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Existence and uniqueness of solutions for fractional boundary value problems with p-Laplacian operator

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for a fractional boundary value problem involving the p-Laplacian operator. Our analysis relies on some properties of the Green function and the Guo-Krasnoselskii fixed point theorem and the Banach contraction mapping principle. Two examples are given to illustrate our theoretical results.

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1 Introduction

Fractional differential equations have excited, in the past decades, a considerable interest both in mathematics and in applications. They were used in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines; for instance, see [1–7]. On the other hand, for studying the turbulent flow in a porous medium, Leibenson [8] introduced the model of a differential equation with the p-Laplacian operator. Since then, differential equations with a p-Laplacian operator are widely applied in different fields of physics and natural phenomena; for examples, see [9–12] and the references therein.

The topic of fractional-order boundary value problems with the p-Laplacian operator has been intensively studied by several researchers in the recent years. We refer the reader to [13–17] and the references therein.

Chen et al. [18] studied the existence of solutions for the boundary value problem of the fractional p-Laplacian equation

$$\begin{cases} {}^{c}D_{0^{+}}^{\beta}(\varphi_{p}({}^{c}D_{0^{+}}^{\alpha}x(t))) = f(t, x(t), D_{0^{+}}^{\alpha}x(t)), & 0 < t < 1, \\ {}^{c}D_{0^{+}}^{\alpha}x(0) = {}^{c}D_{0^{+}}^{\alpha}x(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, ${}^{c}D_{0^{+}}^{\alpha}$ is the Caputo fractional derivative of order α, $\varphi_{p}(s) = |s|^{p-2}s$, $p > 1$, and $f : [0,1] \times \mathbb{R}^{2} \to \mathbb{R}$ is a continuous function. Obviously, φ_{p} is invertible, and its inverse operator is φ_{q}, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Arifi et al. [19] investigated the following nonlinear fractional boundary value problem with p-Laplacian operator:

$$\begin{cases} D_{a^+}^{\beta}(\varphi_p(D_{a^+}^{\alpha}u(t))) + \chi(t)\varphi_p(u(t)) = 0, & a < t < b, \\ u(a) = u'(a) = u'(b) = 0, & D_{a^+}^{\alpha}u(a) = D_{a^+}^{\alpha}u(b) = 0, \end{cases} \tag{1.2}$$

where $2 < \alpha \le 3$, $1 < \beta \le 2$, $p > 1$, $D_{a^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α, and $\chi : [a, b] \to \mathbb{R}$ is a continuous function. Necessary conditions for the existence of nontrivial solutions to (1.2) were given.

In our recent paper [20], we consider the following fractional boundary value problem with mixed fractional derivative and p-Laplacian operator

$$\begin{cases} D_{a^+}^{\beta}(\varphi_p({}^cD_{a^+}^{\alpha}u(t))) = k(t)f(u(t)), & a < t < b, \\ u'(a) = {}^cD_{a^+}^{\alpha}u(a) = 0, & u(b) = {}^cD_{a^+}^{\alpha}u(b) = 0, \end{cases} \tag{1.3}$$

where $1 < \alpha, \beta \le 2$, $p > 1$, and $k : [a, b] \to \mathbb{R}$ is a continuous function. Under some assumptions on the nonlinear term f, the existence of positive solutions to (1.3) was obtained, and two Lyapunov-type inequalities were established.

Motivated by the works mentioned, in this paper, we investigate the existence of positive solutions and the uniqueness of a solution for the following boundary value problem of fractional differential equation with p-Laplacian operator:

$$\begin{cases} (\varphi_p(D_{0^+}^{\alpha}u(t)))' + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = D_{0^+}^{\alpha}u(0) = 0, & {}^cD_{0^+}^{\beta}u(0) = {}^cD_{0^+}^{\beta}u(1) = 0, \end{cases} \tag{1.4}$$

where $0 < \beta \le 1$, $2 < \alpha < 2 + \beta$, $D_{0^+}^{\alpha}$ and ${}^cD_{0^+}^{\beta}$ are the Riemann-Liouville fractional derivative and Caputo fractional derivative of orders α, β, respectively, $p > 1$, and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The paper is organized as follows. In Section 2, we briefly introduce some necessary basic knowledge and definitions about fractional calculus theory. In Section 3, we write (1.4) as an equivalent integral equation, and then, under some assumptions on the nonlinear term f, we establish three theorems on the existence of nontrivial positive solutions and uniqueness of a solution for FBVP (1.4) by means of the Guo-Krasnoselskii fixed point theorem and the Banach contraction mapping principle, respectively. Finally, in Section 4, we give two examples to show the effectiveness of the results obtained.

2 Preliminaries

In this section, we introduce some concepts and results of fractional calculus. For more details, we refer to [2, 3].

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$\left(I_{0^+}^{\alpha}f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) \, ds,$$

where Γ denotes the gamma function.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$\left(D_{0^+}^\alpha f\right)(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \, ds,$$

where $n = [\alpha] + 1$.

Definition 2.3 The Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$\left({}^c D_{0^+}^\alpha f\right)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds,$$

where $n = [\alpha] + 1$.

We now state some properties of fractional operators.

Lemma 2.4 ([3]) *Let* $\alpha, \beta \in \mathbb{R}^+$ *and* $u \in L_1[0,1]$. *Then* $I_{0^+}^\alpha I_{0^+}^\beta u(x) = I_{0^+}^{\alpha+\beta}$ *almost everywhere on* $[0,1]$.

Lemma 2.5 ([3]) *Let* $\alpha \in \mathbb{R}^+$ *and* $u \in C[0,1]$. *Then* ${}^c D_{a^+}^\alpha I_{a^+}^\alpha u(x) = u(x)$.

Lemma 2.6 ([3]) *If* $\alpha > 0$, $n = [\alpha] + 1$, *and* $u \in AC^n[0,1]$, *then* $I_{0^+}^\alpha {}^c D_{0^+}^\alpha u(x) = u(x) - \sum_{k=0}^{n-1} \frac{x^k}{k!} u^{(k)}(0)$.

Lemma 2.7 ([21]) *Let* X *be a Banach space, and let* $P \subset X$ *be a cone. Let* Ω_1 *and* Ω_2 *be bounded open subsets of* X *with* $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, *and let* $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ *be a completely continuous operator such that*

(i) $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \leq \|u\|$ *for any* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for any* $u \in P \cap \partial\Omega_2$.

Then, T has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

Let $E = C[0,1]$ be endowed with the norm $\|x\| = \max_{t \in [0,1]} |x(t)|$.

We now consider the following boundary value problem:

$$\begin{cases} (\varphi_p(D_{0^+}^\alpha u(t)))' + h(t) = 0, & 0 < t < 1, \\ u(0) = D_{0^+}^\alpha u(0) = 0, & {}^c D_{0^+}^\beta u(0) = {}^c D_{0^+}^\beta u(1) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1 *Let* $h \in C[0,1] \cap L[0,1]$, $0 < \beta \leq 1$, *and* $2 < \alpha < 2 + \beta$. *Then* $u \in C[0,1]$ *is a solution of* (3.1) *if and only if*

$$u(t) = \int_0^1 G(t,s) \varphi_q\left(\int_0^s h(\tau) \, d\tau\right) ds, \tag{3.2}$$

where $G(t,s)$ *is Green's function given by*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \\ \frac{(1-s)^{\alpha-\beta-1} t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.3}$$

Proof Integrating the first equation of (3.1) on $[0, t]$, by the boundary condition $D_{0^+}^\alpha u(0) = 0$ we have that

$$\varphi_p\left(D_{0^+}^\alpha u(t)\right) = \varphi_p\left(D_{0^+}^\alpha u(0)\right) - \int_0^t h(s)\, ds = -\int_0^t h(s)\, ds,$$

which implies that

$$D_{0^+}^\alpha u(t) = -\varphi_q\left(\int_0^t h(s)\, ds\right).$$

By Lemma 2.6 we have that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\left(\int_0^s h(\tau)\, d\tau\right) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \tag{3.4}$$

By the boundary value condition $u(0) = 0$ we have $c_3 = 0$. Moreover, from Lemmas 2.4 and 2.5 we can easily obtain

$$
\begin{aligned}
{}^c D_{0^+}^\beta u(t) = &-\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)\, d\tau\right) ds \\
&+ c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + c_2 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} t^{\alpha-\beta-2}.
\end{aligned}
\tag{3.5}
$$

By (3.5) and ${}^c D_{0^+}^\beta u(0) = 0$ we have $c_2 = 0$. On the other hand,

$$0 = {}^c D_{0^+}^\beta u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)\, d\tau\right) ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)},$$

which yields that

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)\, d\tau\right) ds. \tag{3.6}$$

Substituting $c_2 = c_3 = 0$ and (3.6) into (3.4), we can obtain that the solution of (3.1) is

$$u(t) = \int_0^1 G(t, s) \varphi_q\left(\int_0^s h(\tau)\, d\tau\right) ds, \tag{3.7}$$

where Green's function $G(t, s)$ is as in (3.3). The proof is completed. □

Lemma 3.2 *The function $G(t, s)$ has the following properties*:
(1) $G(t, s) > 0$ *for* $t, s \in (0, 1)$,
(2) $\beta t^{\alpha-1} s(1-s)^{\alpha-\beta-1} \le \Gamma(\alpha) G(t, s) \le (\alpha-1) t^{\alpha-2} s(1-s)^{\alpha-\beta-1}$ *for* $t, s \in [0, 1]$.

Proof First, we prove that (1) holds. Since

$$(1-s)^{\alpha-\beta-1} \ge \left(1 - \frac{s}{t}\right)^{\alpha-\beta-1} \ge \left(1 - \frac{s}{t}\right)^{\alpha-1}, \quad 0 \le s \le t \le 1,$$

we have that

$$(1 - s)^{\alpha - \beta - 1} t^{\alpha - 1} \geq (t - s)^{\alpha - 1}.$$

Thus, we easily obtain that $G(t, s) > 0$ for $s, t \in (0, 1)$.

We now will prove that (2) holds. If $0 \leq s \leq t \leq 1$, then

$$\begin{aligned}
\Gamma(\alpha) G(t, s) &= (1 - s)^{\alpha - \beta - 1} t^{\alpha - 1} - (t - s)^{\alpha - 1} \leq (1 - s)^{\alpha - \beta - 1} \big[t^{\alpha - 1} - (t - s)^{\alpha - 1} \big] \\
&= (1 - s)^{\alpha - \beta - 1} (\alpha - 1) \int_{t - s}^{t} x^{\alpha - 2} \, dx \\
&\leq (\alpha - 1)(1 - s)^{\alpha - \beta - 1} t^{\alpha - 2} \big[t - (t - s) \big] \\
&= (\alpha - 1) s (1 - s)^{\alpha - \beta - 1} t^{\alpha - 2}.
\end{aligned}$$

On the other hand, we have

$$\Gamma(\alpha) G(t, s) \geq \beta t^{\alpha - 1} s (1 - s)^{\alpha - \beta - 1}. \tag{3.8}$$

The proof of (3.8) is the same as that of (2.6) of Lemma 2.7 in [22], and here we omit it.

When $0 \leq t \leq s \leq 1$, we get

$$\begin{aligned}
\Gamma(\alpha) G(t, s) &= (1 - s)^{\alpha - \beta - 1} t^{\alpha - 1} \leq (1 - s)^{\alpha - \beta - 1} s t^{\alpha - 2} \\
&\leq (\alpha - 1) t^{\alpha - 2} s (1 - s)^{\alpha - \beta - 1}.
\end{aligned}$$

On the other hand, we have

$$\Gamma(\alpha) G(t, s) = (1 - s)^{\alpha - \beta - 1} t^{\alpha - 1} \geq \beta t^{\alpha - 1} s (1 - s)^{\alpha - \beta - 1}.$$

The proof is completed. □

Define the cone $P \subset E = C[0, 1]$ by

$$P = \left\{ x \in E : x(t) \geq \frac{\beta}{\alpha - 1} t^{\alpha - 1} \|x\|, \text{ for all } t \in [0, 1] \right\}.$$

Theorem 3.3 *Let $0 < \beta \leq 1$, $2 < \alpha < 2 + \beta$, and $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+ = [0, +\infty)$ be a continuous function. Suppose that there exist two positive constants $r_2 > r_1 > 0$ such that the following assumptions are satisfied:*

(H1) $f(t, x) \geq \rho \varphi_p(r_1)$ *for* $(t, x) \in [0, 1] \times [0, r_1]$,
(H2) $f(t, x) \leq \omega \varphi_p(r_2)$ *for* $x \in [0, 1] \times [0, r_2]$,

where

$$\rho = \varphi_p \left(\frac{\Gamma(\alpha) 2^{\alpha - 1}}{\beta B(q + 1, \alpha - \beta)} \right)$$

and

$$\omega = \varphi_p \left(\frac{\Gamma(\alpha)}{(\alpha - 1) B(q + 1, \alpha - \beta)} \right).$$

Then FBVP (1.4) *has at least one nontrivial positive solution u belonging to E such that* $r_1 \le \|u\| \le r_2$.

Proof From Lemma 3.1 we know that $u \in C[0,1]$ is a solution of (1.4) if and only if u is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds.$$

Let $T : P \to E$ be the operator defined by

$$Tu(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds.$$

For any $u \in P$, we have by Lemma 3.2 that

$$
\begin{aligned}
Tu(t) &\ge \frac{1}{\Gamma(\alpha)}\beta \int_0^1 t^{\alpha-1}s(1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&= \frac{\beta}{\alpha-1}t^{\alpha-1}\int_0^1 \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&\ge \frac{\beta}{\alpha-1}t^{\alpha-1}\int_0^1 \max_{t\in[0,1]} G(t,s)\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&\ge \frac{\beta}{\alpha-1}t^{\alpha-1}\max_{t\in[0,1]}\int_0^1 G(t,s)\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&= \frac{\beta}{\alpha-1}t^{\alpha-1}\|Tu\|,
\end{aligned}
$$

which implies that $T : P \to P$. Using the Arzelà-Ascoli theorem, we can prove that $T : P \to P$ is completely continuous. Let $\Omega_i = \{u \in P : \|u\| \le r_i\}$, $i = 1, 2$. From (H1) and Lemmas 3.1 and 3.2 we obtain for $t \in [\frac{1}{2}, 1]$ and $u \in P \cap \partial\Omega_1$ that

$$
\begin{aligned}
(Tu)(t) &\ge \int_0^1 \min_{t\in[\frac{1}{2},1]} G(t,s)\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&\ge \frac{1}{\Gamma(\alpha)}\int_0^1 \min_{t\in[\frac{1}{2},1]} \beta t^{\alpha-1}s(1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&= \frac{\beta}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s f(\tau, u(\tau))\, d\tau\right) ds \\
&\ge \frac{\beta}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^s \rho\varphi_p(r_1)\, d\tau\right) ds \\
&= \frac{\beta}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-\beta-1}\varphi_q(s)\, ds \cdot \varphi_q(\rho)r_1 \\
&= \frac{\beta}{2^{\alpha-1}\Gamma(\alpha)}\int_0^1 s^q(1-s)^{\alpha-\beta-1}\, ds \cdot \varphi_q(\rho)r_1 \\
&= \frac{\beta}{2^{\alpha-1}\Gamma(\alpha)}B(q+1, \alpha-\beta) \cdot \varphi_q(\rho)r_1 = \|u\|.
\end{aligned}
$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$. On the other hand, from (H2) and Lemmas 3.1 and 3.2 we have

$$
\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q\left(\int_0^s f\left(\tau, u(\tau)\right) d\tau\right) ds \\
&\leq \max_{t\in[0,1]} \int_0^1 \frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-2} s(1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s f\left(\tau, u(\tau)\right) d\tau\right) ds \\
&= \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s f\left(\tau, u(\tau)\right) d\tau\right) ds \\
&\leq \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s \omega\varphi_p(r_2) d\tau\right) ds \\
&= \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-\beta-1} \varphi_q(s) ds \cdot \varphi_q(\omega)r_2 \\
&= \frac{\alpha-1}{\Gamma(\alpha)} B(q+1, \alpha-\beta) \cdot \varphi_q(\omega)r_2 = \|u\|
\end{aligned}
$$

for $u \in P \cap \partial\Omega_2$. Thus, by Lemma 2.7 we have that the operator T has a fixed point in $u \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution for FBVP (1.4). The proof is completed. □

Lemma 3.4 ([12]) *The p-Laplacian operator has the following properties*:

   (i) *If $1 < p < 2$, $xy > 0$ and $|x|, |y| \geq m > 0$, then*

$$
\left|\varphi_p(x) - \varphi_p(y)\right| \leq (p-1)m^{p-2}|x-y|.
$$

   (ii) *If $p > 2$ and $|x|, |y| \leq M$, then*

$$
\left|\varphi_p(x) - \varphi_p(y)\right| \leq (p-1)M^{p-2}|x-y|.
$$

Now we are in position to prove the uniqueness of a solution for FBVP (1.4).

Theorem 3.5 *Assume that $0 < \beta \leq 1$, $2 < \alpha < 2 + \beta$, $1 < p < 2$, and the following conditions are satisfied*:

(H3) *For all $r > 0$, there exists a nonnegative function $h_r \in L[0,1]$ with $0 < \int_0^1 h_r(t)\,dt \leq M$ (a positive constant) such that*

$$
\left|f(t,u)\right| \leq h_r(t), \quad \forall (t,u) \in (0,1] \times [-r,r];
$$

(H4) *There exists a constant $k > 0$ such that*

$$
\left|f(t,u) - f(t,v)\right| \leq k|u-v|, \quad \forall t \in [0,1], u,v \in \mathbb{R}.
$$

If

$$
0 < k < \frac{1}{(\alpha-1)(q-1)M^{q-2}B(3, \alpha-\beta)}, \tag{3.9}
$$

then FBVP (1.4) *has a unique solution in $C[0,1]$.*

Proof By (H3) we have that

$$\left| \int_0^t f\big(s, u(s)\big)\, ds \right| \le \int_0^1 \big| f\big(s, u(s)\big) \big|\, ds \le \int_0^1 h_r(s)\, ds \le M, \quad t \in [0, 1]. \tag{3.10}$$

By Lemmas 3.1 and 3.4(i) and by (3.10) we obtain

$$\big| (Tu)(t) - (Tv)(t) \big|$$

$$= \left| \int_0^1 G(t, s) \varphi_q \left(\int_0^s f\big(\tau, u(\tau)\big) \right) d\tau - \int_0^1 G(t, s) \varphi_q \left(\int_0^s f\big(\tau, v(\tau)\big) \right) ds \right|$$

$$\le \int_0^1 G(t, s) \left| \varphi_q \left(\int_0^s f\big(\tau, u(\tau)\big)\, d\tau \right) - \varphi_q \left(\int_0^s f\big(\tau, v(\tau)\big)\, d\tau \right) \right| ds$$

$$\le (q - 1) M^{q-2} \int_0^1 G(t, s) \left| \int_0^s f\big(\tau, u(\tau)\big)\, d\tau - \int_0^s f\big(\tau, v(\tau)\big)\, d\tau \right| ds$$

$$\le (q - 1) M^{q-2} \int_0^1 G(t, s) \int_0^s \big| f\big(\tau, u(\tau)\big) - f\big(\tau, v(\tau)\big) \big|\, d\tau\, ds$$

$$\le (q - 1) M^{q-2} \int_0^1 G(t, s) \int_0^s k \| u - v \|\, d\tau\, ds$$

$$\le (q - 1) k M^{q-2} \| u - v \| \int_0^1 (\alpha - 1) t^{\alpha-2} s^2 (1 - s)^{\alpha-\beta-1}\, ds$$

$$= (\alpha - 1)(q - 1) k M^{q-2} t^{\alpha-2} B(3, \alpha - \beta) \| u - v \|$$

$$\le (\alpha - 1)(q - 1) k M^{q-2} B(3, \alpha - \beta) \| u - v \|$$

$$= L_1 \| u - v \|,$$

where $L_1 = (\alpha - 1)(q - 1) k M^{q-2} B(3, \alpha - \beta)$. From condition (3.9) we know that $0 < L_1 < 1$. Hence, by means of the Banach contraction mapping principle we obtain that T has a unique fixed point in E, that is, that FBVP (1.4) has a unique solution. The proof is completed. □

Theorem 3.6 *Assume that $0 < \beta \le 1$, $2 < \alpha < 2 + \beta$, and $p > 2$ and that* (H4) *and the following condition holds*:

(H5) *There exist constants $\lambda > 0$ and $0 < \delta < \frac{\alpha - 2}{2 - q}$ such that*

$$f(t, u) \ge \lambda \delta t^{\delta - 1}, \quad \forall (t, u) \in (0, 1] \times \mathbb{R}.$$

If

$$0 < k < \frac{1}{(\alpha - 1)(q - 1)\lambda^{q-2} B(3, \alpha - \beta)}, \tag{3.11}$$

then FBVP (1.4) *has a unique solution in $C[0, 1]$.*

Proof By (H5) we have

$$\int_0^t f(s, u)\, ds \ge \int_0^t \lambda \delta s^{\delta-1}\, ds = \lambda t^\delta, \quad \forall (t, u) \in [0, 1] \times \mathbb{R}. \tag{3.12}$$

Obviously, for any $u, v \in E$, we have $|(Tu)(0) - (Tv)(0)| = 0$. For any $t \in (0, 1]$, by Lemmas 3.2 and 3.4(ii) and by (3.12) we get

$$
\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \int_0^1 G(t,s) \left| \varphi_q \left(\int_0^s f(\tau, u(\tau)) \, d\tau \right) - \varphi_q \left(\int_0^s f(\tau, v(\tau)) \, d\tau \right) \right| ds \\
&\leq (q-1)(\lambda t^\delta)^{q-2} \int_0^1 G(t,s) \left| \int_0^s f(\tau, u(\tau)) \, d\tau - \int_0^s f(\tau, v(\tau)) \, d\tau \right| ds \\
&\leq (q-1)(\lambda t^\delta)^{q-2} \int_0^1 G(t,s) \int_0^s k |u(\tau)) - v(\tau))| \, d\tau \, ds \\
&\leq (q-1) k (\lambda t^\delta)^{q-2} \|u - v\| \int_0^1 (\alpha - 1) t^{\alpha-2} s^2 (1-s)^{\alpha-\beta-1} \, ds \\
&\leq (\alpha - 1)(q-1) k (\lambda t^\delta)^{q-2} t^{\alpha-2} \|u - v\| \int_0^1 s^2 (1-s)^{\alpha-\beta-1} \, ds \\
&= (\alpha - 1)(q-1) k \lambda^{q-2} t^{(q-2)\delta + \alpha - 2} B(3, \alpha - \beta) \|u - v\| \\
&\leq (\alpha - 1)(q-1) k \lambda^{q-2} B(3, \alpha - \beta) \|u - v\|,
\end{aligned}
$$

which implies that

$$
\|Tu - Tv\| \leq L_2 \|u - v\|,
$$

where $L_2 = (\alpha - 1)(q - 1) k \lambda^{q-2} B(3, \alpha - \beta)$. By condition (3.11) we obtain that $0 < L_2 < 1$. Thus, $T : E \to E$ is a contraction mapping. Using the Banach contraction mapping principle, we obtain that T has a unique fixed point in E. Hence, FBVP (1.4) has a unique solution. The proof is completed. $\qquad \square$

Similarly, we have the following:

Theorem 3.7 *Assume that* $0 < \beta \leq 1$, $2 < \alpha < 2 + \beta$, *and* $p > 2$ *and that* (H4) *and the following condition holds*:

(H6) *There exist constants* $\lambda > 0$ *and* $0 < \delta < \frac{\alpha - 2}{2 - q}$ *such that*

$$
f(t, u) \leq -\lambda \delta t^{\delta-1}, \quad \forall (t, u) \in (0, 1] \times \mathbb{R}.
$$

If k satisfies (3.11), *then FBVP* (1.4) *has a unique solution in* $C[0, 1]$.

4 Examples

In this section, we present some examples to illustrate our main results obtained in the previous section.

Example 4.1 We consider the fractional boundary value problem

$$
\begin{cases}
|D_{0^+}^{\frac{7}{3}} u(t)|^{\frac{1}{2}} D_{0^+}^{\frac{7}{3}} u(t) + (t^2 + 2) e^u = 0, & 0 < t < 1, \\
u(0) = D_{0^+}^{\frac{7}{3}} u(0) = 0, \qquad {}^c D_{0^+}^{0.7} u(0) = {}^c D_{0^+}^{0.7} u(1) = 0.
\end{cases}
\tag{4.1}
$$

Obviously, FBVP (4.1) can be regarded as FBVP (1.4) with $p = \frac{5}{2}$, $\alpha = \frac{7}{3}$, $\beta = 0.7$, and

$$f(t, u) = (t^2 + 2)e^u, \quad \forall (t, u) \in [0, 1] \times \mathbb{R}.$$

By a simple computation we obtain $q = \frac{5}{3}$,

$$\rho = \varphi_{\frac{5}{2}}\left(\frac{\Gamma(\frac{7}{3})2^{\frac{4}{3}}}{0.7 \cdot B(\frac{8}{3}, \frac{49}{30})}\right) = 148.986, \quad \text{and} \quad \omega = \varphi_{\frac{5}{2}}\left(\frac{\Gamma(\frac{7}{3})}{\frac{4}{3} \cdot B(\frac{8}{3}, \frac{49}{30})}\right) = 14.169.$$

We choose $r_1 = \frac{1}{20}$ and $r_2 = \frac{2}{3}$. Then we can obtain

$$f(t, u) = (t^2 + 2)e^u \geq 2 > \rho\varphi_{\frac{5}{2}}(r_1) = 1.669 \quad \text{for } (t, u) \in [0, 1] \times [0, r_1];$$

$$f(t, u) = (t^2 + 2)e^u \leq 3e^{\frac{2}{3}} < \omega\varphi_{\frac{5}{2}}(r_2) = 7.713 \quad \text{for } (t, u) \in [0, 1] \times [0, r_2].$$

Hence, by Theorem 3.3, FBVP (4.1) has at least one nontrivial positive solution u in E such that $\frac{1}{20} \leq \|u\| \leq \frac{2}{3}$.

Example 4.2 Consider the following fractional boundary value problem:

$$(4.2) \quad \begin{cases} |D_{0+}^{\frac{5}{2}}u(t)|^{-\frac{1}{3}}D_{0+}^{\frac{5}{2}}u(t) + (1 + 2t^2)\arctan(\frac{2}{3}(u + 1)) = 0, & 0 < t < 1, \\ u(0) = D_{0+}^{\frac{5}{2}}u(0) = 0, \qquad {}^c D_{0+}^{0.8}u(0) = {}^c D_{0+}^{0.8}u(1) = 0. \end{cases}$$

FBVP (4.2) can be regarded as FBVP (1.4) with $p = \frac{5}{3}$, $\alpha = \frac{5}{2}$, $\beta = 0.8$, and

$$f(t, u) = (1 + 2t^2)\arctan\left(\frac{2}{3}(u + 1)\right), \quad \forall (t, u) \in [0, 1] \times \mathbb{R}.$$

It is easy to see that, for any $r > 0$,

$$|f(t, u)| \leq h_r(t) \quad \text{for } (t, u) \in [0, 1] \times [-r, r],$$

where $h_r(t) = (1 + 2t^2)\arctan(\frac{2}{3}(r + 1))$. Obviously, $h_r \in L[0, 1]$ and $\int_0^1 h_r(s)\, ds \leq \int_0^1 \frac{\pi}{2}(1 + 2s^2)\, ds = \frac{5\pi}{6} := M$. Moreover, we have

$$|f(t, u) - f(t, v)| = (1 + 2t^2)\left|\arctan\left(\frac{2}{3}(u + 1)\right) - \arctan\left(\frac{2}{3}(v + 1)\right)\right|$$

$$\leq \frac{2}{3}(1 + 2t^2)|u - v| \leq 2|u - v| \quad \text{for } t \in [0, 1], u, v \in \mathbb{R}.$$

Let $k = 2$. We have

$$0 < k < \frac{1}{(\alpha - 1)(q - 1)M^{q-2}B(3, \alpha - \beta)} = 2.332,$$

where $q = \frac{5}{2} > 2$. Thus, by Theorem 3.5, FBVP (4.2) has a unique nontrivial solution.

5 Conclusion

In this paper, we obtain an equivalent integral equation for a class of fractional boundary value problem with p-Laplacian operator. Using the properties of the corresponding Green function and Guo-Krasnosel'skii fixed point theorem on cones, we obtain the existence of positive solutions to problem (1.4). Moreover, applying the properties of the p-Laplacian operator and the Banach contraction mapping principle, we get some uniqueness results of solutions. Finally, we provide two examples to illustrate the main results.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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