Jaradat et al. *Advances in Difference Equations* (2017) 2017:376
DOI 10.1186/s13662-017-1431-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Construction and solitary wave solutions of two-mode higher-order Boussinesq-Burger system

Ali Jaradat[1*], Mohd Salmi Md Noorani[1], Marwan Alquran[2] and Husein M Jaradat[3]

*Correspondence: alijaradat81@yahoo.com
[1]School of Mathematical Sciences, University Kebangsaan Malaysia, Bangi, 43600, Malaysia
Full list of author information is available at the end of the article

Abstract
A new nonlinear partial differential system called two-mode higher-order Boussinesq-Burgers system is established. We aim to use the simplified bilinear method to find the necessary constraint conditions that guarantee the existence of both regular and singular multiple soliton solutions of the model. To study the correctness of the obtained results, we use the hyperbolic-tangent expansion method as an alternative technique to investigate more possible solutions.

Keywords: two-mode higher-order Boussinesq-Burgers system; simplified Hirota method; N-soliton solutions; hyperbolic tangent expansion

1 Introduction

Nonlinear evolution equations have been used as the models to describe a variety of engineering phenomena and science such as optical fibers, chemical kinetics, fluid dynamics, mathematical biology, chemical reactions, plasma waves and others [1–4]. The study of nonlinear equations is an interesting topic in solitary waves theory. Many methods are established to find exact solutions for nonlinear partial differential equations (PDEs) such as Lie symmetries method, truncated Painleve expansion method, the inverse scattering method, Hirota's bilinear method, Darboux transformation, Backlund transformation, simplified Hirota's method, trigonometric-function series method, modified mapping method, modified (G'/G)-expansion method, tanh-coth expansion method, Jacobi elliptic function expansion method, first integral method and others (see [5–30] and [31–40]). Also, important developments for searching for analytical solitary wave solutions for PDEs can be found in [41–50].

The most integrable systems describe unidirectional waves such as the KP equation, Burgers, KdV, mKdV equations and a higher-order Boussinesq-Burgers equation, where these equations are first order PDE in time and model only the right-moving in the positive x-direction. However, the Boussinesq equation models both left- and right-going waves and second-order in time. The two-mode equations are analogous to the two-mode Boussinesq equation and hence are represented by a new nonlinear partial differential equation of second-order in time.

A higher-order Boussinesq-Burgers equation was introduced by Jin-Ming and Yao-Ming (2011) [2] in the form

$$\begin{aligned} &v_t - 3av^2 v_x + \frac{3}{2}a(vw)_x - \frac{1}{4}av_{xxx} = 0, \\ &w_t + \frac{3}{2}aww_x - 3a(v^2w)_x + 3av_x v_{xx} + \frac{3}{2}avv_{xxx} - \frac{1}{4}aw_{xxx} = 0, \end{aligned} \tag{1}$$

where a is a non-zero arbitrary constant.

Wazwaz [51–53] and Korsunsky [54] proposed the scaled form of the two-mode equation as follows:

$$u_{tt} - c^2 u_{xx} + \left(\frac{\partial}{\partial t} - a_1 c\frac{\partial}{\partial x}\right)N(u, u_x, \ldots) + \left(\frac{\partial}{\partial t} - a_2 c\frac{\partial}{\partial x}\right)L(u_{kx}) = 0, \tag{2}$$

where $c > 0$ is the phase velocities, $|a_1| \leq 1$, $|a_2| \leq 1$, a_2 is the dispersion parameter, a_1 is the parameter of nonlinearity, $N(u, u_x, \ldots)$ is the nonlinear term and $L(u_{kx})$ is the linear term in the equation, $k \geq 2$. Based on this argument, Jaradat *et al.* [55–57] used the same sense as Korsunsky [54] and Wazwaz [51–53] to establish the two-mode coupled system of n equations

$$\begin{aligned} 0 &= (u_i)_{tt} - c^2(u_i)_{xx} + \left(\frac{\partial}{\partial t} - a_1 c\frac{\partial}{\partial x}\right)N_i\big(u_1, \ldots, u_n, (u_1)_x, \ldots, (u_n)_x, \ldots\big) \\ &\quad + \left(\frac{\partial}{\partial t} - a_2 c\frac{\partial}{\partial x}\right)L_i\big((u_i)_{kx}\big), \quad i = 2, 3, \ldots, n. \end{aligned} \tag{3}$$

To establish the two-mode higher-order Boussinesq-Burgers equation (TM-ho-BBE) (1), we have

$$\begin{aligned} &N_1(v, w, v_x, w_x, \ldots) = -3av^2 v_x + \frac{3}{2}a(vw)_x, \\ &L_1(v_{kx}) = -\frac{1}{4}av_{xxx}, \\ &N_2(v, w, v_x, w_x, \ldots) = \frac{3}{2}aww_x - 3a\big(v^2w\big)_x + 3av_x v_{xx} + \frac{3}{2}avv_{xxx}, \\ &L_2(v_{kx}) = -\frac{1}{4}aw_{xxx}. \end{aligned}$$

Thus, using equation (3), (TM-ho-BBE) will have the form

$$\begin{aligned} 0 &= v_{tt} - c^2 v_{xx} - \frac{1}{4}a\left(\frac{\partial}{\partial t} - a_2 c\frac{\partial}{\partial x}\right)v_{xxx} + \left(\frac{\partial}{\partial t} - a_1 c\frac{\partial}{\partial x}\right)\left\{-3av^2 v_x + \frac{3}{2}a(vw)_x\right\}, \\ 0 &= w_{tt} - c^2 w_{xx} - \frac{1}{4}a\left(\frac{\partial}{\partial t} - a_2 c\frac{\partial}{\partial x}\right)w_{xxx} \\ &\quad + \left(\frac{\partial}{\partial t} - a_1 c\frac{\partial}{\partial x}\right)\left\{\frac{3}{2}aww_x - 3a\big(v^2w\big)_x + 3av_x v_{xx} + \frac{3}{2}avv_{xxx}\right\}. \end{aligned} \tag{4}$$

Note that when $c = 0$ and integrating with respect to t, the two-mode higher-order Boussinesq-Burgers equation (TM-ho-BBE) (4) is reduced to the standard higher-order Boussinesq-Burgers equation (1).

The aim of this study is to find the necessary conditions needed for multiple-soliton solutions and singular multiple-soliton solutions to exist for (TM-ho-BBE) by using the simplified form of Hirota's method. Moreover, we determine more exact solutions to this new system by using the Tanh method.

2 Multiple soliton solutions and singular multiple soliton solutions

In this section, we used the simplified bilinear method to find the necessary conditions needed to produce single soliton solutions, singular soliton solutions, multiple soliton solutions and singular multiple soliton solutions of (TM-ho-BBE).

Substituting

$$v(x,t) = w(x,t) = e^{\lambda_i(x,t)}, \quad \lambda_i(x,t) = r_i x - d_i t,$$

into the linear terms of equation (4) and solving the resulting equation gives

$$d_i = \frac{-ar_i^3 \pm r_i\sqrt{a^2 r_i^4 - 16aa_2 cr_i^2 + 64c^2}}{8}.$$

As a result, $\lambda_i(x,t)$ becomes

$$\lambda_i(x,t) = r_i x - \frac{-ar_i^3 \pm r_i\sqrt{a^2 r_i^4 - 16aa_2 cr_i^2 + 64c^2}}{8}t, \quad i = 1,2,\ldots. \tag{5}$$

Now, we propose the solutions to (TM-ho-BBE) (4) in the form

$$v(x,t) = C_1 \frac{\partial}{\partial x}\bigl(\ln h(x,t)\bigr), \tag{6}$$

$$w(x,t) = C_2 \frac{\partial^2}{\partial x^2}\bigl(\ln h(x,t)\bigr), \tag{7}$$

where C_1 and C_2 are constants. To find the one-soliton solution, we assume the auxiliary function $h(x,t)$ to be

$$h(x,t) = 1 + \alpha_1 e^{\lambda_1(x,t)} = 1 + \alpha_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2 r_1^4 - 16aa_2 cr_1^2 + 64c^2}}{8}t}, \tag{8}$$

where $\alpha_1 = \pm 1$. Substitute (6), (7) and (8) into (TM-ho-BBE) (4). Then, solving for numerical values C_1 and C_2, we find that the one-soliton solution of (TM-ho-BBE) (4) exists only if $a_1 = a_2$. Therefore, two sets of solutions are obtained

$$C_1 = \frac{\pm 1}{2}, \tag{9}$$

$$C_2 = \frac{-1}{2}. \tag{10}$$

Then the one-soliton solution of (TM-ho-BBE) (4) is given by

$$v(x,t) = \frac{\pm 1}{2}\frac{\alpha_1 r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2 r_1^4 - 16aa_2 cr_1^2 + 64c^2}}{8}t}}{1 + \alpha_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2 r_1^4 - 16aa_2 cr_1^2 + 64c^2}}{8}t}},$$

$$w(x,t) = \frac{-1}{2}\left(\frac{\alpha_1 r_1^2 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1+\alpha_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right) + \frac{1}{2}\left(\frac{\alpha_1 r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1+\alpha_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right)^2,$$

where

$$\lambda_1(x,t) = r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t.$$

For $\alpha_1 = 1$, the single soliton solution is

$$v(x,t) = \frac{\pm 1}{2}\frac{r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1+e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}} = \frac{\pm 1}{4}r_1\left(1+\tanh\left(\frac{\lambda_1(x,t)}{2}\right)\right), \tag{11}$$

$$w(x,t) = \frac{-1}{2}\left(\frac{r_1^2 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1+e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right) + \frac{1}{2}\left(\frac{r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1+e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right)^2 = \frac{-1}{8}r_1^2\operatorname{sech}^2\left(\frac{\lambda_1(x,t)}{2}\right). \tag{12}$$

For $\alpha_1 = -1$, the singular single soliton solution is

$$v(x,t) = \frac{\mp 1}{2}\frac{r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1-e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}} = \frac{\pm 1}{4}r_1\left(1+\coth\left(\frac{\lambda_1(x,t)}{2}\right)\right), \tag{13}$$

$$w(x,t) = \frac{1}{2}\left(\frac{r_1^2 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1-e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right) + \frac{1}{2}\left(\frac{r_1 e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}{1-e^{r_1 x - \frac{-ar_1^3 \pm r_1\sqrt{a^2r_1^4-16aa_2cr_1^2+64c^2}}{8}t}}\right)^2 = \frac{1}{8}r_1^2\operatorname{csch}^2\left(\frac{\lambda_1(x,t)}{2}\right). \tag{14}$$

To find the two-wave solutions, we assume

$$h(x,t) = 1 + \alpha_1 e^{\lambda_1(x,t)} + \alpha_2 e^{\lambda_2(x,t)} + c_{12}\alpha_1\alpha_2 e^{\lambda_1(x,t)+\lambda_2(x,t)}. \tag{15}$$

Insert (9), (10) and (15) in (4) and solve for c_{12}. Then two-soliton solutions exist only if $a_1 = a_2 = \pm 1$ and

$$c_{12} = 0. \tag{16}$$

This can be generalized to

$$c_{ij} = 0, \quad 1 \leq i < j \leq 3. \tag{17}$$

Also, inserting (16), (15), (9), (10) in (6) and (7) and under the constraint condition $a_1 = a_2 = \pm 1$, we obtain the following two-soliton solutions:

$$\begin{aligned} v(x,t) &= \frac{\pm 1}{2} \frac{r_1 \alpha_1 e^{\lambda_1} + r_2 \alpha_2 e^{\lambda_2}}{1 + \alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2}}, \\ w(x,t) &= \frac{-1}{2} \left(\frac{\alpha_1 r_1^2 e^{\lambda_1} + \alpha_2 r_2^2 e^{\lambda_2}}{1 + \alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2}} - \left(\frac{\alpha_1 r_1 e^{\lambda_1} + r_2 \alpha_2 e^{\lambda_2}}{1 + \alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2}} \right)^2 \right), \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= r_1 x - \frac{-a r_1^3 \pm (a r_1^3 \pm 8 c r_1)}{8} t, \\ \lambda_2 &= r_2 x - \frac{-a r_2^3 \pm (a r_2^3 \pm 8 c r_2)}{8} t. \end{aligned}$$

For $\alpha_1 = \alpha_2 = 1$, the two-soliton solution is

$$\begin{aligned} v(x,t) &= \frac{\pm 1}{2} \frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2}}{1 + e^{\lambda_1} + e^{\lambda_2}}, \\ w(x,t) &= \frac{-1}{2} \left(\frac{r_1^2 e^{\lambda_1} + r_2^2 e^{\lambda_2}}{1 + e^{\lambda_1} + e^{\lambda_2}} - \left(\frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2}}{1 + e^{\lambda_1} + e^{\lambda_2}} \right)^2 \right). \end{aligned}$$

For $\alpha_1 = \alpha_2 = -1$, the two-singular-soliton solution is

$$\begin{aligned} v(x,t) &= \frac{\mp 1}{2} \frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2}}{1 - e^{\lambda_1} - e^{\lambda_2}}, \\ w(x,t) &= \frac{1}{2} \left(\frac{r_1^2 e^{\lambda_1} + r_2^2 e^{\lambda_2}}{1 - e^{\lambda_1} - e^{\lambda_2}} + \left(\frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2}}{1 - e^{\lambda_1} - e^{\lambda_2}} \right)^2 \right). \end{aligned}$$

To find three-soliton solutions, we assume

$$h(x,t) = 1 + \alpha_1 e^{\lambda_1(x,t)} + \alpha_2 e^{\lambda_2(x,t)} + \alpha_3 e^{\lambda_3(x,t)} + c_{123} \alpha_1 \alpha_2 \alpha_3 e^{\lambda_1(x,t) + \lambda_2(x,t) + \lambda_3(x,t)}, \tag{18}$$

where

$$\lambda_i = r_i x - \frac{-a r_i^3 \pm (a r_i^3 \pm 8 c r_i)}{8} t, \quad i = 1, 2, 3.$$

Substituting (6), (7) and (18) into (4) and solving for c_{123}, we find

$$c_{123} = 0.$$

Accordingly, we obtain the following three-wave solutions:

$$v(x,t) = \frac{\pm 1}{2}\frac{\alpha_1 r_1 e^{\lambda_1} + r_2\alpha_2 e^{\lambda_2} + r_3\alpha_3 e^{\lambda_3}}{1+\alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2} + \alpha_3 e^{\lambda_3}},$$
$$w(x,t) = \frac{-1}{2}\left(\frac{r_1^2\alpha_1 e^{\lambda_1} + r_2^2\alpha_2 e^{\lambda_2} + r_3^2\alpha_3 e^{\lambda_3}}{1+\alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2} + \alpha_3 e^{\lambda_3}} - \left(\frac{r_1\alpha_1 e^{\lambda_1} + r_2\alpha_2 e^{\lambda_2} + r_3\alpha_3 e^{\lambda_3}}{1+\alpha_1 e^{\lambda_1} + \alpha_2 e^{\lambda_2} + \alpha_3 e^{\lambda_3}}\right)^2\right).$$

For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the three-wave solution is

$$v(x,t) = \frac{\pm 1}{2}\frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2} + r_3 e^{\lambda_3}}{1+e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_3}},$$
$$w(x,t) = \frac{-1}{2}\left(\frac{r_1^2 e^{\lambda_1} + r_2^2 e^{\lambda_2} + r_3^2 e^{\lambda_3}}{1+e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_3}} - \left(\frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2} + r_3 e^{\lambda_3}}{1+e^{\lambda_1}+e^{\lambda_2}+e^{\lambda_3}}\right)^2\right).$$

For $\alpha_1 = \alpha_2 = \alpha_3 = -1$, the singular three-soliton solution is

$$v(x,t) = \frac{\mp 1}{2}\frac{r_1 e^{\lambda_1} - r_2 e^{\lambda_2} - r_3 e^{\lambda_3}}{1-e^{\lambda_1}-e^{\lambda_2}-e^{\lambda_3}},$$
$$w(x,t) = \frac{1}{2}\left(\frac{r_1^2 e^{\lambda_1} + r_2^2 e^{\lambda_2} + r_3^2 e^{\lambda_3}}{1-e^{\lambda_1}-e^{\lambda_2}-e^{\lambda_3}} + \left(\frac{r_1 e^{\lambda_1} + r_2 e^{\lambda_2} + r_3 e^{\lambda_3}}{1-e^{\lambda_1}-e^{\lambda_2}-e^{\lambda_3}}\right)^2\right).$$

Finally, we reach the fact that (TM-ho-BBE) has N-soliton solutions under the necessary condition $a_1 = a_2 = \pm 1$, where $N \geq 1$ [16, 58]. They are given by

$$v(x,t) = \frac{\pm 1}{2}\frac{\sum_{k=1}^{N} r_k e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1+\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}},$$
$$w(x,t) = \frac{-1}{2}\left(\frac{\sum_{k=1}^{N} r_k^2 e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1+\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}} - \left(\frac{\sum_{k=1}^{N} r_k e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1+\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}\right)^2\right).$$

Also, the singular N-soliton solutions under the same condition are given by

$$v(x,t) = \frac{\mp 1}{2}\frac{\sum_{k=1}^{N} r_k e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1-\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}},$$
$$w(x,t) = \frac{1}{2}\left(\frac{\sum_{k=1}^{N} r_k^2 e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1-\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}} + \left(\frac{\sum_{k=1}^{N} r_k e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}{1-\sum_{k=1}^{N} e^{r_k x - \frac{-ar_k^3 \pm (ar_k^3 \pm 8cr_k)}{8}t}}\right)^2\right).$$

Finally, we would like to present two plots of the obtained solutions of (TM-ho-BBE). Figure 1 is the one-soliton solution and Figure 2 is the two-soliton solution.

3 Alternative method: hyperbolic tangent expansion

In this section, we aim to validate the necessary conditions that guarantee the existence of solutions to (TM-ho-BBE). To achieve this goal, we propose an alternative method, the hyperbolic tangent expansion [59–63], to construct solitary wave solutions of problem (4).

Figure 1 The one-soliton solution of u_1 when $r_1 = 3$, $a = -2$, $c = 1$, $a_1 = a_2 = 1.5$, $C_1 = 0.5$, $C_2 = -0.5$.

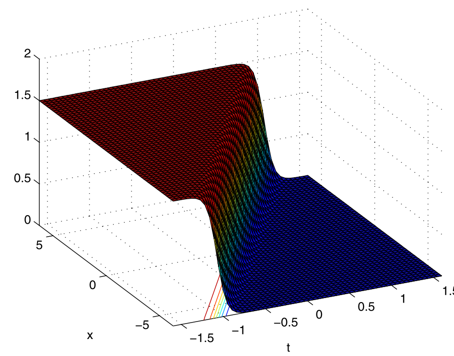
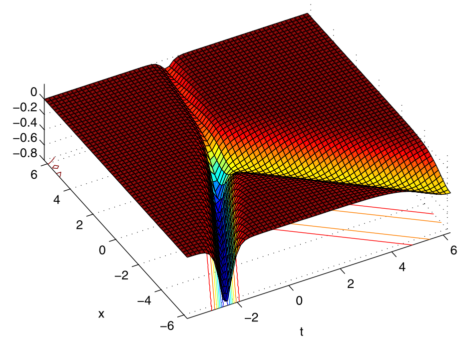


Figure 2 The two-soliton solution of u_2 when $r_1 = 2.5$, $r_2 = 1.5$, $a = -1$, $c = a_1 = a_2 = 1$, $C_1 = 0.5$, $C_2 = -0.5$.



We first consider the wave transform $z = x - \lambda t$ to reduce (4) into the following ordinary differential equations:

$$\begin{aligned} 0 &= (\lambda^2 - c^2)v + \frac{1}{4}a(\lambda + a_2c)v'' + 3a(\lambda + a_1c)\left(\frac{1}{3}v^3 - \frac{1}{2}vw\right), \\ 0 &= (\lambda^2 - c^2)w' + \frac{1}{4}a(\lambda + a_2c)w''' \\ &\quad + 3a(\lambda + a_1c)\left((v^2w)' - \frac{1}{2}ww' - v'v'' - \frac{1}{2}vv'''\right), \end{aligned} \tag{19}$$

where $v = v(z)$ and $w = w(z)$.

Finite polynomials in terms of hyperbolic functions are to be considered as favorable suggested solutions of (19). Thus, we assume that

$$\begin{aligned} v(z) &= \sum_{i=0}^{n_1} c_i Y^i, \\ w(z) &= \sum_{j=0}^{n_2} d_j Y^j, \end{aligned} \tag{20}$$

where

$$Y = Y(z) = \tanh(\mu z). \tag{21}$$

Balancing the behavior of Y in the highest derivative against its counterpart within the nonlinear terms appearing in (19) leads to $n_1 + 2 = 3n_1 = n_1 + n_2$ or $n_2 + 3 = 2n_1 + 3 = 2n_2 + 1 = 2n_1 + n_2 + 1$. Thus, $n_1 = 1$, $n_2 = 2$. Accordingly, the solution of (19) in deterministic form is

$$\begin{aligned} v(z) &= c_0 + c_1 Y, \\ w(z) &= d_0 + d_1 Y + d_2 Y^2. \end{aligned} \tag{22}$$

The task now is to determine the values of the parameters c_0, c_1, d_0, d_1, d_2 and λ, μ. To achieve this, we first substitute (22) in (19), then we collect all coefficients of powers of Y in the resulting equations and set them to zero. Finally, we solve the prescribed algebraic system to retrieve the values of the required parameters. The algebraic calculations have been executed by *Mathematica* and the following obtained solutions have been verified as well. The first solution of (4) is

$$\begin{aligned} v(x,t) &= \mp\mu \tanh\bigl(\mu(x-\lambda_1 t)\bigr), \\ w(x,t) &= -\mu^2 + \mu^2 \tanh^2\bigl(\mu(x-\lambda_1 t)\bigr), \end{aligned} \tag{23}$$

where $\lambda_1 = \frac{1}{2}(-a\mu^2 \mp \sqrt{4c^2 - 4aa_2c\mu^2 + a^2\mu^4})$.

The second solution is

$$\begin{aligned} v(x,t) &= \mp\frac{\mu}{2} \tanh\bigl(\mu(x-\lambda_2 t)\bigr), \\ w(x,t) &= -\frac{\mu^2}{2} + \frac{\mu^2}{2} \tanh^2\bigl(\mu(x-\lambda_2 t)\bigr), \end{aligned} \tag{24}$$

where $\lambda_2 = \frac{1}{8}(-a\mu^2 - \sqrt{64c^2 - 16aa_2c\mu^2 + a^2\mu^4})$.

The third solution is

$$\begin{aligned} v(x,t) &= \mp\frac{\mu}{2} \mp \frac{\mu}{2} \tanh\bigl(\mu(x-\lambda_k t)\bigr), \\ w(x,t) &= -\frac{\mu^2}{2} + \frac{\mu^2}{2} \tanh^2\bigl(\mu(x-\lambda_k t)\bigr), \quad k = 3,4, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \lambda_3 &= \frac{1}{2}\bigl(-a\mu^2 - \sqrt{4c^2 - 4aa_2c\mu^2 + a^2\mu^4}\bigr), \\ \lambda_4 &= \frac{1}{2}\bigl(-a\mu^2 + \sqrt{4c^2 - 4aa_2c\mu^2 + a^2\mu^4}\bigr). \end{aligned}$$

4 Conclusion

In this paper, we established a new nonlinear two-mode higher-order Boussinesq-Burgers equation (TM-ho-BBE). By using a simplified form of Hirota's method we reached the following two facts:

- Single soliton and singular soliton solutions exist for (TM-ho-BBE) if $a_1 = a_2$.
- Multiple soliton and multiple singular soliton solutions exist only under the condition $a_1 = a_2 = \mp 1$.

Also, we applied an alternative method called hyperbolic tangent expansion to validate the necessary conditions needed for solitary solutions to exist.

Acknowledgements
The first and second authors are financially supported by UKM Grant: DIP-2017-011 and Ministry of Education Malaysia Grant FRGS/1/2017/STG06/UKM/01/1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed and performed by all authors. All authors read and approved the final manuscript.

Author details
[1]School of Mathematical Sciences, University Kebangsaan Malaysia, Bangi, 43600, Malaysia. [2]Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box(3030), Irbid, 22110, Jordan. [3]Department of Mathematics, Al al-Bayt University, Mafraq, Jordan.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 22 September 2017 Accepted: 23 November 2017 Published online: 02 December 2017

References
1. Chen, A, Li, X: Darboux transformation and soliton solutions for Boussinesq-Burgers equation. Chaos Solitons Fractals **27**, 43-49 (2006)
2. Jin-Ming, Z, Yao-Ming, Z: The Hirota bilinear method for the coupled Burgers equation and the high-order Boussinesq-Burgers equation. Chin. Phys. B **20**, Article ID 010205 (2011)
3. Kamchatnov, A, Kraenkel, A, Umarov, B: Asymptotic soliton train solutions of Kaup-Boussinesq equations. Wave Motion **38**, 355-365 (2003)
4. Mhlanga, I, Khalique, CM: Exact solutions of generalized Boussinesq-Burgers equations and (2 + 1)-dimensional Davey-Stewartson equations. J. Appl. Math. **2012**, Article ID 389017 (2012)
5. Khalique, CM: Exact solutions and conservation laws of a coupled integrable dispersionless system. Filomat **26**, 957-964 (2012)
6. Matveev, V, Salle, M: Darboux Transformation and Solitons. Springer, Berlin (1991)
7. Wang, Z, Chen, A: Explicit solutions of Boussinesq-Burgers equation. Chin. Phys. **16**, 1233-1238 (2007)
8. Wazwaz, AM: Two integrable extensions of the Kadomtsev-Petviashvili equation. Cent. Eur. J. Phys. **91**, 49-56 (2011)
9. Wazwaz, AM: Solitons and singular solitons for a variety of Boussinesq-like equations. Ocean Eng. **53**, 1-5 (2012)
10. Wazwaz, AM: Multiple kink solutions and multiple singular kink solutions for two systems of coupled Burgers' type equations. Commun. Nonlinear Sci. Numer. Simul. **14**, 2962-2970 (2009)
11. Wazwaz, AM: A study on the (2 + 1)-dimensional and the (2 + 1)-dimensional higher-order Burgers equations. Appl. Math. Lett. **25**, 1495-1499 (2012)
12. Wazwaz, AM: Combined equations of the Burgers hierarchy: multiple kink solutions and multiple singular kink solutions. Phys. Scr. **82**, Article ID 025001 (2010)
13. Wazwaz, AM: Kinks and travelling wave solutions for Burgers-like equations. Appl. Math. Lett. **38**, 174-179 (2014)
14. Wazwaz, AM: Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities. Nonlinear Dyn. **83**, 591-596 (2016)
15. Wazwaz, AM: Multiple kink solutions for two coupled integrable (2 + 1)-dimensional systems. Appl. Math. Lett. **58**, 1-6 (2016)
16. Hirota, R: Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. **27**, 1192-1194 (1971)
17. Hirota, R: Exact N-soliton solutions of a nonlinear wave equation. J. Math. Phys. **14**, 805-809 (1973)
18. Jaradat, HM, Al-Shara', S, Awawdeh, F, Alquran, M: Variable coefficient equations of the Kadomtsev-Petviashvili hierarchy: multiple soliton solutions and singular multiple soliton solutions. Phys. Scr. **85**, Article ID 035001 (2012)
19. Jaradat, HM, Awawdeh, F, Al-Shara', S, Alquran, M, Momani, S: Controllable dynamical behaviors and the analysis of fractal Burgers hierarchy with the full effects of inhomogeneities of media. Rom. J. Phys. **60**(3-4), 324-343 (2015)
20. Awawdeh, F, Jaradat, HM, Al-Shara', S: Applications of a simplified bilinear method to ion-acoustic solitary waves in plasma. Eur. Phys. J. D **66**, 1-8 (2012)
21. Awawdeh, F, Al-Shara', S, Jaradat, HM, Alomari, AK, Alshorman, R: Symbolic computation on soliton solutions for variable coefficient quantum Zakharov-Kuznetsov equation in magnetized dense plasmas. Int. J. Nonlinear Sci. Numer. Simul. **15**(1), 35-45 (2014)
22. Wazwaz, AM: Multiple soliton solutions for the (2 + 1)-dimensional asymmetric Nizhnik Novikov Veselov equation. Nonlinear Anal. **72**, 1314-1318 (2010)
23. Wazwaz, AM: Multiple-soliton solutions for the Boussinesq equation. Appl. Math. Comput. **192**, 479-486 (2007)
24. Jaradat, HM: New solitary wave and multiple soliton solutions for the time-space fractional Boussinesq equation. Ital. J. Pure Appl. Math. **36**, 367-376 (2016)
25. Alsayyed, O, Jaradat, HM, Jaradat, MMM, Mustafa, Z, Shatat, F: Multi-soliton solutions of the BBM equation arisen in shallow water. J. Nonlinear Sci. Appl. **9**(4), 1807-1814 (2016)
26. Jaradat, HM: Dynamic behavior of traveling wave solutions for a class for the time-space coupled fractional kdV system with time-dependent coefficients. Ital. J. Pure Appl. Math. **36**, 945-958 (2016)
27. Alquran, M, Jaradat, HM, Al-Shara', S, Awawdeh, F: A new simplified bilinear method for the N-soliton solutions for a generalized FmKdV equation with time-dependent variable coefficients. Int. J. Nonlinear Sci. Numer. Simul. **16**, 259-269 (2015)
28. Jaradat, HM: Dynamic behavior of traveling wave solutions for new couplings of the Burgers equations with time-dependent variable coefficients. Adv. Differ. Equ. **2017**, Article ID 167 (2017). doi:10.1186/s13662-017-1223-1

29. Jaradat, HM, Al-Shara', S, Jaradat, MMM, Mustafa, Z, Alsayyed, O, Alquran, M, Abohassan, KM, Momani, S: New solitary wave and multiple soliton solutions for the time-space coupled fractional mKdV system with time-dependent coefficients. J. Comput. Theor. Nanosci. **13**(12), 1-8 (2016)
30. Jaradat, HM, Syam, M, Alquran, M: Necessary conditions of coupled mkdV-BLMP system for multiple-soliton solutions to exist. Alex. Eng. J. (2017). doi:10.1016/j.aej.2017.06.012
31. Zhang, ZY: New exact traveling wave solutions for the nonlinear Klein-Gordon equation. Turk. J. Phys. **32**, 235-240 (2008)
32. Zhang, ZY, Liu, ZH, Miao, XJ, Chen, YZ: New exact solutions to the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity. Appl. Math. Comput. **216**, 3064-3072 (2010)
33. Zhang, ZY, Li, YX, Liu, ZH, Miao, XJ: New exact solutions to the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity via modified trigonometric function series method. Commun. Nonlinear Sci. Numer. Simul. **16**(8), 3097-3106 (2011)
34. Zhang, ZY, Zhang, YH, Gan, XY, Yu, DM: A note on exact traveling wave solutions for the Klein-Gordon-Zakharov equations. Z. Naturforsch. A **67**(3-4), 167-172 (2014)
35. Zhang, ZY, Liu, ZH, Miao, XJ, Chen, YZ: Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity. Phys. Lett. A **375**, 1275-1280 (2011)
36. Zhang, ZY, Gan, XY, Yu, DM: Bifurcation behavior of the traveling wave solutions of nonlinear the perturbed nonlinear Schrodinger equation with Kerr law nonlinearity. Z. Naturforsch. A **66**(12), 721-727 (2014)
37. Zhang, ZY, Zhong, J, Dou, S, Liu, J, Peng, D, Gao, T: A new method to construct traveling wave solutions for the Klein-Gordon-Zakharov equations. Rom. J. Phys. **58**(7-8), 766-777 (2013)
38. Zhang, ZY, Huang, J, Zhong, J, Dou, SS, Liu, J, Peng, D, Gao, T: The extended (G'/G)-expansion method and travelling wave solutions for the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity. Pramana **82**(6), 1011-1029 (2014)
39. Zhang, ZY, Zhong, J, Dou, SS, Liu, J, Peng, D, Gao, T: Abundant exact traveling wave solutions for the Klein-Gordon-Zakharov equations via the tanh-coth expansion method and Jacobi elliptic function expansion method. Rom. J. Phys. **58**(7-8), 749-765 (2013)
40. Zhang, ZY, Zhong, J, Dou, SS, Liu, J, Peng, D, Gao, T: First integral method and exact solutions to nonlinear partial differential equations arising in mathematical physics. Rom. Rep. Phys. **65**(4), 1155-1169 (2013)
41. Khater, AH, Callebaut, DK, Malfliet, W, Seadawy, AR: Nonlinear dispersive Rayleigh-Taylor instabilities in magnetohydrodynamic flows. Phys. Scr. **64**, 533-547 (2001)
42. Khater, AH, Callebaut, DK, Seadawy, AR: Nonlinear dispersive instabilities in Kelvin-Helmholtz magnetohydrodynamic flows. Phys. Scr. **67**, 340-349 (2003)
43. Khater, AH, Callebaut, DK, Seadawy, AR: General soliton solutions of an n-dimensional complex Ginzburg-Landau equation. Phys. Scr. **62**, 353-357 (2000)
44. Khater, AH, Callebaut, DK, Helal, MA, Seadawy, AR: Variational method for the nonlinear dynamics of an elliptic magnetic stagnation line. Eur. Phys. J. D **39**, 237-245 (2006)
45. Khater, AH, Callebaut, DK, Seadawy, AR: General soliton solutions for nonlinear dispersive waves in convective type instabilities. Phys. Scr. **74**, 384-393 (2006)
46. Seadawy, AR: New exact solutions for the KdV equation with higher order nonlinearity by using the variational method. Comput. Math. Appl. **62**, 3741-3755 (2011)
47. Helal, MA, Seadawy, AR: Benjamin-Feir instability in nonlinear dispersive waves. Comput. Math. Appl. **64**, 3557-3568 (2012)
48. Seadawy, AR: Solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili dynamic equation in dust-acoustic plasmas. Pramana J. Phys. **89**(3), Article ID 49 (2017)
49. Seadawy, AR: Modulation instability analysis for the generalized derivative higher order nonlinear Schrodinger equation and its the bright and dark soliton solutions. J. Electromagn. Waves Appl. **31**(14), 1353-1362 (2017)
50. Seadawy, AR: Traveling-wave solutions of a weakly nonlinear two-dimensional higher-order Kadomtsev-Petviashvili dynamical equation for dispersive shallow-water waves. Eur. Phys. J. Plus **132**(29), 1-13 (2017)
51. Wazwaz, AM: Multiple soliton solutions and other exact solutions for a two-mode KdV equation. Math. Methods Appl. Sci. **40**, 2277-2283 (2017). doi:10.1002/mma.4138
52. Wazwaz, AM: A two-mode Burgers equation of weak shock waves in a fluid: multiple kink solutions and other exact solutions. Int. J. Appl. Comput. Math. **3**, 3977-3985 (2017). doi:10.1007/s40819-016-0302-4
53. Wazwaz, AM: Two-mode fifth-order KdV equations: necessary conditions for multiple-soliton solutions to exist. Nonlinear Dyn. **87**, 1685-1691 (2017). doi:10.1007/s11071-016-3144-z
54. Korsunsky, SV: Soliton solutions for a second-order KdV equation. Phys. Lett. A **185**, 174-176 (1994)
55. Syam, M, Jaradat, H, Alquran, M: A study on the two-mode coupled modified Korteweg-de Vries using the simplified bilinear and the trigonometric-function methods. Nonlinear Dyn. **90**(2), 1363-1371 (2017)
56. Jaradat, HM, Syam, M, Alquran, M: A two-mode coupled Korteweg-de Vries: multiple-soliton solutions and other exact solutions. Nonlinear Dyn. **90**(1), 371-377 (2017)
57. Jaradat, HM: Two-mode coupled Burgers equation: multiple-kink solutions and other exact solutions. Alex. Eng. J. (2017). doi:10.1016/j.aej.2017.06.014
58. Hirota, R: Exact solution of the modified Korteweg-de Vries equation for multiple collisions of solitons. J. Phys. Soc. Jpn. **33**, 1456-1458 (1972)
59. Shukri, S, Al-Khaled, K: The extended tanh method for solving systems of nonlinear wave equations. Appl. Math. Comput. **217**(5), 1997-2006 (2010)
60. Qawasmeh, A, Alquran, M: Reliable study of some new fifth-order nonlinear equations by means of (G'/G)-expansion method and rational sine-cosine method. Appl. Math. Sci. **8**(120), 5985-5994 (2014)
61. Qawasmeh, A, Alquran, M: Soliton and periodic solutions for $(2 + 1)$-dimensional dispersive long water-wave system. Appl. Math. Sci. **8**(50), 2455-2463 (2014)
62. Alquran, M, Al-Khaled, K: The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type. Phys. Scr. **84**, Article ID 025010 (2011)
63. Alquran, M, Al-Khaled, K: Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations. Phys. Scr. **83**, Article ID 065010 (2011)