Al-Omari *Advances in Difference Equations* (2018) 2018:7
https://doi.org/10.1186/s13662-017-1457-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# Certain results related to the $N$-transform of a certain class of functions and differential operators

Shrideh K Al-Omari*

*Correspondence:
s.k.q.alomari@fet.edu.jo
Department of Physics and Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman, Jordan

**Abstract**

In this paper, we aim to investigate $q$-analogues of the natural transform on various elementary functions of special type. We obtain results associated with classes of $q$-convolution products, Heaviside functions, $q$-exponential functions, $q$-hyperbolic functions and $q$-trigonometric functions as well. Further, we give definitions and derive results involving some $q$-differential operators.

**Keywords:** $q$-differential operator; $q$-trigonometric function; $q$-convolution product; $q$-natural transform; $q$-hyperbolic function; Heaviside function

## 1 Introduction

The subject of fractional calculus has gained noticeable importance and popularity due to its mainly demonstrated applications in diverse fields of science and engineering problems. Much of developed theories of fractional calculus were based upon the familiar Riemann-Liouville activity related to the $q$-calculus has been noticed due to applications of this theory in mathematics, statistics and physics.

Jackson in [1] presented a precise definition and developed the $q$-calculus in a systematic way. Thereafter, some remarkable integral transforms were associated with different analogues in a $q$-calculus context. Among those examined integrals, we recall the $q$-Laplace integral operator [2–5], the $q$-Sumudu integral operator [6–9], the Weyl fractional $q$-integral operator [10], the $q$-wavelet integral operator [11], the $q$-Mellin type integral operator [12], Mangontarum transform [13], Widder potential transform [14], $E_{2;1}$-transform [15] and a few others. In this paper, we give some $q$-analogues of some recently investigated integral transform, named the natural transform, and obtain some new desired $q$-properties.

In the following section, we present some notations and terminologies from the $q$-calculus. In Section 3, we give the definition and properties of the natural transform. In Section 4, we give the definition of the analogue $N_q$ of the natural transform and apply it to some functions of special type. In more details, Sections 5 to 7 are associated with the application of the analogue $N_q$ to Heaviside functions, convolutions and differentiations. The remaining two sections investigate the $N^q$ analogue of the natural transform on some elementary functions with similar approach.

## 2 Definitions and preliminaries

We present some usual notions and notations used in the $q$-calculus [16–18]. Throughout this paper, we assume $q$ to be a fixed number satisfying $0 < q < 1$. The $q$-calculus begins with the definition of the $q$-analogue $d_q f(x)$ of the differential of functions

$$d_q f(x) = f(qx) - f(x). \tag{1}$$

Having said this, we immediately get the $q$-analogue of the derivative of $f(x)$, called its $q$-derivative,

$$(D_q f)(x) := \frac{d_q f(x)}{d_q x} := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \tag{2}$$

$(D_q f)(0) = \acute{f}(0)$ provided $\acute{f}(0)$ exists. If $f$ is differentiable, then $(D_q f)(x)$ tends to $\acute{f}(0)$ as $q$ tends to 1. It can be seen that the $q$-derivative satisfies the following $q$-analogue of the Leibniz rule:

$$D_q\bigl(f(x)g(x)\bigr) = g(x)D_q f(x) + f(qx)D_q g(x). \tag{3}$$

The $q$-Jackson integrals are defined by [1, 12]

$$\int_0^x f(x)\, d_q x = (1-q)x \sum_0^\infty f\bigl(xq^k\bigr)q^k, \tag{4}$$

$$\int_0^\infty f(x)\, d_q x = (1-q)x \sum_{-\infty}^\infty f\bigl(q^k\bigr)q^k, \tag{5}$$

provided the sum converges absolutely.

The $q$-Jackson integral in a generic interval $[a, b]$ is given by [1]

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x. \tag{6}$$

The improper integral is defined in the way that

$$\int_0^{\frac{\infty}{A}} f(x)\, d_q x = (1-q) \sum_{-\infty}^\infty f\left(\frac{q^k}{A}\right)\frac{q^k}{A}, \tag{7}$$

and, for $n \in Z$, we have

$$\int_0^{\frac{\infty}{q^n}} f(x)\, d_q x = \int_0^\infty f(x)\, d_q x. \tag{8}$$

The $q$-integration by parts is defined for functions $f$ and $g$ by

$$\int_0^b g(x)D_q f(x)\, d_q x = f(b)g(b) - f(a)g(a) - \int_a^b f(qx)D_q g(x)\, d_q x. \tag{9}$$

For $x \in C$, the $q$-shifted factorials are defined by $(x;q)_0 = 1; (x;q)_t = \frac{(x;q)_\infty}{(xq^t;q)_\infty}; (x;q)_n = \prod_0^{n-1}(1 - xq^k)$, and hence

$$(x;q)_\infty = \prod_{k=0}^{\infty}\left(1 - xq^k\right). \tag{10}$$

The $q$-analogues of $x$ and $\infty$ are respectively defined by

$$[x] = \frac{1-q^x}{1-q}, \qquad [\infty] = \frac{1}{1-q}. \tag{11}$$

The $q$-analogues of the exponential function of the first and second kind are respectively given as follows:

$$E_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{([n]_q)!} \quad (x \in C) \tag{12}$$

and

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{([n]_q)!} \quad \left(|x| < 1\right), \tag{13}$$

where $([n]_q)! = [n]_q[n-1]_q \cdots [2]_q[1]_q$, $[n] = \frac{1-q^n}{1-q} = q^{n-1} + \cdots + q + 1$.

However, in view of the product expansions, $(e_q(x))^{-1} = E_q(-x)$, which explains the need for both analogues of the exponential function.

The $q$-derivative of $E_q(x)$ is $D_q E_q(x) = E_q(qx)$, whereas the $q$-derivative of $e_q(x)$ is $D_q e_q(x) = e_q(x)$, $e_q(0) = 1$.

The gamma and beta functions satisfy the $q$-integral representations

$$\left.\begin{array}{l} \Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx)\, d_q x \\ B_q(t;s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x \quad (t,s > 0) \end{array}\right\} \tag{14}$$

and are related to the identities $B_q(t;s) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}$ and $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$. Due to (12) and (13), the $q$-analogues of sine and cosine functions of the second and first kind are respectively given as

$$\left.\begin{array}{l} \sin^q(at) = \sum_0^\infty (-1)^n \frac{(at)^{2n+1}}{([2n+1]_q)!}; \\ \cos^q(at) = \sum_0^\infty (-1)^n \frac{(at)^{2n}}{([2n]_q)!}; \\ \sin_q(at) = \sum_0^\infty (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{([2n+1]_q)!} a^{2n+1} t^{2n+1}; \\ \cos_q(at) = \sum_0^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{([2n]_q)!} a^{2n} t^{2n}. \end{array}\right\} \tag{15}$$

## 3 The *N*-transform or the natural transform

The natural transform of a function $f(x)$, $x \in (0,\infty)$, was proposed by Khan and Khan [19] as an extension to Laplace and Sumudu transforms to solve some fluid flow problems. Later, Silambarasan and Belgacem [20] derived certain electric field solutions to

Maxwell's equation in conducting media. In [21], the author applied the natural transform to some ordinary differential equations and some space of Boehmians. For further investigations of the natural transform, one can refer to [22] and [23]. Over the set $A$, where $A = \{f(t)|\exists\tau_1,\tau_2,M > 0, |f(t)| < Me^{t/\tau j}, t \in (-1)^j \times [0,\infty)|\}$, the $N$-transform of $f(t)$ is given as [19, (1)],

$$(Nf)(u;v) = \int_0^\infty f(ut)\exp(-vt)\,dt \quad (u,v > 0). \tag{16}$$

Provided the integral above exists, it is so easy to see that

$$(Nf)(1;v) = (Lf)(v), \qquad (Nf)(u;1) = (Sf)(u), \tag{17}$$

where $Sf$ and $Lf$ are respectively the Sumudu and Laplace transforms of $f$. Moreover, the natural-Laplace and natural-Sumudu dualities are given as [21, (5), (6)],

$$(Nf)(u;v) = \int_0^\infty \frac{1}{u} f(t)\exp\left(-\frac{vt}{u}\right)dt \tag{18}$$

and

$$(Nf)(u;v) = \int_0^\infty \frac{1}{v} f\left(\frac{ut}{v}\right)\exp(-t)\,dt, \tag{19}$$

respectively.

Further, from (18) and (19), it can be easily observed that

$$(Nf)(u;v) = \frac{1}{u}(Lf)\left(\frac{u}{v}\right), \qquad (Nf)(u;v) = \frac{1}{v}(Sf)\left(\frac{u}{v}\right). \tag{20}$$

The natural transform of some known functions is given as follows [21, p. 731]:

(i) $N(a)(u;v) = \frac{1}{v}$, where $a$ is a constant;
(ii) $N(\delta)(u;v) = \frac{1}{v}$, where $\delta$ is the delta function;
(iii) $N(e^{at})(u;v) = \frac{1}{v-au}$, $a$ is a constant.

## 4 The $N_q$ analogue of the natural transform

Hahn [24] and later Ucar and Albayrak [4] defined the $q$-analogues of the first and second type of Laplace transforms by means of the $q$-integrals

$$L_q(f;s) = \frac{1}{1-q}\int_0^{\frac{1}{s}} E_q(qst)f(t)\,d_qt, \tag{21}$$

$${}_qL(f;s) = \frac{1}{1-q}\int_0^\infty e_q(-st)f(t)\,d_qt. \tag{22}$$

On the other hand, the $q$-analogues of the Sumudu transform of the first and second type were defined by [6, 7]

$$S_q(f;s) = \frac{1}{(1-q)_s}\int_0^s E_q\left(\frac{q}{s}t\right)f(t)\,d_qt, \tag{23}$$

$$ {}_qS(f;s) = \frac{1}{1-q}\int_0^\infty e_q\left(-\frac{t}{s}\right)f(t)\,d_qt, \tag{24}$$

where $s \in (-\tau_1, \tau_2)$ and $f$ is a function of the set $A$, $A = \{f(t)|\exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0,\infty)\}$. Now, we are in a position to demonstrate our results as follows: the $q$-analogue of the natural transform of the first kind is

$$ (N_qf)(u;v) := \bigl(N_qf;(u,v)\bigr) := \int_0^\infty \frac{1}{u}f(t)E_q\left(-q\frac{vt}{u}\right)d_qt, \tag{25}$$

provided the function $f(t)$ is defined on $A$ and $u$ and $v$ are the transform variables. The $q$-series representation of (25) can be written as

$$ (N_qf)(u;v) = \frac{1}{(1-q)u}\sum_{k\in Z} q^kf\bigl(q^k\bigr)E_q\left(-q^{k+1}\frac{v}{u}\right), \tag{26}$$

which, by (10), can be put into the form

$$ (N_qf)(u;v) = \frac{(q;q)_\infty}{(1-q)^n}\sum_{k\in Z}\frac{q^kf(q^k)}{(-\frac{v}{u};q)_{k+1}}. \tag{27}$$

Let us now discuss some general applications of $N_q$ to some elementary functions.

**Theorem 1** *Let $\alpha \in R$. Then we have*

$$ \bigl(N_qt^\alpha\bigr)(u;v) = \frac{u^\alpha}{v^{\alpha+1}}\Gamma(\alpha+1). \tag{28}$$

*Proof* By setting the variables, we get

$$ \bigl(N_qt^\alpha\bigr)(u;v) = \frac{u^\alpha}{v^{\alpha+1}}\int_0^\infty t^\alpha E_q(-qt)\,d_qt = \frac{u^\alpha}{v^{\alpha+1}}\Gamma_q(\alpha+1).$$

The theorem hence follows. □

The direct consequence of (28) is that

$$ \bigl(N_qt^n\bigr)(u;v) = \frac{u^n}{v^{n+1}}\bigl([n]_q\bigr)!. \tag{29}$$

**Theorem 2** *Let $a$ be a positive real number. Then we have*

$$ \bigl(N_qe_q(at)\bigr)(u;v) = \frac{1}{v-au}, \quad au < v. \tag{30}$$

*Proof* By aid of the $e_q$ analogue of the $q$-exponential function, we write

$$ \bigl(N_qe_q(at)\bigr)(u;v) = \int_0^\infty \frac{1}{u}e_q(at)E_q\left(-q\frac{v}{u}t\right)d_qt. \tag{31}$$

In series representation, (31) can be written as

$$ \bigl(N_qe_q(at)\bigr)(u;v) = \sum_0^\infty \frac{a^n}{u([n]_q)!}t^nE^q\left(-q\frac{v}{u}t\right)d_qt. \tag{32}$$

By (29), (32) gives a geometric series expansion, and hence we have $(N_q e_q(at))(u;v) = \frac{1}{v}\sum_0^\infty (\frac{au}{v})^n = \frac{1}{v-au}, au < v$.

This finishes the proof of the lemma. □

**Theorem 3** *Let a be a positive real number. Then we have*

$$\bigl(N_q E_q(at)\bigr)(u;v) = \frac{1}{v}\sum_0^\infty q\frac{n(n-1)}{2}\left(\frac{au}{v}\right)^n. \tag{33}$$

*Proof* In view of the $E_q$ analogue of the exponential function, we indeed get

$$\bigl(N_q(at)\bigr)(u;v) = \sum_0^\infty \frac{a_n q^{\frac{n(n-1)}{2}}}{([n]_q)!}\frac{1}{u}\int_0^\infty t^n E_q\left(-q\frac{v}{u}t\right)d_q t. \tag{34}$$

The parity of (29) gives $(N_q E_q(at))(u;v) = \frac{1}{v}\sum q^{\frac{n(n-1)}{2}}(\frac{au}{v})^n$.

This completes the proof of the theorem.

The hyperbolic $q$-cosine and $q$-sine functions are given as $\cosh^q t = \frac{e_q(t)+e_q(-t)}{2}$ and $\sinh^q t = \frac{e_q(t)-e_q(-t)}{2}$. Hence, as a corollary of Theorem 2, we have

$$\bigl(N_q \cosh^q t\bigr)(u;v) = \frac{v}{v^2+a^2u^2}, \quad au < v, \tag{35}$$

$$\bigl(N_q \sinh^q t\bigr)(u;v) = \frac{au}{v^2+a^2u^2}, \quad au < v. \tag{36}$$

□

**Theorem 4** *Let a be a positive real number. Then we have* $(N_q \cos^q at)(u;v) = \frac{v}{v^2-a^2u^2}$ *provided* $au < v$.

*Proof* On account of (15) and (29), we obtain

$$\begin{aligned}\bigl(N_q \cos^q at\bigr)(u;v) &= \sum_0^\infty \frac{a^{2n}}{u([2n]_q)!}\int_0^\infty t^{2n}E_q\left(-q\frac{v}{u}t\right)d_q t \\ &= \frac{1}{v}\sum_0^\infty\left(\frac{au}{v}\right)^{2n}.\end{aligned}$$

For $au < v$, the geometric series converges to the sum

$$\bigl(N_q \cos^q at\bigr)(u;v) = \frac{v}{v^2-a^2u^2} \tag{37}$$

provided $au < v$.

This finishes the proof. □

Similarly, by (15) and (29), we deduce that

$$\bigl(N_q \sin^q at\bigr)(u;v) = \frac{au}{v^2-a^2u^2}, \quad au < v. \tag{38}$$

## 5 $N_q$ and $q$-differentiation

In this section, we discuss some $q$-differentiation formulae. On account of (12), we derive the following differentiation result.

**Lemma 5** *Let $u, v > 0$. Then we have*

$$D_q E_q\left(-q\frac{u}{v}t\right) = \frac{v}{u}\sum_0^\infty (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n. \tag{39}$$

*Proof* By using the $q$-representation of $E_q$ in (12), we write

$$\begin{aligned} D_q E_q\left(-q\frac{v}{u}t\right) &= D_q \sum_0^\infty \frac{(-1)^{\frac{n(n-1)}{2}}}{([n]_q)!}\left(\frac{qv}{u}\right)^n t^n \\ &= \sum_1^\infty (-1)^n \frac{q^{\frac{(n+1)(n+2)}{2}}}{([n-1]_q)!} q^n \frac{v^n}{u^n} t^{n-1} \\ &= \sum_1^\infty (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^{n+1}}{u^{n+1}} t^n. \end{aligned}$$

Hence, it follows that $D_q E_q(-q\frac{v}{u}t) = \frac{v}{u}\sum_0^\infty (-1)^{n+1} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n$. This finishes the proof. □

The natural transform of the $q$-derivative $D_q f$ can be written as follows.

**Theorem 6** *Let $u, v > 0$, then we have*

$$N_q\big(D_q f(t)\big)(u;v) = -f(0) + \frac{v}{u} N_q(f)(u;v). \tag{40}$$

*Proof* Using the idea of $q$-integration by parts and the formula in (9), we write

$$\begin{aligned} N_q\big(D_q f(t)\big)(u;v) &= \int_0^\infty D_q f(t) E_q\left(-q\frac{v}{u}t\right) d_q t \\ &= f(t) D_q E_q\left(-q\frac{v}{u}t\right)\Big|_0^\infty - \int_0^\infty f(qt) D_q E_q\left(-q\frac{v}{u}t\right) d_q t. \end{aligned}$$

The parity of Lemma 5, gives

$$\begin{aligned} N_q\big(D_q f(t)\big)(u;v) &= -f(0) - \int_0^\infty f(qt) \frac{v}{u} \sum_0^\infty \frac{(-1)^{n+1}}{([n]_q)!} \\ &= -f(0) + \frac{v}{u}\int_0^\infty f(qt) \sum_0^\infty \frac{(-1)^n}{([n]_q)!} q^{\frac{(n+1)(n+2)}{2}} \frac{v^n}{u^n} t^n \, d_q t. \end{aligned}$$

Changing the variables as $qt = y$ and $t^n = q^{-n} y^n$ implies

$$N_q\big(D_q f(t)\big)(u;v) = -f(0) + \frac{v}{u}\int_0^\infty f(t) \sum_0^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{([n]_q)!} \frac{v^n}{u^n} t^n \, d_q t. \tag{41}$$

By virtue of (12), (41) yields $N_q(D_q f(t))(u;v) = -f(0) + \frac{v}{u}\int_0^\infty f(t)E_q(-q\frac{v}{u}t)\,d_qt$. Hence, $N_q(D_q f(t))(u;v) = -f(0) + \frac{v}{u}N_q(f)(u;v)$. This finishes the proof. □

Now we extend Theorem 6 to the $n$th derivative as in the following theorem.

**Theorem 7** *Let $u, v > 0$ and $n \in Z^+$. Then we have*

$$N_q\bigl(D_q^n f(t)\bigr)(u;v) = \frac{v^n}{u^n}\bigl(N_q(f)\bigr)(u,v) - \sum_{i=0}^{n-1}\left(\frac{u}{v}\right)^{n-1-i} D_q^i f(0). \tag{42}$$

*Proof* On account of Theorem 6, we can write $N_q(D_q^2 f(t))(u;v) = -D_q f(0) + \frac{v}{u}(-f(0) + \frac{v}{u}(N_q f)(u;v))$. Hence we have obtained

$$N_q\bigl(D_q^2 f(t)\bigr)(u;v) = -D_q f(0) - \frac{v}{u}f(0) + \frac{v^2}{u^2}(N_q f)(u;v). \tag{43}$$

Proceeding as in (43), we obtain $N_q(D_q^n f(t))(u;v) = \frac{v^2}{u^2}N_q(f)(u;v) - \sum_{i=0}^{n-1}(\frac{v}{u})^{n-1-i}D_q^i f(0)$.
This finishes the proof. □

## 6 $N_q$ of $q$-convolutions

Let functions $f$ and $g$ be given in the form $f(t) = t^\alpha$ and $g(t) = t^{\beta-1}$ for $\alpha, \beta > 0$. We define the $q$-convolution of $f$ and $g$ as

$$(f * g)_q(t) = \int_0^t f(\tau)g(t - q\tau)\,d_qt. \tag{44}$$

The $q$-convolution theorem can be read as follows.

**Theorem 8** *Let $\alpha, \beta > 0$. Then we have*

$$N_q\bigl((f * g)_q\bigr)(u;v) = u^2 N_q\bigl(t^\alpha\bigr)(u,v)N_q\bigl(t^{\beta-1}\bigr)(u;v). \tag{45}$$

*Proof* By aid of (44) and (14), we get $N_q((f * g)_q)(u;v) = \frac{B_q(\alpha+1,\beta)}{u}\int_0^\infty t^{\alpha+\beta}E_q(-q\frac{vt}{u})\,d_qt$. Hence, by (28) we obtain

$$N_q\bigl((f * g)_q\bigr)(u;v) = \Gamma_q(\alpha+1)\Gamma_q(\beta)\frac{u^{\alpha+\beta+1}}{v^{\alpha+\beta+1}}. \tag{46}$$

Motivation on (46) gives $N_q((f*g)_q)(u;v) = u^2(N_q t^\alpha)(u,v)(N_q t^{\beta-1})(u;v)$. The proof is therefore completed. □

In a similar way, we extend the $q$-convolution to functions of a power series form.

**Theorem 9** *Let $f(t) = \sum_0^\infty a_i t^{\alpha i}$ and $g(t) = t^{\beta-1}$. Then we have $N_q((f * g)_q)(u;v) = u^2(N_q f)(u;v)(N_q g)(u;v)$.*

*Proof* Under the hypothesis of the theorem and Theorem 8, we write

$$\begin{aligned} N_q\big((f*g)_q\big)(u;v) &= \sum_0^\infty a_i N_q\big(\big(t^{\alpha i} * t^{\beta-1}\big)_q\big)(u;v) \\ &= \sum_0^\infty a_i \big(N_q t^{\alpha i}\big)(u;v)\big(N_q t^{\beta-1}\big)(u;v) \\ &= u^2 (N_q f)(u;v)(N_q g)(u;v). \end{aligned}$$

Hence, the proof of the theorem is finished. □

## 7 $N_q$ and Heaviside functions

The Heaviside function is defined by

$$\acute{u}(t-a) = \begin{cases} 1, & t \ge a, \\ 0, & 0 \le t < a, \end{cases} \tag{47}$$

where $a$ is a real number.

In this part of the paper, we establish the following theorem.

**Theorem 10** *If $\acute{u}$ denotes the Heaviside function and $u, v > 0$, then we have*

$$N_q\big(\acute{u}(t-a)\big)(u;v) = \frac{1}{v} E_q\left(-\frac{v}{u} a\right). \tag{48}$$

*Proof* By (47) we have $N_q(\acute{u}(t-a))(u;v) = \frac{1}{u}\int_a^\infty E_q(-q\frac{v}{u}t)\, d_q t$. On account of (21), we get

$$\begin{aligned} N_q\big(\acute{u}(t-a)\big)(u;v) &= \frac{1}{v} - \frac{1}{u}\int_0^a \sum_0^\infty \frac{q^{\frac{n(n-1)}{2}}}{([n]_q)!}\left(-q\frac{v}{u}t\right)^n d_q t \\ &= \frac{1}{v} - \frac{1}{u}\sum_0^\infty (-1)^n \frac{q^{\frac{n(n-1)}{2}}}{([n]_q)!} q^n \frac{v^n}{u^n}\int_0^a t^n\, d_q t. \end{aligned}$$

Upon integrating and using simple calculation, the above equation reveals

$$\begin{aligned} N_q\big(\acute{u}(t-a)\big)(u;v) &= \frac{1}{v} - \frac{1}{u}\sum_0^\infty \frac{q^{\frac{n(n-1)}{2}}}{[n]_q!}\frac{v^n}{u^n}\frac{a^{n+1}}{[n+1]_q} \\ &= \frac{1}{v} + \frac{1}{u}\sum_0^\infty (-1)^{n+1}\frac{q^{n+1\frac{n}{2}}}{[n+1]_q!}\frac{v^n}{u^n}a^{n+1} \\ &= \frac{1}{v} + \frac{1}{v}\sum_0^\infty (-1)^{n+1}\frac{q^{\frac{(n+1)n}{2}}}{[n+1]_q!}\frac{v^{n+1}}{u^{n+1}}a^{n+1}. \end{aligned}$$

This can be written as $N_q(\acute{u}(t-a))(u;v) = \frac{1}{v} + \frac{1}{v}\sum_1^\infty (-1)^m \frac{q^{m(m-1)}}{([m]_q)!}\frac{v^m}{u^m}a^m$. Starting the summation from 0 gives

$$N_q\big(\acute{u}(t-a)\big)(u;v) = \frac{1}{v}\sum_1^\infty (-1)^m \frac{q^{\frac{m(m-1)}{2}}}{([m]_q)!}\frac{v^m}{u^m}a^m = \frac{1}{v}E_q\left(\frac{-v}{u}a\right).$$ □

## 8 The $q$-analogue of the second kind

Over the set $A = \{f(t)|\exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, t \in (-1)^j \times [0,\infty)\}$, the $q$-analogue of the $N$-transform of the second type is expressed as

$$\left(N^q f\right)(u;v) = \int_0^\infty \frac{1}{u} f(t) e_q\left(-\frac{v}{u}t\right) d_q t. \tag{49}$$

As the $q$-analogue of the gamma function of the second kind is defined as

$$\gamma_q(t) = \int_0^\infty x^{t-1} e_q(-x)\, d_q x, \tag{50}$$

it follows that

$$\gamma_q(1) = 1, \gamma_q(t+1) = q^{-t}[t]_q \gamma_q(t), \quad \gamma_q(n) = q^{\frac{n(n-1)}{2}} \Gamma_q(n), \tag{51}$$

$\Gamma_q$ being the $q$-analogue of the gamma function of the first kind.

In the remainder of this section, we derive certain results analogous to the ones we have obtained in the previous chapters.

**Lemma 11** *Let $\alpha > -1$. Then we have*

$$\left(N^q t^\alpha\right)(u;v) = \frac{u^\alpha}{v^{\alpha+1}} \gamma_q(\alpha+1). \tag{52}$$

*In particular,*

$$\left(N^q t^n\right)(u;v) = \frac{u^n}{v^{n+1}} q^{\frac{-n(n-1)}{2}} \left([n_q]\right)!. \tag{53}$$

*Proof* Let $\alpha > -1$. Then, by the change of variables, we have

$$\left(N^q t^\alpha\right)(u;v) = \frac{1}{u} \int_0^\infty t^\alpha e_q\left(\frac{-vt}{u}\right) d_q t = \frac{u^\alpha}{v^{\alpha+1}} \int_0^\infty t^\alpha e_q(-t)\, d_q t.$$

By aid of (50), we get $(N^q t^\alpha)(u;v) = \frac{u^\alpha}{v^{\alpha+1}} \gamma_q(\alpha+1)$. Proof of the second part of the theorem follows from (51). Hence, we have completed the proof of the theorem. □

**Theorem 12** *Let $a \in R, a > 0$, then we have*

$$\left(N^q e_q(at)\right)(u;v) = \frac{1}{uv} \sum_0^\infty \frac{a^n u^n}{v^n} q^{\frac{-n(n-1)}{2}}. \tag{54}$$

*Proof* By (13) we write

$$\left(N^q e_q(at)\right)(u;v) = \frac{1}{u} \int_0^\infty e_q(at) e_q\left(-\frac{v}{u}t\right) d_q t = \frac{1}{u} \sum_0^\infty \frac{a^n}{([n]_q)!} \int_0^\infty t^n e_q\left(-\frac{v}{u}t\right) d_q t.$$

By aid of Theorem 2, we get $(N^q e_q(at))(u;v) = \frac{1}{uv} \sum_0^\infty \frac{a^n u^n}{v^n} q^{\frac{-n(n-1)}{2}}$.

This completes the proof of the theorem. □

**Theorem 13** *Let $a > 0, a \in R$, then we have*

$$\big(N^q E_q(at)\big)(u;v) = \frac{1}{u(v-au)}, \quad au < v. \tag{55}$$

*Proof* After simple calculations and using Theorem 2, we obtain

$$\begin{aligned}\big(N^q e_q(at)\big)(u;v) &= \frac{1}{u}\int_0^\infty E_q(at)e_q\left(-\frac{v}{u}t\right)d_qt \\ &= \frac{1}{u}\sum_0^\infty \frac{q^{\frac{n(n-1)}{2}}}{([n]_q)!}a^n \int_0^\infty t^n e_q\left(-\frac{v}{u}t\right)d_qt \\ &= \frac{1}{uv}\sum_0^\infty a^n \frac{u^n}{v^n}.\end{aligned}$$

Since the above series determines a geometric series, it reads $(N^q e_q(at))(u;v) = \frac{1}{uv}\frac{1}{1-\frac{au}{v}} = \frac{1}{u(v-au)}, au < v$.

Hence the theorem is proved. □

The $N^q$ transform of $\cos^q$ and $\sin^q$ is given as follows.

**Theorem 14** *Let $a > 0$. Then we have*

$$\big(N^q \cos^q(at)\big)(u;v) = \frac{1}{u}\sum_0^\infty (-1)^n a^{2n}\frac{u^{2n}}{v^{2n}}q^{-2n\frac{(2n-1)}{2}}. \tag{56}$$

*Proof* Using the definition of $\cos^q$, we write

$$\begin{aligned}\big(N^q \cos^q(at)\big)(u;v) &= \frac{1}{u}\int_0^\infty \cos^q(at)e_q\left(-\frac{v}{u}t\right)d_qt \\ &= \frac{1}{u}\sum_0^\infty (-1)^n \frac{a^{2n}}{([2n]_q)!}\int_0^\infty t^{2n}e^q\left(-\frac{v}{u}t\right)d_qt \\ &= \frac{1}{u}\sum_0^\infty (-1)^n a^{2n}\frac{u^{2n}}{v^{2n}}q^{-2n\frac{(2n-1)}{2}}.\end{aligned}$$

This completes the proof of the theorem. □

The $N^q$ transform of $\sin^q(at)$ is given as follows.

**Theorem 15** *Let $a > 0$, then we have*

$$\big(N^q \sin^q(at)\big)(u;v) = \frac{1}{uv}\sum_0^\infty (-1)^n a^{2n+1}\frac{u^{2n+1}}{v^{2n+1}}q^{-2n\frac{(2n-1)}{2}}. \tag{57}$$

The proof of Theorem 15 follows from a proof similar to that of Theorem 14.

## 9 $N^q$ of $q$-differentiation

Before we start investigations, we first assert that

$$D_q e_q\left(-\frac{v}{u}t\right) = -\frac{v}{u} e_q\left(-\frac{v}{u}t\right). \tag{58}$$

In detail, this can be interpreted as

$$\begin{aligned} D_q e_q\left(-\frac{v}{u}t\right) &= \sum_0^\infty \frac{(-1)^n}{([n]_q)!}\frac{v^n}{u^n} D_q t^n \\ &= \sum_1^\infty \frac{(-1)^n}{([n-1]_q)!}\frac{v^n}{u^n} t^{n-1} \\ &= \sum_0^\infty \frac{(-1)^{n+1}}{([n]_q)!}\frac{v^{n+1}}{u^{n+1}} t^n \\ &= -\frac{v}{u}\sum_0^\infty \frac{(-1)^n}{([n]_q)!}\frac{v^n}{u^n} t^n = -\frac{v}{u} e_q\left(-\frac{v}{u}t\right). \end{aligned}$$

This proves the above assertion.

Hence we prove the following theorem.

**Theorem 16** *Let* $u, v > 0$. *Then we have*

$$\left(N^q D_q f(t)\right)(u;v) = -f(0) - \int_0^\infty f(qt) D_q e_q\left(-\frac{v}{u}t\right) d_q t. \tag{59}$$

*Proof* By (58), the above equation gives

$$\begin{aligned} \left(N^q D_q f(t)\right)(u;v) &= \int_0^\infty D_q f(t) e_q\left(-\frac{v}{u}t\right) d_q t \\ &= -f(0) + \frac{v}{u}\int_0^\infty f(qt) e_q\left(-\frac{v}{u}t\right) d_q t \\ &= -f(0) + \frac{v}{u}q^{-1}\int_0^\infty f(t) e_q\left(-\frac{v}{u}q^{-1}t\right) d_q t. \end{aligned}$$

By setting variables, we have $(N^q D_q f(t))(u;v) = -f(0) + \frac{v}{u}q^{-1}(N^q f)(q^{-1}v;u)$.

This completes the proof.

Now, we extend (59) to have

$$\begin{aligned} \left(N^q D_q^2 f\right)(u;v) &= \left(N^q D_q(D_q f)\right)(u;v) \\ &= -D_q f(0) + \frac{v}{u}q^{-1}\left(N^q D_q f\right)\left(q^{-1}v;u\right) \\ &= -D_q f(0) + \frac{v}{u}q^{-1}\left(-f(0) + \frac{v}{u}q^{-1}\left(N^q f\right)\left(q^{-2}v;u\right)\right) \\ &= -D_q f(0) - \left(\frac{v}{u}\right)q^{-1} f(0) + \left(\frac{v}{u}\right)q^{2-2}\left(N^q f\right)\left(q^{-2}v;u\right). \end{aligned}$$

Proceeding to the $n$th derivative, we get $(N^q D_q^n f)(u;v) = (\frac{v}{u})^n q^{-n}(N^q f)(q^{-n}v;u) - \sum_{i=0}^{n-1}(\frac{v}{u})^{n-1-i} D_q^i f(0)$.

This completes the proof of the theorem. □

## 10 Conclusion

The $q$-analogues of the $N$-transform have been applied to various classes of exponential functions, trigonometric functions, hyperbolic functions and certain differential operators as well. As the $N$-transform defines an extension to Laplace and Sumudu transforms, our results generalize the results existing in the literature.

**Acknowledgements**
The author would like to express many thanks to the anonymous referees for their corrections and comments on this manuscript.

**Funding**
Not applicable.

**Availability of data and materials**
Not applicable.

**Competing interests**
The author declares that he has no competing interests.

**Authors' contributions**
The author read and approved the final manuscript.

## Publisher's Note



Received: 19 September 2017 Accepted: 22 December 2017 Published online: 10 January 2018

**References**

1. Jackson, FH: On $q$-definite integrals. Quart. J. Pure Appl. Math. **41**, 193-203 (1910)
2. Abdi, WH: On $q$-Laplace transforms. Proc. Natl. Acad. Sci., India **29**, 389-408 (1961)
3. Purohit, SD, Kalla, SL: On $q$-Laplace transforms of the $q$-Bessel functions. Fract. Calc. Appl. Anal. **10**(2), 189-196 (2007)
4. Ucar, F, Albayrak, D: On $q$-Laplace type integral operators and their applications. J. Differ. Equ. Appl. **18**(6), 1001-1014 (2011)
5. Yadav, RK, Purohit, SD: On $q$-Laplace transforms of certain q-hypergeometric polynomials. Proc. Natl. Acad. Sci., India **76**(A) III, 235-242 (2006)
6. Albayrak, D, Purohit, SD, Ucar, F: On $q$-Sumudu transforms of certain $q$-polynomials. Filomat **27**(2), 413-429 (2013)
7. Albayrak, D, Purohit, SD, Ucar, F: On $q$-analogues of Sumudu transform. An. Ştiinţ. Univ. 'Ovidius' Constanţa **21**(1), 239-260 (2013)
8. Albayrak, D, Ucar, F, Purohit, SD: Certain inversion and representation formulas for $q$-Sumudu transforms. Hacet. J. Math. Stat. **43**(5), 699-713 (2014)
9. Ucar, F: $q$-Sumudu transforms of $q$-analogues of Bessel functions. Sci. World J. **2014**, Article ID 327019 (2014)
10. Yadav, R, Purohit, SD: On applications of Weyl fractional $q$-integral operator to generalized basic hypergeometric functions. Kyungpook Math. J. **46**, 235-245 (2006)
11. Fitouhi, A, Bettaibi, N: Wavelet transforms in quantum calculus. J. Nonlinear Math. Phys. **13**(3), 492-506 (2006)
12. Fitouhi, A, Bettaibi, N: Applications of the Mellin transform in quantum calculus. J. Math. Anal. Appl. **328**, 518-534 (2007)
13. Al-Omari, SKQ: On $q$-analogues of the Mangontarum transform for certain $q$-Bessel functions and some application. J. King Saud Univ., Sci. **28**, 375-379 (2016)
14. Albayrak, D, Purohit, SD, Ucar, F: On $q$-integral transforms and their applications. Bull. Math. Anal. Appl. **4**(2), 103-115 (2012)
15. Salem, A, Ucar, F: The $q$-analogue of the $E_{2;1}$-transform and its applications. Turk. J. Math. **40**(1), 98-107 (2016)
16. Gasper, G, Rahman, M: Basic Hypergeometric Series. Encyclopedia Math. Appl., vol. 35. Cambridge Univ. Press, Cambridge (1990)
17. Jackson, FH: On a $q$-definite integrals. Quart. J. Pure Appl. Math. **41**, 193-203 (1910)
18. Kac, VG, Cheung, P: Quantum Calculus. Universitext. Springer, New York (2002)
19. Khan, ZH, Khan, WA: N-transform properties and applications. NUST J. Eng. Sci. **1**, 127-133 (2008)
20. Silambarasan, R, Belgacem, FBM: Applications of the natural transform to Maxwell's equations. In: Prog. Electromagn. Res. Sym. Proceed., Suzhou, China, 12-16 September 2011
21. Al-Omari, SKQ: On the application of the natural transforms. Int. J. Pure Appl. Math. **85**(4), 729-744 (2013)
22. Belgacem, FBM, Silambarasan, R: Theoretical investigations of the natural transform. In: Prog. Electromagn. Res. Sym. Proceed., Suzhou, China, 12-16 September 2011

23. Belgacem, FBM, Silambarasan, R: Maxwell's equations solutions through the natural transform. Math. Engin. Sci. Aerospace **3**(3), 313-323 (2012)
24. Hahn, W: Beitrage zur theorie der heineschen reihen, die 24 Integrale der hypergeometrischen $q$-Diferenzengleichung, das $q$-analog on der Laplace transformation. Math. Nachr. **2**, 340-379 (1949)