|\beta|\bigr)^2E\sup_{0\le t\le t_1}\biggl|x^m\biggl(t-\frac{1}{n}\biggr)-x^m\biggl(t-\frac{1}{m}\biggr)\biggr|^2\biggr).\end{aligned}\tag{3.10}$$

Since $\rho(\cdot)$ is concave, we have $\rho(a+b)\le\rho(a)+\rho(b)$. Then Lemma 3.3 yields

$$\begin{aligned}&E\sup_{0\le t\le t_1}\bigl|x^n(t)-x^m(t)\bigr|^2\\&\le\frac{3}{(1-|\alpha|-|\beta|)^2}\biggl(2(T+4)\int_0^{t_1}\rho\biggl(E\biggl|x^n\biggl(s-\frac{1}{n}\biggr)-x^m\biggl(s-\frac{1}{n}\biggr)\biggr|^2\biggr)\,ds\\&\quad+2(T+4)\int_0^{t_1}\rho\biggl(E\biggl|x^m\biggl(s-\frac{1}{n}\biggr)-x^m\biggl(s-\frac{1}{m}\biggr)\biggr|^2\biggr)\,ds\end{aligned}$$

$$+ \big(|\alpha| + |\beta|\big)^2 E \sup_{0\le t\le t_1} \left| x^m\left(t - \frac{1}{n}\right) - x^m\left(t - \frac{1}{m}\right)\right|^2 \Bigg)$$
$$\le \frac{6(T+4)}{(1-|\alpha|-|\beta|)^2} \left(\int_0^{t_1} \rho\left(E\left| x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{n}\right)\right|^2\right) ds \right.$$
$$\left. + \int_0^{t_1} \rho\left(\bar{C}_2\left(\frac{1}{m} - \frac{1}{n}\right)\right) ds \right) + \frac{3(|\alpha|+|\beta|)^2}{(1-|\alpha|-|\beta|)^2} \bar{C}_2\left(\frac{1}{m} - \frac{1}{n}\right), \tag{3.11}$$

where

$$\int_0^{t_1} \rho\left(E\left| x^n\left(s - \frac{1}{n}\right) - x^m\left(s - \frac{1}{n}\right)\right|^2\right) ds$$
$$\le \int_0^{t_1} \rho\left(E \sup_{0\le\sigma\le s}\left| x^n\left(\sigma - \frac{1}{n}\right) - x^m\left(\sigma - \frac{1}{n}\right)\right|^2\right) ds$$
$$\le \int_0^{t_1} \rho\left(E \sup_{-\frac{1}{n}\le v\le s-\frac{1}{n}} \left| x^n(v) - x^m(v)\right|^2\right) ds$$
$$\le \int_0^{t_1} \rho\left(E \sup_{-\frac{1}{n}\le v\le 0} \left| x^n(v) - x^m(v)\right|^2 + E \sup_{0\le v\le s} \left| x^n(v) - x^m(v)\right|^2\right) ds$$
$$\le \rho\big(2E\|\xi\|^2\big)T + \int_0^{t_1} \rho\left(E \sup_{0\le s\le t} \left| x^n(s) - x^m(s)\right|^2\right) dt. \tag{3.12}$$

Inserting (3.12) into (3.11), we obtain that

$$E \sup_{0\le t\le t_1} \left| x^n(t) - x^m(t)\right|^2$$
$$\le \frac{6(T+4)}{(1-|\alpha|-|\beta|)^2} \int_0^{t_1} \rho\left(E \sup_{0\le s\le t} \left| x^n(s) - x^m(s)\right|^2\right) dt$$
$$+ \frac{3}{(1-|\alpha|-|\beta|)^2} C(m,n), \tag{3.13}$$

where

$$C(m,n) = 2(T+4)T\rho\big(2E\|\xi\|^2\big) + 2(T+4)T\bar{C}_2\rho\left(\frac{1}{m} - \frac{1}{n}\right)$$
$$+ \big(|\alpha| + |\beta|\big)^2 \bar{C}_2\left(\frac{1}{m} - \frac{1}{n}\right).$$

Then the Bihari inequality gives that

$$E \sup_{0\le t\le t_1} \left| x^n(t) - x^m(t)\right|^2 \le G^{-1}\left(G\left(\frac{3C(m,n)}{(1-|\alpha|-|\beta|)^2}\right) + \frac{6(T+4)T}{(1-|\alpha|-|\beta|)^2}\right),$$

where $G(t) = \int_1^t \frac{ds}{\rho(s)}$. Obviously, G is a strictly increasing function, then G has an inverse function which is strictly increasing, and $G^{-1}(-\infty) = 0$. Note that when $m, n \to \infty$, then $\frac{3C(m,n)}{(1-|\alpha|-|\beta|)^2} \to 0$. Recalling $\int_{0^+} \frac{ds}{\rho(s)} = \int_{0^+} \frac{s}{k^2(s)}\, ds = \infty$, we have

$$G\left(\frac{3C(m,n)}{(1-|\alpha|-|\beta|)^2}\right) \to -\infty$$

and

$$G^{-1}\left[G\left(\frac{3C(m,n)}{(1-|\alpha|-|\beta|)^2}\right)+\frac{6(T+4)T}{(1-|\alpha|-|\beta|)^2}\right]\to 0.$$

We therefore have

$$\begin{aligned}&\limsup_{n,m\to\infty} E\Bigl(\sup_{0\le t\le t_1}\left|x^n(t)-x^m(t)\right|^2\Bigr)\\&\quad\le \limsup_{n,m\to\infty} G^{-1}\left[G\left(\frac{3C(m,n)}{(1-|\alpha|-|\beta|)^2}\right)+\frac{6(T+4)T}{(1-|\alpha|-|\beta|)^2}\right]=0,\end{aligned}\tag{3.14}$$

which implies that $\{x^n(t)\}_{n\ge 1}$ is a Cauchy sequence. Denote the limit by $x(t)$. Letting $m\to\infty$ in (3.14) yields

$$\lim_{n\to\infty} E\sup_{0\le t\le T}\left|x^n(t)-x(t)\right|^2=0.$$

Similar to (3.13), (3.14), we can show that $x(t)$ is a unique solution of equation (2.1) under non-Lipschitz conditions. Then the proof is completed. □

Remark 3.2 To see the generality of our results, let us give a few examples of the function $k(\cdot)$. Let $\delta\in(0,1)$ be sufficiently small, define

$$k_1(u)=\begin{cases}0, & u=0,\\ u\sqrt{\log(u^{-1})}, & u\in(0,\delta],\\ \delta\sqrt{\log(\delta^{-1})}, & u\in[\delta,+\infty],\end{cases}$$

and

$$k_2(u)=\begin{cases}0, & u=0,\\ u\sqrt{\log(u^{-1})}\log\log(u^{-1}), & u\in(0,\delta],\\ \delta\sqrt{\log(\delta^{-1})}\log\log(\delta^{-1}), & u\in[\delta,+\infty].\end{cases}$$

They are all concave non-decreasing functions satisfying $\int_{0^+}\frac{u}{k_i^2(u)}\,du=\infty$, $i=1,2$.

Remark 3.3 In particular, if we let $k(u)=ku$, $u\ge 0$, we see that the Lipschitz condition (2.3) is a special case of our proposed condition (3.1). In other words, we obtain a more general result than Theorem 2.1.

Remark 3.4 In fact, our theories developed can be applied to study doubly perturbed stochastic differential equations with jumps (DPSDEwJs) and doubly perturbed stochastic differential equations with Markovian switching (DPSDEwMS) respectively. If $\alpha=\beta=0$, then DPSDEwJs and DPSDEwMS will become SDEs with jumps and SDEs with Markovian switching which were investigated by [23–34]. Similarly, we can also give the Carathéodory approximate solution and show that the Carathéodory approximate solution converges to the solution of SDEs with jumps and SDEs with Markovian switching under our assumptions.

Acknowledgements
The authors would like to thank the editor and the anonymous referee for their careful reading and valuable comments. This work was supported by the Leverhulme Trust (RF-2015-385), the Royal Society Wolfson Research Merit Award (WM160014), the NSFC (Nos. 11401261, 11471071) and '333 High-level Personnel Training Project' of Jiangsu Province.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
[1]College of Information Sciences and Technology, Donghua University, Shanghai, China. [2]School of Mathematics and Information Technology, Jiangsu Second Normal University, Nanjing, China. [3]Department of Applied Mathematics, Donghua Univerisity, Shanghai, China. [4]Department of Mathematics and Statistics, University of Strathclyde, Glasgow, UK.

Publisher's Note

Received: 25 July 2017 Accepted: 15 January 2018 Published online: 24 January 2018

References

1. Norris, J, Rogers, L, Williams, D: Self-avoiding random walk: a Brownian motion model with local time drift. Probab. Theory Relat. Fields **74**, 271-287 (1987)
2. Toth, B: Generalized Ray-Knight theory and limit theorems for self-interacting random walks on Z^1. Ann. Probab. **24**, 1324-1367 (1996)
3. Chaumont, L, Doney, R: Some calculations for doubly perturbed Brownian motion. Stoch. Process. Appl. **85**, 61-74 (2000)
4. Chaumont, L, Doney, RA, Hu, Y: Upper and lower limits of doubly perturbed Brownian motion. Ann. Inst. Henri Poincaré Probab. Stat. **36**, 219-249 (2000)
5. Davis, B: Weak limits of perturbed random walks and the equation $Y_t = B_t + \alpha \max_{0\le s\le t} Y_s + \beta \min_{0\le s\le t} Y_s$. Ann. Probab. **24**, 2007-2023 (1997)
6. Doney, RA: Some calculations for perturbed Brownian motion. In: Séminaire de Probabilités XXXII. Lecture Notes in Math., pp. 231-236. Springer, Berlin (1998)
7. Le Gall, J, Yor, M: Enlacements du mouvement brownien autour des courbes de léspace. Trans. Am. Math. Soc. **317**, 687-722 (1990)
8. Perman, M, Werner, W: Perturbed Brownian motions. Probab. Theory Relat. Fields **108**, 357-383 (1997)
9. Werner, W: Some remarks on perturbed Brownian motion. In: Seminaire de Probabilites XXIX. Lecture Notes in Math. Springer, Berlin (1995)
10. Doney, RA, Zhang, T: Perturbed Skorohod equations and perturbed reflected diffusion processes. Ann. Inst. Henri Poincaré Probab. Stat. **41**, 107-121 (2005)
11. Hu, L, Ren, Y: Doubly perturbed neutral stochastic functional equations. J. Comput. Appl. Math. **231**, 319-326 (2009)
12. Luo, J: Doubly perturbed jump-diffusion processes. J. Math. Anal. Appl. **351**, 147-151 (2009)
13. Coddington, EA, Levinson, N: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)
14. Bell, DR, Mohammed, SEA: On the solution of stochastic ordinary differential equations via small delays. Stoch. Int. J. Probab. Stoch. Process. **28**, 293-299 (1989)
15. Mao, X: Approximate solutions for a class of stochastic evolution equations with variable delays. Numer. Funct. Anal. Optim. **12**, 525-533 (1991)
16. Mao, X: Approximate solutions for a class of stochastic evolution equations with variable delays II. Numer. Funct. Anal. Optim. **15**, 65-76 (1994)
17. Turo, J: Carathéodory approximation solutions to a class of stochastic functional differential equations. Appl. Anal. **61**, 121-128 (1996)
18. Liu, K: Carathéodory approximate solutions for a class of semilinear stochastic evolution equations with time delays. J. Math. Anal. Appl. **220**, 349-364 (1998)
19. Mao, X: Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients. Stoch. Process. Appl. **58**, 281-292 (1995)
20. Yamada, T: On the successive approximation of solutions of stochastic differential equations. J. Math. Kyoto Univ. **21**, 501-515 (1981)
21. Mao, X: Stochastic Differential Equations and Their Applications. Horwood, Chichester (2007)
22. Bihari, I: A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations. Acta Math. Acad. Sci. Hung. **7**, 71-94 (1956)
23. Applebaum, D, Siakalli, M: Asymptotic stability of stochastic differential equations driven by Levy noise. J. Appl. Probab. **46**, 1116-1129 (2009)
24. Hu, L, Mao, X, Zhang, L: Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations. IEEE Trans. Autom. Control **58**, 2319-2332 (2013)
25. Luo, J, Liu, K: Stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps. Stoch. Process. Appl. **118**, 864-895 (2008)
26. Li, X, Mao, X, Shen, Y: Approximate solutions of stochastic differential delay equations with Markovian switching. J. Differ. Equ. Appl. **16**, 195-207 (2010)
27. Mao, X: Stability of stochastic differential equations with Markovian switching. Stoch. Process. Appl. **79**, 45-67 (1999)

28. Wang, B, Zhu, Q: Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems. Syst. Control Lett. **105**, 55-61 (2017)
29. Xi, F: On the stability of a jump-diffusions with Markovian switching. J. Math. Anal. Appl. **341**, 588-600 (2008)
30. Yin, G, Xi, F: Stability of regime-switching jump diffusions. SIAM J. Control Optim. **48**, 4525-4549 (2010)
31. Yuan, C, Bao, J: On the exponential stability of switching-diffusion processes with jumps. Q. Appl. Math. **71**, 311-329 (2013)
32. Zhu, Q: Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching, I. Int. J. Control **90**, 1703-1712 (2017)
33. Zhu, Q, Zhang, Q: *p*th moment exponential stabilization of hybrid stochastic differential equations by feedback controls based on discrete time state observations with a time delay. IET Control Theory Appl. **11**, 1992-2003 (2017)
34. Zhu, Q: Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise. J. Math. Anal. Appl. **416**, 126-142 (2014)