Baseri et al. *Advances in Difference Equations* (2017) 2017:346
DOI 10.1186/s13662-017-1401-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Normalized Bernstein polynomials in solving space-time fractional diffusion equation

A Baseri, E Babolian and S Abbasbandy*

*Correspondence:
abbasbandy@yahoo.com
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract

In this paper, we solve a time-space fractional diffusion equation. Our methods are based on normalized Bernstein polynomials. For the space domain, we use a set of normalized Bernstein polynomials and for the time domain, which is a semi-infinite domain, we offer an algebraic map to make the rational normalized Bernstein functions. This study uses Galerkin and collocation methods. The integrals in the Galerkin method are established with Chebyshev interpolation. We implemented the proposed methods for some examples that are presented to demonstrate the theoretical results. To confirm the accuracy, error analysis is carried out.

Keywords: rational normalized Bernstein functions (RNBF); normalized Bernstein polynomials (NBP); time-space fractional diffusion equation; error analysis; collocation and Galerkin methods

1 Introduction

Fractional calculus allows mathematicians and engineers better modeling of a wide class of systems with anomalous dynamic behavior and better understanding of the facets of both physical phenomena and artificial processes. Hence the mathematical models derived from differential equations with noninteger/fractional order derivatives or integrals are becoming a fundamental research issue for scientists and engineers [1].

The fractional advection-diffusion equation is presented as a useful approach for the description of transport dynamics in complex systems. The time, space and time-space fractional advection-diffusion equations have been used to describe important physical phenomena that occur in amorphous, colloid, geophysical and geological processes [2].

Bernstein polynomials play an important role in many branches of mathematics, such as probability, approximation theory and computer-aided geometric design [3]. Also, in recent decades, the authors discovered some new analytic properties and some applications for these polynomials. For example, the rate of convergence of these polynomials derived by Cheng [4] for a certain class of functions. Farouki [5] showed that the Bernstein polynomial basis is an optimal stable basis among non-negative bases on the desired interval. Alshbool [6] approximated solutions of fractional-order differential equations with estimation error by using fractional Bernstein polynomials. Also, he applied a new mod-

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ification of the Bernstein polynomial method called multistage Bernstein polynomials to solve fractional order stiff systems [7].

In this paper, the normalized Bernstein polynomials with collocation and Galerkin methods are applied to turn the problem into an algebraic system. The fractional diffusion equation with variable coefficients is considered

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + b(x,t)\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + s(x,t), \quad 0 < x < b, t \geq 0. \tag{1}$$

In this equation $u(x,t)$ is the unknown function, $s(x,t)$ is called the source-term and $d(x,t)$ and $b(x,t)$ are the diffusion and advection coefficients, $1 < \alpha \leq 2$ and $0 < \gamma \leq 1$ are the orders of space fractional derivatives and $0 < \beta \leq 1$ is the time fractional order. We also consider the initial condition

$$u(x,0) = u_0(x), \quad 0 < x < b, \tag{2}$$

and the boundary conditions

$$u(0,t) = g_0(t), \qquad u(b,t) = g_1(t), \quad t \geq 0. \tag{3}$$

The existence and uniqueness of solutions for such problems are guaranteed in [8, 9]. Note that problem (1) for $\alpha = 2$, $\beta = 1$ and $\gamma = 1$ is the classical advection-diffusion equation. Problem (1) is discussed from the view of some scholars like Gao and Sun [10] who propose a numerical algorithm based on the finite difference method for $\gamma = 1$, $\alpha = 2$ and Deng [4] who discussed in the case $\beta = 1$ a finite element method with Riemann-Liouville space fractional derivatives.

The structure of this paper is as follows. In Section 2, some definitions are presented. In Section 3, normalized Bernstein polynomials and their required properties are given. In Section 4, we introduce the rational Bernstein functions and also describe some useful properties of these basis functions. In Section 5, the relation between the Legendre polynomials and orthonormal Bernstein is demonstrated. In Section 6, to estimate the integrals, we introduce a rational Chebyshev interpolation. In Section 7, Galerkin and collocation methods to approximate the unknown function $u(x,t)$ are applied. In Section 8, we offer error bounds for integer and fractional derivatives. To show the effectiveness of the proposed method, we report our numerical findings in Section 9; and finally, in Section 10, we give a brief conclusion.

2 Fundamentals of fractional calculus

There are various definitions for fractional derivative and integration. The widely-used definition for a fractional derivative is the Caputo sense and a fractional integration is the Riemann-Liouville definition.

Definition 2.1 ([9]) The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as follows:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x), \quad \alpha > 0, x > 0, \tag{4}$$

$$J^0 f(x) = f(x). \tag{5}$$

Definition 2.2 ([9]) The Caputo definition of a fractional derivative operator is given by

$$D^{\alpha}f(x) = J^{m-\alpha}D^{m}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt, \tag{6}$$

where $m-1<\alpha\leq m$, $m\in\mathbb{N}$, $x>0$. It has the following two basic properties:

$$\begin{aligned} &D^{\alpha}J^{\alpha}f(x) = f(x),\\ &J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}\bigl(0^{+}\bigr)\frac{x^{k}}{k!}, \quad x>0. \end{aligned} \tag{7}$$

For simplicity, we denote $\frac{\partial^{\alpha_i}}{\partial z^{\alpha_i}}$ by $D_z^{\alpha_i}$ which α_i denotes the Caputo derivative with respect to z, note that α_i may be β, γ or α and z can be x or t.

According to Definition 2.2, the Caputo time and space fractional derivatives of the function u are given as follows.

Definition 2.3 ([11]) The Caputo time-fractional derivative operator of order $\beta>0$ is defined as

$$D_t^{\beta}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_0^t (t-\xi)^{m-\beta-1}\frac{\partial^m u(x,\xi)}{\partial \xi^m}\,d\xi, & \text{if } m-1<\beta<m,\\ \frac{\partial^m u(x,t)}{\partial t^m}, & \beta=m\in\mathbb{N}, \end{cases}$$

and the space-fractional derivative operator of order $\alpha>0$ is defined as

$$D_x^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-\theta)^{m-\alpha-1}\frac{\partial^m u(\theta,t)}{\partial \theta^m}\,d\theta, & \text{if } m-1<\alpha<m,\\ \frac{\partial^m u(x,t)}{\partial x^m}, & \alpha=m\in\mathbb{N}. \end{cases}$$

One of the properties of fractional operators is linearity, i.e.,

$$\xi\bigl(\eta f(x)+\lambda g(x)\bigr) = \eta\xi\bigl(f(x)\bigr)+\lambda\xi\bigl(g(x)\bigr), \tag{8}$$

where ξ is the Riemann-Liouville fractional integral operator or Caputo fractional derivative and η and λ are real numbers.

3 Bernstein polynomials

The Bernstein polynomials of degree n on the interval $[0,1]$ are defined as

$$B_{i,n}(x) = \binom{n}{i}x^{i}(1-x)^{n-i}, \quad i=0,1,2,\ldots,n. \tag{9}$$

According to [12], Bernstein polynomials form a complete basis over the interval $[0,1]$ where they can be produced by the recursive relation

$$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x), \quad i=0,\ldots,n, \tag{10}$$

with $B_{-1,n-1}(x)=0$ and $B_{n,n-1}(x)=0$. Any Bernstein polynomial of degree n can be written in terms of the power basis directly calculated by using the binomial expansion of $(1-x)^{n-i}$

as follows:

$$B_{i,n}(x) = \sum_{j=i}^{n} (-1)^{j-i} \binom{n}{i} \binom{n-i}{j-i} x^j, \quad i = 0, \dots, n. \tag{11}$$

The product of Bernstein polynomials is defined as

$$B_{i,n}(x) B_{j,m}(x) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{m+n}{j+i}} B_{i+j,m+n}(x), \quad i = 0, \dots, n, j = 0, \dots, m, \tag{12}$$

and the definite integral of Bernstein polynomials is defined by

$$\int_0^1 B_{i,n}(x)\, dx = \frac{1}{n+1}, \quad i = 0, \dots, n. \tag{13}$$

The interesting properties of Bernstein polynomials can be found in [13]. By applying the Gram-Schmidt process on the set of Bernstein polynomials of degree n, the explicit representation of orthonormal Bernstein polynomials is obtained as follows [14]:

$$b_{i,n}(x) = \left(\sqrt{2(n-i)+1}\right)(1-x)^{n-i} \sum_{k=0}^{i} (-1)^k \binom{2n+1-k}{i-k} \binom{i}{k} x^{i-k}. \tag{14}$$

In addition, the orthonormal polynomials can be written in terms of non-orthonormal Bernstein functions

$$b_{i,n}(x) = \left(\sqrt{2(n-i)+1}\right) \sum_{k=0}^{i} (-1)^k \frac{\binom{2n+1-k}{i-k}\binom{i}{k}}{\binom{n-k}{i-k}} B_{i-k,n-k}(x). \tag{15}$$

With regard to the weight function $w_s(x) = 1$ on the interval $[0,1]$, these polynomials have the orthogonality relation

$$\int_0^1 b_{i,n}(x) b_{j,n}(x) w_s(x)\, dx = \delta_{ij}. \tag{16}$$

A function square integrable in $[0,1]$ may be approximated in terms of the normalized Bernstein basis (14). In practice, the first $(n+1)$ term Bernstein polynomials are considered [14]

$$f(x) \simeq \sum_{j=0}^{n} c_{j,n} b_{j,n}(x) = C^T \Phi(x), \tag{17}$$

where

$$\begin{aligned} &c_{j,n} = (f, b_{j,n}) = \int_0^1 f(x) b_{j,n}(x)\, dx, \quad j = 0, 1, \dots, n, \\ &C = [c_{0,n}, c_{1,n}, \dots, c_{n,n}]^T, \\ &\Phi(x) = \left[b_{0,n}(x), b_{1,n}(x), \dots, b_{n,n}(x)\right]^T. \end{aligned} \tag{18}$$

4 Rational normalized Bernstein functions

For the problems with a semi-infinite domain, we offer an algebraic map according to [15] to make a new class of functions which are called rational normalized Bernstein functions shown in the following form:

$$x = \frac{t}{t+L} \leftrightarrow t = \frac{xL}{1-x},$$

with $x \in [0,1]$, $t \in [0,+\infty)$ and L is a constant parameter. Here we take $L = 1$. For every fixed L, the semi-infinite interval $[0,+\infty)$ into $[0,1]$ is mapped by the presented algebraic map. Thus, a great variety of the new basis sets $R_{i,n}(t)$ which are the images under the change-of-coordinate of normalized Bernstein polynomials $\phi(t) = \frac{t}{t+L}$ are generated. The rational normalized Bernstein functions are denoted by

$$R_{i,n}(t) = b_{i,n}\big(\phi(t)\big). \tag{19}$$

Let $\Lambda = \{t | 0 \le t < \infty\}$ and consider that the non-negative function $w_r : [0,+\infty) \to \mathbb{R}$ is defined by $w_r(t) = w_s(t)\frac{d}{dt}\phi(t) = \frac{L}{(t+L)^2}$ on Λ [15]. The Banach space $L^2_{w_r}(\Lambda)$ is defined as follows:

$$L^2_{w_r} = \big\{f : \Lambda \to \mathbb{R} | f \text{ is measurable and } \|f\|_{L^2_{w_r}} < \infty\big\}, \tag{20}$$

where

$$\|f\|_{L^2_{w_r}} = \left(\int_0^\infty \big|f(t)\big|^2 w_r(t)\,dt\right)^{1/2}. \tag{21}$$

The orthogonality of the rational normalized Bernstein functions on $L^2_{w_r}(\Lambda)$ is given by

$$(R_{i,n}, R_{j,n})_{w_r} = \int_0^\infty R_{i,n}(t)R_{j,n}(t)w_r(t)\,dx = \delta_{ij}. \tag{22}$$

Any function $f \in L^2_{w_r}(\Lambda)$ may be approximated by rational normalized Bernstein functions as follows:

$$f(t) = \sum_{j=0}^{n} k_{j,n}R_{j,n}(t) = K^T\Psi(t), \tag{23}$$

where

$$k_{j,n} = (f, R_{j,n})_{w_r} = \int_\Lambda f(t)R_{j,n}(t)w_r(t)\,dt, \quad j = 0,1,\dots,n, \tag{24}$$

$$K = [k_{0,n}, k_{1,n}, \dots, k_{n,n}]^T, \tag{25}$$

$$\Psi(t) = \big[R_{0,n}(t), R_{1,n}(t), \dots, R_{n,n}(t)\big]^T. \tag{26}$$

5 Relation between Legendre and normalized Bernstein polynomials

First, we provide a brief description of Legendre polynomials, and then we try to find the relation between the Legendre polynomials and the normalized Bernstein polynomials by

introducing the transformation matrices W and G. The Legendre polynomials constitute an orthonormal basis on the interval $[0,1]$ as follows:

$$\begin{aligned} &L_0(x) = 1, \qquad L_1(x) = 2x - 1, \\ &L_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} L_i(x) - \frac{i}{i+1} L_{i-1}(x), \quad i = 1, 2, \ldots. \end{aligned} \tag{27}$$

The orthogonality relation is

$$\int_0^1 L_i(x) L_j(x)\, dx = \begin{cases} 0 & \text{for } i \neq j, \\ \frac{1}{2i+1} & \text{for } i = j. \end{cases} \tag{28}$$

A polynomial $P_n(x)$ of degree n can be expressed as

$$P_n(x) = \sum_{j=0}^{n} s_j L_j(x) = S^T \varphi(x), \tag{29}$$

where the shifted Legendre coefficient vector S and the shifted Legendre vector $\varphi(x)$ are given by

$$S = [s_0, s_1, \ldots, s_n]^T, \qquad \varphi(x) = \left[L_0(x), L_1(x), \ldots, L_n(x)\right]^T. \tag{30}$$

Now, the transformation of the Legendre polynomials into nth degree orthonormal Bernstein basis functions on $[0,1]$ can be written as follows:

$$L_k(x) = \sum_{i=0}^{n} w_{k,i} b_{i,n}(x), \quad k = 0, 1, \ldots, n. \tag{31}$$

The coefficients $w_{k,i}$, $k, i = 0, 1, \ldots, n$, form an $(n+1) \times (n+1)$ basis conversion matrix W

$$w_{k,j} = \int_0^1 L_k(x) b_{j,n}(x)\, dx. \tag{32}$$

$L_k(x)$ can be written as [16]

$$L_k(x) = \sum_{j=0}^{n} h_{k,j} B_{j,n}(x), \tag{33}$$

where

$$h_{k,j} = \frac{1}{\binom{n}{j}} \sum_{i=\max(0,j+k-n)}^{\min(j,k)} (-1)^{k+j} \binom{k}{i} \binom{k}{i} \binom{n-k}{j-i}.$$

To get $w_{k,j}$, replace $b_{i,n}(x)$ and $L_k(x)$ with (15) and (33), respectively, then by using (12), (13), we have

$$w_{k,j} = \sum_{j=0}^{n} \sum_{k_1=0}^{i} h_{k,j} \sqrt{2(n-i)+1} (-1)^{k_1} \frac{\binom{n}{j}\binom{2n+1-k_1}{i-k_1}\binom{i}{k_1}}{(2n-k_1+1)\binom{2n-k_1}{i+j-k_1}}, \tag{34}$$

where the matrix form of this relation is

$$\varphi(x) = W\Phi(x). \tag{35}$$

We write the transformation of the orthonormal Bernstein basis functions into Legendre polynomials on $[0,1]$ as

$$b_{k,n}(x) = \sum_{i=0}^{n} g_{k,i} L_i(x), \quad k = 0,1,\dots,n. \tag{36}$$

The coefficients $g_{k,i}$, $k,i = 0,1,\dots,n$, form an $(n+1)\times(n+1)$ basis conversion matrix G

$$g_{k,j} = (2j+1)\int_0^1 b_{k,n}(x)L_j(x)\,dx. \tag{37}$$

Again, for orthonormal Bernstein put (15) and for non-orthonormal Bernstein use the following relation [16]:

$$B_{k,n}(x) = \sum_{j=0}^{n} l_{k,j} L_j(x)\,dx, \tag{38}$$

where

$$l_{k,j} = \frac{2j+1}{n+j+1}\binom{n}{k}\sum_{i=0}^{j}(-1)^{j+i}\frac{\binom{j}{i}\binom{j}{i}}{\binom{n+j}{k+i}}, \quad k,j = 0,1,\dots,n.$$

The properties of Legendre polynomials lead to

$$g_{k,j} = (2j+1)^2\sqrt{2(n-k)+1}\sum_{p=0}^{k}(-1)^p\frac{\binom{2n+1-p}{k-p}\binom{k}{p}}{\binom{n-p}{k-p}}l_{k-p,i}. \tag{39}$$

The matrix form of the equation can be expressed as follows:

$$\Phi(x) = G\varphi(x). \tag{40}$$

6 Rational Chebyshev interpolation approximation

The Gauss-Radau integration is introduced in [17, 18]. Further rational Chebyshev-Gauss-Radau points are defined in [19]. Let $\Upsilon_n = \operatorname{span}\{R_0, R_1, \dots, R_n\}$ be the space of rational Chebyshev functions and

$$x_j = \cos\left(\frac{2j\pi}{2n+1}\right), \quad j = 0,1,\dots,n, \tag{41}$$

be the $(n+1)$ Chebyshev-Gauss-Radau points in $[-1,1]$, thus we define

$$t_j = L\frac{1+x_j}{1-x_j}, \quad j = 0,1,\dots,n, \tag{42}$$

as rational Chebyshev-Gauss-Radau nodes. The relations between rational Chebyshev orthogonal systems and rational Gauss-Radau integrations are as follows:

$$\int_0^{\infty} u(t)w(t)\,dt = \int_{-1}^{1} u\left(L\frac{1+x}{1-x}\right)\rho(x)\,dx = \sum_{j=0}^{n} u(t_j)w_j, \tag{43}$$

where $\rho(x) = \frac{1}{\sqrt{1-x^2}}$, $w_0 = \frac{\pi}{2n+1}$ and $w_j = \frac{\pi}{n+1}$, $1 \le j \le n$.

7 Function approximation by normalized Bernstein polynomials and rational normalized Bernstein functions

Generally, we approximate any real-valued function $u(x,t)$ defined on $[0,b] \times [0,+\infty)$ by Bernstein polynomials and rational Bernstein functions as [20, 21]

$$u(x,t) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \lambda_{ij} b_{i,n}(x) R_{j,n}(t), \tag{44}$$

where

$$\lambda_{ij} = \int_0^b \int_0^{+\infty} u(x,t) b_{i,n}(x) w_s(x) R_{j,n}(t) w_r(t)\,dt\,dx, \quad i = 0,1,\dots,n, j = 0,1,\dots,m.$$

Let the approximation of $u(x,t)$ be obtained by truncating the series (44) as

$$u_{n,m}(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{m} \lambda_{ij} b_{i,n}(x) R_{j,m}(t) = \Phi^T(x) U \Psi(t), \tag{45}$$

where $\Phi(x)$ is $(n+1)$ column vector defined in (18) and $\Psi(t)$ is $(m+1)$ vector corresponding in (26), $U = [\lambda_{ij}]$ is a matrix of size $(n+1) \times (m+1)$. In most cases, we take $n = m$.

We can also approximate the derived functions by applying the linear property (8) as follows:

$$D_x^{\alpha} u(x,t) \approx \left(D_x^{\alpha}\Phi(x)\right)^T U \Psi(t), \tag{46}$$

$$D_t^{\beta} u(x,t) \approx \Phi(x)^T U \left(D_t^{\beta}\Psi(t)\right). \tag{47}$$

For calculating $D_x^{\alpha}\Phi(x)$ and $D_t^{\beta}\Psi(t)$, use the Caputo definition

$$D_x^{\alpha} b_{i,n}(x) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^x \frac{1}{(x-z)^{\alpha+1-\lceil\alpha\rceil}} D_z^{\lceil\alpha\rceil}\left(b_{i,n}(z)\right) dz, \tag{48}$$

and

$$D_t^{\beta} R_{j,n}(t) = \frac{1}{\Gamma(\lceil\beta\rceil - \beta)} \int_0^t \frac{1}{(t-z)^{\beta+1-\lceil\beta\rceil}} D_z^{\lceil\beta\rceil}\left(R_{j,n}(z)\right) dz. \tag{49}$$

$D_x^{\alpha}\Phi(x)$ and $D_t^{\beta}\Psi(t)$ are $[D_x^{\alpha} b_{i,n}(x)]$ and $[D_t^{\beta} R_{j,n}(t)]$ vectors, respectively. Note when $0 \le t \le 1$, we use the usual normalized Bernstein polynomials because these functions are defined in $[0,1]$. It means $\psi(t) = \Phi(t)$.

7.1 Collocation method

Now, we consider problem (1) with initial and boundary conditions (2) and (3), by using Eqs. (44)-(47). Problem (1) becomes

$$\begin{aligned} H(x,t) = {} & \Phi(x)^T U D_t^{\beta}\Psi(t) - d(x,t)\big(D_x^{\alpha}\Phi(x)\big)^T U\Psi(t) \\ & - b(x,t)\big(D_x^{\gamma}\Phi(x)\big)^T U\Psi(t) - s(x,t) = 0. \end{aligned} \tag{50}$$

Relation (50) gives $(n^2 - n)$ independent equations:

$$H(x_i,t_j) = 0, \quad i = 1,\dots,n-1, j = 1,\dots,n, \tag{51}$$

where x_j are shifted-Chebyshev-Gauss-Radau points in $[0,b]$ and t_j are rational Chebyshev-Gauss-Radau nodes. We can also approximate the initial and boundary conditions in (2), (3) as follows:

$$\rho(x) = \Phi(x)^T U\Psi(0) - u_0(x) = 0, \quad 0 < x < b, \tag{52}$$

$$\Omega_1(t) = \Phi(0)^T U\Psi(t) - g_0(t) = 0, \tag{53}$$

$$\Omega_2(t) = \Phi(b)^T U\Psi(t) - g_1(t) = 0, \quad t \geq 0. \tag{54}$$

By choosing n equations $\rho(x) = 0$ and $\Omega_i(t) = 0$ $(i = 1,2)$, we get $(3n+1)$ more equations, i.e.,

$$\rho(x_i) = 0, \quad i = 0,\dots,n, \tag{55}$$

$$\Omega_k(t_j) = 0, \quad k = 1,2 j = 1,\dots,n. \tag{56}$$

Equations (51), (55), (56) give a system of $(n+1)^2$ equations, which can be solved for u_{ij}, $i,j = 0,1,\dots,n$.

7.2 Galerkin method

To formulate the Galerkin method, we take the inner product with basis polynomials

$$\big(\big(H(x,t),\phi(x)\big)_{w_s},\psi(t)\big)_{w_r} = 0.$$

We multiply (50) by $b_{i,n}(x)$ and $R_{j,n}(t)$, integrate the resulting equation over $[0,b]$ and $[0,+\infty)$:

$$\begin{aligned} p[i,j] := {} & \int_0^b \int_0^{+\infty} H(x,t) b_{i,n}(x) R_{j,n}(t) w_s(x) w_r(t)\,dt\,dx \\ = {} & 0, \quad i = 1,\dots,n-1, j = 1,\dots,n. \end{aligned} \tag{57}$$

We use the initial and boundary conditions (52)-(54) with an inner product

$$\big(\rho(x),\phi(x)\big)_{w_s} = 0 \quad \Rightarrow \quad p[i,1] := \big(\rho(x), b_{i,n}(x)\big)_{w_s} = 0, \quad i = 0,1,\dots,n, \tag{58}$$

$$\big(\Omega_k(t),\psi(t)\big)_{w_r} = 0 \quad \Rightarrow \quad p[k,j] := \big(\Omega_k(t), R_{j,n}(t)\big)_{w_r} = 0, \quad k = 1,2, j = 1,2,\dots,n. \tag{59}$$

Equations (57)-(59) are solved with Chebyshev-Gauss-Radau integration in the infinite interval $(0,+\infty)$, so we have a system of $(n+1)^2$ equations, which can be solved for u_{ij}, $i,j = 0,1,\dots,n$.

8 Error estimates

We begin this section with the basic error bound for an integer derivative, and then we refer to fractional derivatives, which are important for the main result (see [22] for more details). So, for fractional time-space fractional diffusion equations like Eq. (1), an approach to the convergence analysis of the Bernstein method is presented.

Definition 8.1 Suppose that $\mathbb{P}_N = \operatorname{span}\{b_{0,N}, b_{1,N}, \dots, b_{N,N}\}$. We define $\Pi_N : L^2(I) \to \mathbb{P}_N$ the L^2-orthogonal projection such that

$$(\Pi_N u - u, v) = 0, \quad \forall v \in \mathbb{P}_N,$$

where

$$(\Pi_N u)(x) = \sum_{j=0}^{N} a_j b_{j,N}(x).$$

A general procedure in error analysis is to compare the numerical solution u_N with an orthogonal projection $\Pi_N u$ of the exact solution u and use the inequality

$$\|u - u_N\| \le \|u - \Pi_N u\| + \|\Pi_N u - u_N\|.$$

We know (from Theorem 3.14 of [15]) that $\Pi_N u$ is the best polynomial approximation of u, so we just need to measure the truncation error $\|u - \Pi_N u\|$. We introduce the Sobolev space

$$H^m(I) = \left\{ u : \frac{\partial^k u}{\partial x^k} \in L^2(I), 0 \le k \le m \right\}, \quad m \in \mathbb{N}.$$

To measure the truncation error $\Pi_N u - u$, the inner product, the norm and the semi-norm are equipped as

$$(u,v)_{H^m} = \sum_{j=0}^{m} \left(\frac{\partial^j u}{\partial x^j}, \frac{\partial^j v}{\partial x^j} \right),$$

$$\|u\|_{H^m} = (u,u)_{H^m}^{1/2}, \qquad |u|_{H^m} = \left\| \frac{\partial^m u}{\partial x^m} \right\|.$$

Theorem 8.1 *Let $0 \le l < m_1 \le N+1$. For any $u \in H^{m_1}(I)$,*

$$\left\| \partial_x^l (u - \Pi_N u) \right\| \le C_1 \sqrt{\frac{(N-m_1+1)!}{(N-l+1)!}} (N+m_1)^{(l-m_1)/2} \|u\|_{H^{m_1}},$$

where C_1 is constant.

Proof According to (36), we have

$$(\Pi_N u)(x) = \sum_{j=0}^{N} a_j b_{j,N}(x) = \sum_{j=0}^{N} a_j \sum_{i=0}^{N} g_{j,i} L_i(x)$$
$$= \sum_{i=0}^{N} \left(\sum_{j=0}^{N} a_j g_{j,i} \right) L_i(x) = \sum_{i=0}^{N} q_i L_i(x),$$

where $q_i = \sum_{j=0}^{N} a_j g_{j,i}$. This means that $\Pi_N u$ is also the best polynomial approximation of u in $L^2(I)$ for Legendre polynomials, and due to [15] (see Section 3.5, Theorem 3.35 of [15]) for Legendre polynomials, we have

$$\left\| \partial_x^l (u - \Pi_N u) \right\| \leq C_2 \sqrt{\frac{(N - m_1 + 1)!}{(N - l + 1)!}} (N + m_1)^{(l - m_1)/2} \left\| \partial_x^{m_1} u \right\|,$$

where C_2 is constant. By using the definition $H^{m_1}(I)$, we have

$$\left\| \partial_x^l (u - \Pi_N u) \right\| \leq C_2 \sqrt{\frac{(N - m_1 + 1)!}{(N - l + 1)!}} (N + m_1)^{(l - m_1)/2} \|u\|_{H^{m_1}}.$$

Because of the relation between Legendre and Bernstein in Section 5, we just have $C_1 = cC_2$. □

It is shown that a valid projection property for any basis of the space, and in fact any element, means that they are basis-free. So, we do not present the proof for other theorems.

Through examining the temporal domain in the interval $\Lambda = (0, +\infty)$, we also have definitions and theorems where they are fundamental results with the mapped orthonormal Bernstein approximations as follows.

Definition 8.2 Define the L^2-orthogonal projection $\hat{\Pi}_M : L^2(\Lambda) \to \mathbb{P}_M$ supposing that the space $\tilde{H}^m(\Lambda) := L^2_{w_r}$ and $\mathbb{P}_M = \operatorname{span}\{R_{0,M}, R_{1,M}, \ldots, R_{M,M}\}$ such that

$$(\hat{\Pi}_M u - u, v) = 0, \quad \forall v \in \mathbb{P}_M,$$

where

$$(\hat{\Pi}_M u)(t) = \sum_{j=0}^{M} k_j R_{j,M}(t).$$

For this space $\tilde{H}^m(\Lambda)$,

$$\tilde{H}^m(\Lambda) = \left\{ u : \frac{\partial^k u}{\partial x^k} \in L^2(\Lambda), 0 \leq k \leq m \right\}, \quad m \in \mathbb{N},$$

the norm and the semi-norm are defined as

$$\|u\|_{\tilde{H}^m} = \left(\sum_{j=0}^{m} \left\| \frac{\partial^j u}{\partial t^j} \right\|^2 \right)^{1/2}, \qquad |u|_{\tilde{H}^m} = \left\| \frac{\partial^m u}{\partial t^m} \right\|.$$

Theorem 8.2 ([15]) *Let $l = 0$ or 1 and $l \le m_2 \le M+1$, for any $u \in \tilde{H}^{m_2}(\Lambda)$, we have*

$$\left\| \partial_t^l (u - \hat{\Pi}_M u) \right\| \le C_3 \sqrt{\frac{(M-m_2+1)!}{(M-l+1)!}} (M+m_2)^{(l-m_2)/2} \|u\|_{\tilde{H}^{m_2}},$$

where C_3 is constant.

Definition 8.3 We define

$$H^{r,s}(\Omega) = H^s\big(I_t; H^r(I_x)\big) = \left\{ u \in L^2(\Omega) \Big| \frac{\partial^{i+j} u}{\partial x^i \, \partial t^j} \in L^2(\Omega), 0 \le i \le r, 0 \le j \le s \right\}$$

as a Hilbert space, and in this space the inner product and norm are defined as

$$(u,v)_{r,s} = \sum_{i=0}^{r} \sum_{j=0}^{s} \int_0^{+\infty} \int_0^1 \frac{\partial^{i+j} u(x,t)}{\partial x^i \, \partial t^j} \frac{\partial^{i+j} v(x,t)}{\partial x^i \, \partial t^j} \, dx \, dt,$$

$$\|u\|_{H^{r,s}} = \left(\sum_{i=0}^{r} \sum_{j=0}^{s} \left\| \frac{\partial^{i+j} u(x,t)}{\partial x^i \, \partial t^j} \right\|_{L^2(\Omega)}^2 \right)^{1/2},$$

where $\Omega = I_x \times I_t = (0,1) \times (0,+\infty)$ is the two-dimensional domain.

The space of the measurable functions $u : \Omega \to \mathbb{R}$ is considered, and we denote $H^{r,0} = L^2(I_t; H^r(I_x))$ such that

$$\|u\|_{H^{r,0}} = \left(\int_0^{+\infty} \|u(\cdot,t)\|_{H^r(I_x)}^2 \, dt \right)^{1/2} < +\infty.$$

On the other hand, $H^{0,s}$ for any positive integer s can be defined as

$$H^{0,s} = H^s\big(I_t; L^2(I_x)\big) = \left\{ u \in L^2(\Omega) \Big| \frac{\partial^j u}{\partial t^j} \in L^2(\Omega), 0 \le j \le s \right\},$$

with the given norm

$$\|u\|_{H^{0,s}} = \left(\sum_{j=0}^{s} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Theorem 8.3 *Consider $\Pi_{N,M} = \operatorname{span}\{b_{i,N}(x) R_{j,M}(t), i = 0,1,\dots,N, j = 0,1,\dots,M\}$ if $\Pi_{N,M} u$ is the projection of u upon $\mathbb{P}_{N,M}$, i.e.,*

$$(\Pi_{N,M} u)(x,t) = u_{N,M}(x,t),$$

which means $\Pi_{N,M} u(x,t)$ is the best polynomial approximation of $u_{N,M}(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} \lambda_{ij} b_{i,N}(x) R_{j,M}(t)$ in $L^2(\Omega)$. For any function $u \in L^2(\Omega)$, there are constants C_6,

C_7 for all $0 \le m_1 \le N+1$ and $0 \le m_2 \le M+1$, where $m_1, m_2 \in \mathbb{N}$,

$$\|u - \Pi_{N,M}u\|_{L^2(\Omega)} \le C_6\sqrt{\frac{(N-m_1+1)!}{(N+1)!}}(N+m_1)^{-m_1/2}\|u\|_{H^{m_1,0}} + C_7\sqrt{\frac{(M-m_2+1)!}{(M+1)!}}(M+m_2)^{-m_2/2}\|u\|_{H^{0,m_2}}.$$

Proof One-dimensional orthogonal projections defined in Definitions 8.1 and 8.2 are assumed to be Π_N, $\hat{\Pi}_M$. Then

$$\Pi_{N,M}u = \Pi_N\hat{\Pi}_M u.$$

Given Theorems 8.1 and 8.2, we have

$$\begin{aligned}\|u - \Pi_{N,M}u\|_{L^2(\Omega)} &\le \|u - \Pi_N u\|_{L^2(\Omega)} + \left\|\Pi_N(u - \hat{\Pi}_M u)\right\|_{L^2(\Omega)} \\ &\le \|u - \Pi_N u\|_{L^2(\Omega)} + C_5\|u - \hat{\Pi}_M u\|_{L^2(\Omega)} \\ &\le C_6\sqrt{\frac{(N-m_1+1)!}{(N+1)!}}(N+m_1)^{-m_1/2}\|u\|_{H^{m_1,0}} \\ &\quad + C_7\sqrt{\frac{(M-m_2+1)!}{(M+1)!}}(M+m_2)^{-m_2/2}\|u\|_{H^{0,m_2}},\end{aligned}$$

where C_5 is constant and $C_7 = C_5C_3$. □

The error function $e(x,t)$ of the approximation $u_{N,M}(x,t)$ for the exact solution $u(x,t)$ of Eq. (1) is defined as

$$e(x,t) = u(x,t) - u_{N,M}(x,t), \tag{60}$$

and corresponding with the best approximation, we have

$$e_{N,M}(x,t) = u(x,t) - \Pi_{N,M}u(x,t), \tag{61}$$

where, according to Theorem 8.3, when $N, M \to \infty$, $\|e_{N,M}\| \to 0$; and consequently,

$$\big(\|e\| = \|u - u_{N,M}\| \le \|u - \Pi_{N,M}u\| + \|\Pi_{N,M}u - u_{N,M}\|\big) \to 0.$$

We have also error bounds for the fractional derivatives as follows.

Theorem 8.4 *Suppose $u \in L^2(\Omega)$, if $n-1 < \alpha \le n$, $n = \lceil\alpha\rceil$, and $n < m_1 < N+1$, $m_1 \in \mathbb{N}$, then we have*

$$\left\|D_x^\alpha u - D_x^\alpha(\Pi_{N,M}u)\right\|_{L^2(\Omega)} \le \frac{C_8\sqrt{\frac{(N-m_1+1)!}{(N-n+1)!}}(N+m_1)^{(n-m_1)/2}}{(\Gamma[n-\alpha+1])^2}\left\|u(x,t) - \big(\Pi_{N,M}u(x,t)\big)\right\|^2_{H^{m_1,0}(\Omega)},$$

where C_8 is constant.

Proof Due to [23], we have

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

By using Definitions 2.1 and 2.2,

$$\begin{aligned}
&\left\| D_x^\alpha u - D_x^\alpha (\Pi_{N,M} u) \right\|^2_{L^2(\Omega)} \\
&\quad = \left\| J_x^{n-\alpha} \left(D_x^n u(x,t) - D_x^n \left(\Pi_{N,M} u(x,t) \right) \right) \right\|^2_{L^2(\Omega)} \\
&\quad = \left\| \frac{1}{x^{1+\alpha-n}\Gamma[n-\alpha]} * \left(D_x^n u(x,t) - D_x^n \left(\Pi_{N,M} u(x,t) \right) \right) \right\|^2_{L^2(\Omega)} \\
&\quad \leq \left\| \frac{1}{x^{1+\alpha-n}\Gamma[n-\alpha]} \right\|^2_1 \left\| D_x^n u(x,t) - D_x^n \left(\Pi_{N,M} u(x,t) \right) \right\|^2_{L^2(\Omega)},
\end{aligned}$$

and applying Theorem 8.1,

$$\begin{aligned}
&\left\| D_x^\alpha u - D_x^\alpha (\Pi_{N,M} u) \right\|^2_{L^2(\Omega)} \\
&\quad \leq \left(\frac{1}{\Gamma[n-\alpha+1]} \right)^2 C_8 \sqrt{\frac{(N-m_1+1)!}{(N-n+1)!}} (N+m_1)^{(n-m_1)/2} \\
&\qquad \times \left\| u(x,t) - \Pi_{N,M} u(x,t) \right\|^2_{H^{m_1,0}(\Omega)},
\end{aligned}$$

the proof is complete. □

So when $N, M \to \infty$, $(\|D_x^\alpha(e_{N,M})\| = \|D_x^\alpha u - D_x^\alpha(\Pi_{N,M} u)\|) \to 0$.

Theorem 8.5 *Suppose* $u \in L^2(\Omega)$, *and* $h - 1 < \beta \leq h < m_2 \leq M + 1$, $m_2 \in \mathbb{N}$, $h = \lceil \beta \rceil$, *then*

$$\begin{aligned}
&\left\| D_t^\beta u - D_t^\beta (\Pi_{N,M} u) \right\|_{L^2(\Omega)} \\
&\quad \leq \frac{C_9 \sqrt{\frac{(M-m_2+1)!}{(M-h+1)!}} (M+m_2)^{(h-m_2)/2}}{(\Gamma[h-\beta+1])^2} \left\| u(x,t) - \Pi_{N,M} u(x,t) \right\|^2_{H^{0,m_2}(\Omega)},
\end{aligned}$$

where C_9 *is constant*.

Proof The proof is similar to Theorem 8.4. □

This theorem shows that, when $N, M \to \infty$, $(\|D_t^\beta(e_{N,M})\| = \|D_t^\beta u - D_t^\beta(\Pi_{N,M} u)\|) \to 0$.

For the proposed method, the error assessment relying on the residual error function is presented [24].

Take the following problem with the initial and boundary conditions:

$$\begin{aligned}
L\left[u_{N,M}(x,t)\right] &= D_t^\beta u_{N,M}(x,t) - d(x,t) D_x^\alpha u_{N,M}(x,t) - b(x,t) D_x^\gamma u_{N,M}(x,t) \\
&= s(x,t) + R_{N,M}(x,t),
\end{aligned} \tag{62}$$

$$u_{N,M}(x,0) = u_0(x), \qquad u_{N,M}(0,t) = g_0(t), \qquad u_{N,M}(b,t) = g_1(t). \tag{63}$$

$R_{N,M}(x,t)$ is the residual function which is obtained by subtracting (1)-(3) from (62)-(63) as follows:

$$D_t^{\beta} e_{N,M}(x,t) - d(x,t)D_x^{\alpha} e_{N,M}(x,t) - b(x,t)D_x^{\gamma} e_{N,M}(x,t) = -R_{N,M}(x,t), \tag{64}$$

with homogeneous initial and boundary conditions. By using Theorem 8.4 for α, γ and Theorem 8.5 for β, when $N, M \to \infty$, we have $R_{N,M} \to 0$.

9 Numerical examples

In this section, we carry out numerical examples for a time-space fractional diffusion equation of the proposed numerical methods. All our tests are done in Mathematica 9.1. We use the discrete L_∞ error for $x \in [0,1]$ and $t = T$ for different T in examples, which is obtained by suitable source term $s(x,t)$ in (1). The rate of convergence of the coefficient matrix with condition number as compared with the condition number of the Hilbert matrix H is provided with respect to the infinity norm, i.e.,

$$R_\infty = \frac{C_\infty(A)}{C_\infty(H)},$$

that $C_\infty(A) := \mathrm{Cond}_\infty(A)$.

Example 9.1 ([25]) Consider problem (1) with the initial condition $u(x,0) = x^2(1-x)^2$, homogeneous boundary conditions, diffusion and advection coefficients $d(x,t) = 0.001$, $b(x,t) = 2$, respectively. Suppose that the exact solution is $u(x,t) = x^2(1-x)^2e^{-t}$ for suitable source term. For $n = 5$, the normalized Bernstein polynomials are

$$\Phi(x) = \begin{pmatrix} \sqrt{11}(1-x)^5 \\ 3(1-x)^4(-1+11x) \\ \sqrt{7}(1-x)^3(1-20x+55x^2) \\ \sqrt{5}(1-x)^2(-1+27x-135x^2+165x^3) \\ \sqrt{3}(1-x)(1-32x+216x^2-480x^3+330x^4) \\ -1+35x-280x^2+840x^3-1{,}050x^4+462x^5 \end{pmatrix}, \tag{65}$$

and the corresponding rational normalized Bernstein polynomials are

$$\Psi(t) = \begin{pmatrix} \sqrt{11}(1-\frac{t}{t+1})^5 \\ 3(1-\frac{t}{t+1})^4(-1+11\frac{t}{t+1}) \\ \sqrt{7}(1-\frac{t}{t+1})^3(1-20\frac{t}{t+1}+\frac{55t^2}{(t+1)^2}) \\ \sqrt{5}(1-\frac{t}{t+1})^2(-1+\frac{27t}{t+1}-\frac{135t^2}{(t+1)^2}+\frac{165t^3}{(t+1)^3}) \\ \sqrt{3}(1-\frac{t}{t+1})(1-\frac{32t}{t+1}+\frac{216t^2}{(t+1)^2}-\frac{480t^3}{(t+1)^3}+\frac{330t^4}{(t+1)^4}) \\ -1+\frac{35t}{t+1}-\frac{280t^2}{(t+1)^2}+\frac{840t^3}{(t+1)^3}-\frac{1{,}050t^4}{(t+1)^4}+\frac{462t^5}{(t+1)^5} \end{pmatrix}. \tag{66}$$

We take U as an unknown matrix $6 * 6$ like

$$\begin{pmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} & u_{0,4} & u_{0,5} \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,5} \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,5} \\ u_{3,0} & u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} & u_{3,5} \\ u_{4,0} & u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} & u_{4,5} \\ u_{5,0} & u_{5,1} & u_{5,2} & u_{5,3} & u_{5,4} & u_{5,5} \end{pmatrix}.$$

Now we approximate the unknown function $u(x,t)$ by Bernstein polynomials and rational Bernstein functions as follows:

$$u_{5,5}(x,t) \simeq \sum_{i=0}^{5}\sum_{j=0}^{5} \lambda_{ij} b_{i,5}(x) R_{j,5}(t) = \Phi^T(x) U \Psi(t). \tag{67}$$

We also approximate the derived functions by (46), (47), where to obtain $D_t^\beta \Psi(t)$, $D_x^\alpha \Phi(x)$ and $D_x^\gamma \Phi(x)$, we must use (48), (49) and calculate $[D_t^\beta R_{j,n}(t)]$, $[D_x^\alpha b_{i,n}(x)]$ and $[D_x^\gamma b_{i,n}(x)]$ vectors of size 6. With replacing (65)-(67) and derived functions in (50), we have

$$\begin{aligned} H(x,t) &= \Phi(x)^T U D_t^\beta \Psi(t) - 0.01\big(D_x^\alpha \Phi(x)\big)^T U \Psi(t) - 2\big(D_x^\gamma \Phi(x)\big)^T U \Psi(t) - s_{\alpha,\beta,\gamma}(x,t) \\ &= 0. \end{aligned} \tag{68}$$

The initial and boundary conditions are also approximated as follows:

$$\rho(x) = \Phi(x)^T U \Psi(0) - x^2(1-x)^2 = 0, \quad 0 < x < 1, \tag{69}$$

$$\Omega_1(t) = \Phi(0)^T U \Psi(t) = 0, \tag{70}$$

$$\Omega_2(t) = \Phi(1)^T U \Psi(t) = 0, \quad t \geq 0. \tag{71}$$

For the collocation method, by substituting collocation points in Eqs. (68)-(71), we have a system of 36 equations, which can be solved for u_{ij}, $i,j = 0,\dots,5$.

To formulate the Galerkin method, we take the inner product with basis polynomials as follows:

$$\Big(\big(H(x,t),\phi(x)\big)_{w_s},\psi(t)\Big)_{w_r} = 0,$$

that leads to

$$p[i,j] := \int_0^1 \int_0^{+\infty} H(x,t) b_{i,5}(x) R_{j,5}(t) w_s(x) w_r(t)\,dt\,dx = 0, \quad i = 1,\dots,4, j = 1,\dots,5,$$

where inner products are solved with Chebyshev-Gauss-Radau integration (43) as follows:

$$p[i,j] = \sum_{k=0}^{5}\sum_{l=0}^{5} H(x_k,t_l) b_{i,5}(x_k) R_{j,5}(t_l) w_k w_l, \quad i = 1,\dots,4, j = 1,\dots,5. \tag{72}$$

Table 1 The maximum absolute error with $\alpha = 1.4$ for Example 9.1, the collocation method

γ	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
0.2	1.15249×10^{-4}	1.74967×10^{-5}	1.71026×10^{-7}	17.209	7.48674×10^{-6}
0.4	1.18497×10^{-4}	2.16571×10^{-5}	3.2801×10^{-7}	15.085	9.01317×10^{-6}
0.6	1.171774×10^{-4}	1.47567×10^{-5}	2.97227×10^{-7}	17.471	7.26192×10^{-6}
0.8	2.22216×10^{-4}	1.93125×10^{-5}	1.94697×10^{-7}	16.473	8.93887×10^{-6}

Table 2 The maximum absolute error with $\gamma = 0.4$ for Example 9.1, the Galerkin method

α	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
1.2	1.18738×10^{-4}	2.17028×10^{-5}	2.28411×10^{-7}	17.082	5.4121×10^{-5}
1.4	1.18497×10^{-4}	2.16751×10^{-5}	2.2801×10^{-7}	16.331	5.41498×10^{-5}
1.6	1.18139×10^{-4}	2.15888×10^{-5}	2.27469×10^{-7}	16.472	5.42155×10^{-5}
1.8	1.17613×10^{-4}	2.14884×10^{-5}	2.26757×10^{-7}	16.256	5.43212×10^{-5}

Initial and boundary conditions are

$$\big(\rho(x), \phi(x)\big)_{w_s} = 0$$

$$\Rightarrow \quad p[i,1] := \big(\rho(x), b_{i,5}(x)\big)_{w_s} = \sum_{k=0}^{5} \rho(x_k) b_{i,5}(x_k) w_k = 0, \quad i = 0,1,\dots,5, \tag{73}$$

$$\big(\Omega_k(t), \psi(t)\big)_{w_r} = 0$$

$$\Rightarrow \quad p[k,j] := \big(\Omega_k(t), R_{j,n}(t)\big)_{w_r} = \sum_{l=0}^{5} \Omega_k(t_l) R_{j,5}(t_l) w_l, \quad k = 1,2, j = 1,2,\dots,5. \tag{74}$$

Finally, Eqs. (72)-(74) give a system of 36 equations, which can be solved for u_{ij}, $i,j = 0,1,\dots,5$. Tables 1 and 2 show the maximum absolute error for collocation and Galerkin methods, respectively, for $n = 5$ and $\beta = 0.6$ and different α and γ.

Example 9.2 ([20, 21]) We consider the fractional diffusion equation (1) with the coefficient functions $d(x,t) = \Gamma(1.2)x^{1.8}$ (where Γ is Euler's gamma function), $b(x,t) = 0$ (hence $\gamma = 0$) with the initial boundary conditions

$$u(x,0) = x^2(1-x), \qquad u(0,t) = u(1,t) = 0.$$

Suppose that the exact solution is $u(x,t) = x^2(1-x)e^{-t}$ for suitable source term. The maximum errors L_∞ for different values of T and n are listed in Table 3 for the collocation method and in Table 4 for the Galerkin method. The results in both tables show that the proposed methods have good approximation accuracy for the relatively long time domain. To investigate the convergence of the methods, we apply our method to different values of fractional orders. Table 5 shows the residual errors for $n = 8$ with $\alpha \in (1,2)$ and $\beta \in (0,1)$ in the collocation method that converge to the exact solution. Also, this converges to the exact solution for different values of α, which has been shown in Figures 1-4. We have a similar situation for different β. The comparisons between the proposed method and the method described by Alavizadeh in [21] are listed in Table 6. The numerical results of this example demonstrate that our methods are more accurate than the Legendre approximation method in [21] at $t = 1$ for $n = 5$.

Table 3 The L_∞ error for Example 9.2 with $\alpha = 1.8$ and $\beta = 1$, the collocation method

n	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
5	5.0×10^{-4}	2.5×10^{-5}	2.0×10^{-7}	4.37	7.82322×10^{-5}
8	1.5×10^{-4}	1.5×10^{-4}	1.2×10^{-7}	12.917	3.52693×10^{-8}
10	6.0×10^{-6}	6.0×10^{-6}	1.2×10^{-7}	21.669	1.60277×10^{-10}
16	4.0×10^{-6}	3.0×10^{-6}	3.5×10^{-10}	74.458	2.32527×10^{-17}

Table 4 The L_∞ error for Example 9.2 with $\alpha = 1.8$ and $\beta = 1$, the Galerkin method

n	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
5	3.0×10^{-3}	1.0×10^{-3}	5.0×10^{-4}	4.868	2.26905×10^{-3}
8	8.0×10^{-4}	2.0×10^{-4}	5.0×10^{-5}	21.416	4.78888×10^{-6}
10	1.5×10^{-4}	1.5×10^{-5}	1.2×10^{-5}	53.039	3.13802×10^{-8}
16	1.5×10^{-6}	1.5×10^{-6}	8.0×10^{-7}	636.265	3.69793×10^{-15}

Table 5 Maximum residual errors for $n = 8$ for Example 9.2, the collocation method

β	α	$T = 2$	$T = 10$	$T = 50$	$T = 100$	R_∞
0.4	2	8.0×10^{-3}	1.4×10^{-2}	7.0×10^{-3}	5.0×10^{-3}	2.64548×10^{-8}
0.6		3.0×10^{-3}	6.0×10^{-3}	2.0×10^{-3}	1.5×10^{-3}	2.57232×10^{-8}
0.8		5.0×10^{-3}	2.0×10^{-3}	5.0×10^{-4}	3.0×10^{-4}	2.85507×10^{-8}
1	1.4	6.0×10^{-2}	1.5×10^{-2}	2.5×10^{-3}	5.0×10^{-5}	9.5567×10^{-8}
	1.6	2.0×10^{-2}	2.5×10^{-3}	4.0×10^{-4}	5.0×10^{-6}	4.49503×10^{-8}
	1.8	1.5×10^{-4}	1.5×10^{-4}	2.0×10^{-5}	1.2×10^{-7}	3.52396×10^{-8}

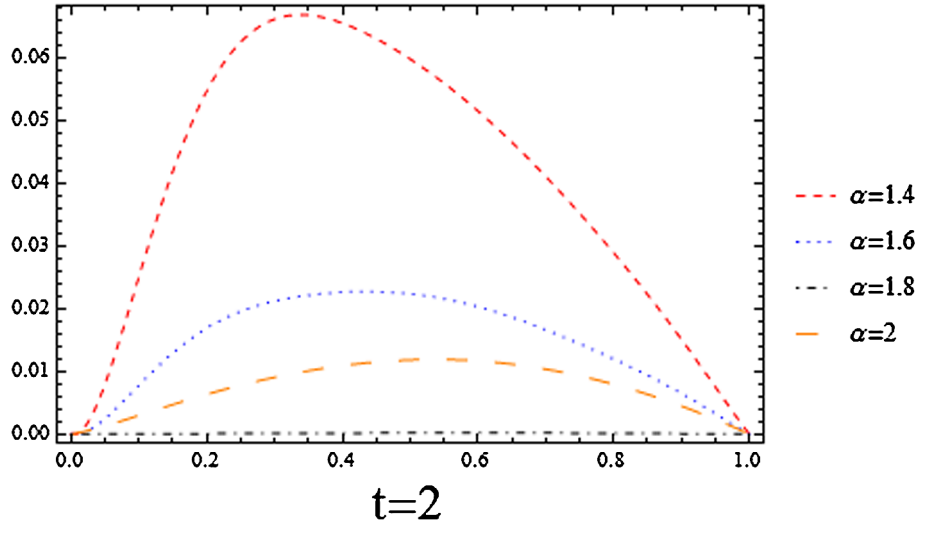


Figure 1 The residual error for Example 9.2 with $\beta = 1$ and $n = 8$ for different α, and the collocation method for $t = 2$.

Example 9.3 ([26]) We consider the fractional diffusion equation (1) with the coefficient functions $d(x,t) = \frac{\Gamma(2.2)}{6}x^{2.8}$, $b(x,t) = 0$ (hence $\gamma = 0$) with the initial boundary conditions

$$u(x,0) = x^3, \qquad u(0,t) = 0, \qquad u(1,t) = \exp(-t).$$

Suppose that the exact solution is $u(x,t) = x^3e^{-t}$ for suitable source term. The maximum errors L_∞ for different values of T and n are listed in Tables 7 and 8 for collocation and

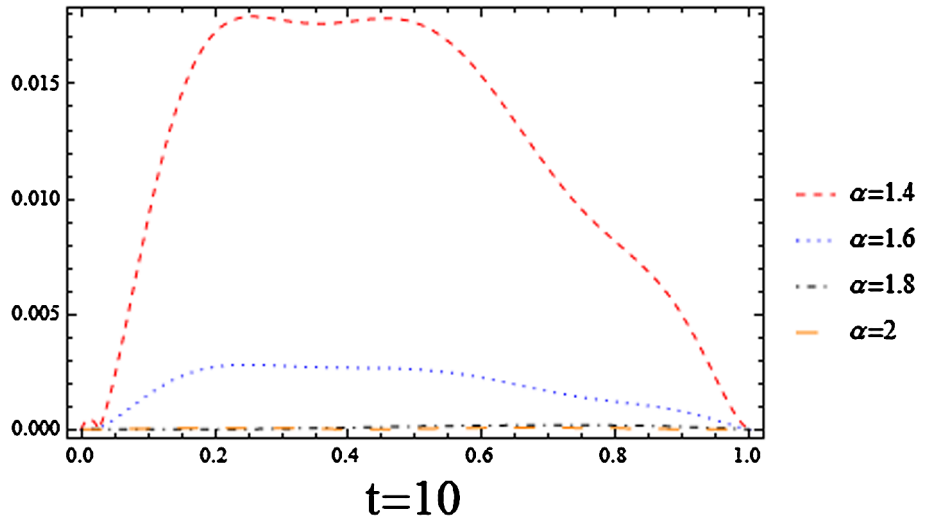


Figure 2 The residual error for Example 9.2 with $\beta = 1$ and $n = 8$ for different α, and the collocation method for $t = 10$.

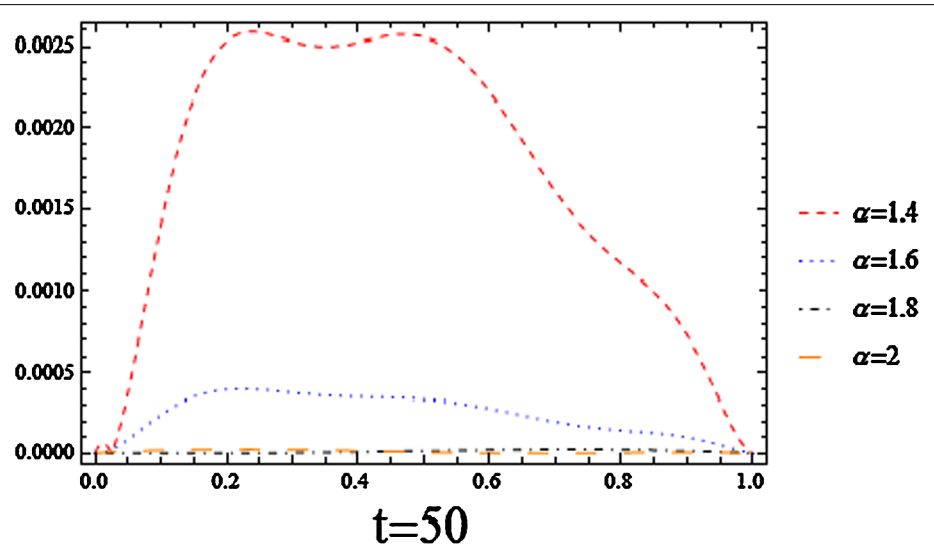


Figure 3 The residual error for Example 9.2 with $\beta = 1$ and $n = 8$ for different α, and the collocation method for $t = 50$.

Galerkin methods. Note that in these tables we provide CPU time (in seconds) consumed in the algorithms for obtaining the numerical solution, and when we compare these together for one problem, we see the collocation method acting in a shorter time compared with the Galerkin method. Table 9 shows the maximum residual errors for $n = 8$ when $\alpha \in (1,2)$ and $\beta \in (0,1)$. The convergence to the exact solution in the Galerkin method is obvious from this table. Also, this converges to the exact solution for different values of β which tends to 1 for the Galerkin method, which has been shown in Figures 5-8. When α tends to 2, we will have a similar figure. For the collocation method in $t = 1$, Table 10 compares the absolute error with the Chebyshev-spectral-Tau method in [26]. This shows our

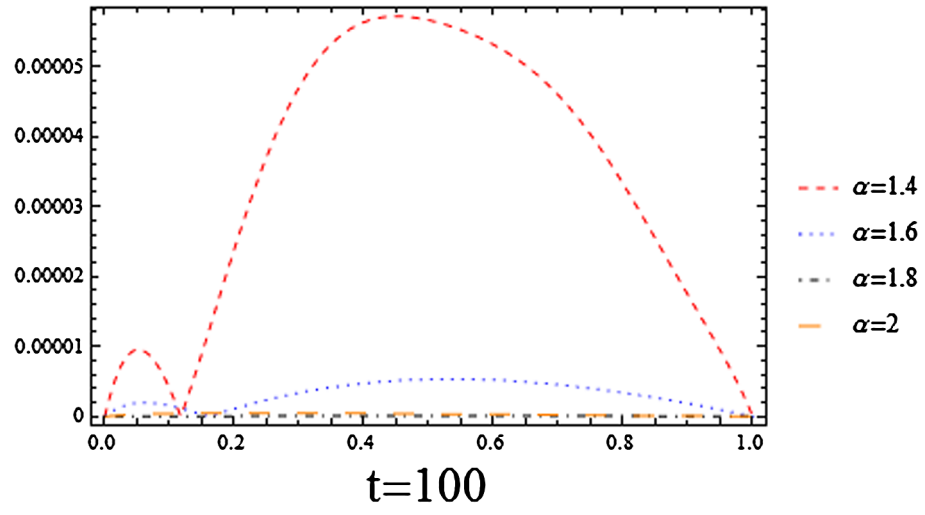


Figure 4 The residual error for Example 9.2 with $\beta = 1$ and $n = 8$ for different α, and the collocation method for $t = 100$.

Table 6 The absolute error at $t = 1$ with $\alpha = 1.8$, $\beta = 1$ and $n = 5$ for Example 9.2

x	Proposed method (collocation)	Proposed method (Galerkin)	Method in [21]
0.1	2.40083×10^{-8}	5.84791×10^{-7}	3.050×10^{-6}
0.2	2.97841×10^{-8}	8.63544×10^{-7}	4.260×10^{-6}
0.3	2.54578×10^{-8}	9.29233×10^{-7}	5.995×10^{-6}
0.4	1.72148×10^{-8}	8.58172×10^{-7}	6.942×10^{-6}
0.5	9.34788×10^{-9}	7.11067×10^{-7}	4.575×10^{-6}
0.6	4.30953×10^{-9}	5.34071×10^{-7}	2.370×10^{-6}
0.7	2.76392×10^{-9}	3.59841×10^{-7}	1.142×10^{-5}
0.8	3.63934×10^{-9}	2.08591×10^{-7}	1.393×10^{-5}
0.9	4.18034×10^{-9}	8.91471×10^{-8}	7.452×10^{-6}

Table 7 The L_∞ error for Example 9.3 with $\alpha = 1.8$ and $\beta = 1$, the collocation method

n	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
5	4.0×10^{-3}	7.0×10^{-4}	1.2×10^{-5}	4.275	2.3301×10^{-4}
8	8.0×10^{-4}	1.0×10^{-3}	5.0×10^{-6}	13.009	8.91987×10^{-7}
10	6.0×10^{-5}	6.0×10^{-5}	4.0×10^{-7}	22.198	7.66056×10^{-9}
16	2.0×10^{-5}	2.0×10^{-5}	2.0×10^{-7}	76.581	9.82014×10^{-16}

Table 8 The L_∞ error for Example 9.3 with $\alpha = 1.8$ and $\beta = 1$, the Galerkin method

n	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
5	1.5×10^{-2}	7.0×10^{-3}	2.5×10^{-3}	5.053	1.267×10^{-3}
8	8.0×10^{-3}	1.5×10^{-3}	7.0×10^{-4}	22.385	2.91652×10^{-6}
10	2.5×10^{-3}	4.0×10^{-4}	3.0×10^{-4}	53.508	1.52719×10^{-8}
16	1.4×10^{-5}	2.0×10^{-5}	6.0×10^{-6}	646.187	5.18714×10^{-16}

Table 9 Maximum residual errors with $n = 8$ for Example 9.3, the Galerkin method

β	α	$T = 2$	$T = 10$	$T = 50$	$T = 100$	R_∞
0.4	1.8	1.0×10^{-1}	1.0×10^{-1}	6.0×10^{-2}	5.0×10^{-2}	1.12744×10^{-7}
0.6		8.0×10^{-2}	6.0×10^{-2}	2.5×10^{-2}	2.5×10^{-2}	3.93491×10^{-7}
0.8		8.0×10^{-2}	3.5×10^{-2}	7.0×10^{-3}	1.0×10^{-2}	1.60004×10^{-6}
1	1.3	4.0×10^{-2}	8.0×10^{-3}	2.0×10^{-3}	1.4×10^{-3}	1.36568×10^{-5}
	1.5	6.0×10^{-2}	8.0×10^{-3}	4.0×10^{-3}	8.0×10^{-3}	2.85066×10^{-5}
	1.8	8.0×10^{-3}	1.5×10^{-3}	4.0×10^{-4}	7.0×10^{-4}	2.91652×10^{-6}

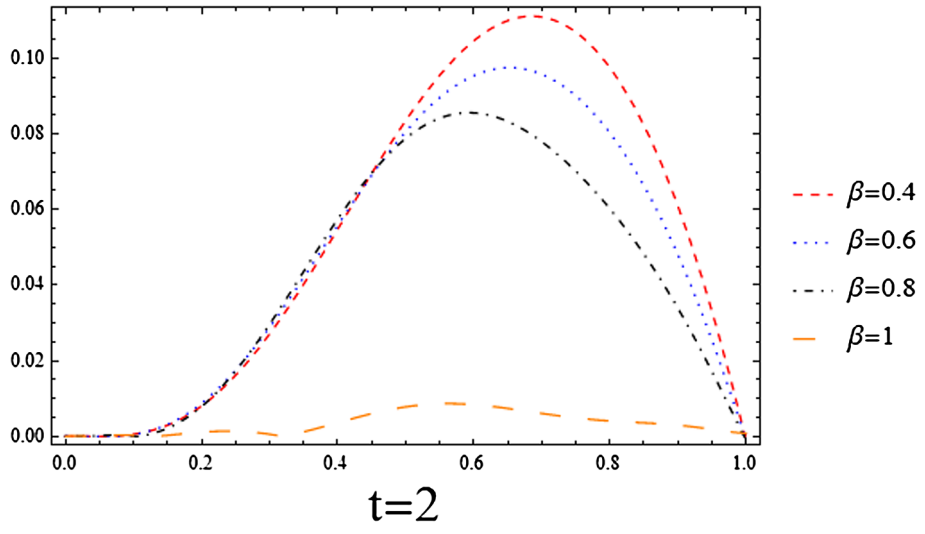


Figure 5 The residual error for Example 9.3 with $\alpha = 1.8$ and $n = 5$ for different β, and the Galerkin method for $t = 2$.

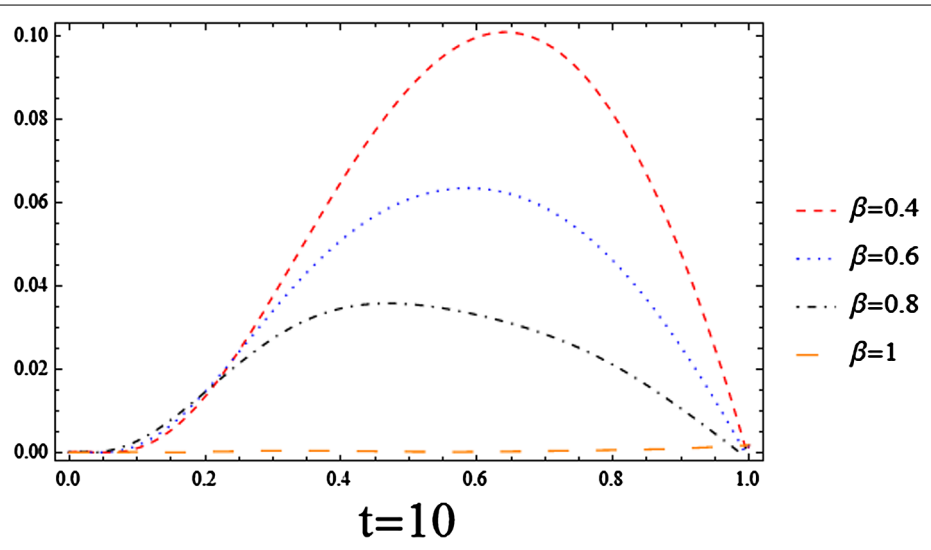


Figure 6 The residual error for Example 9.3 with $\alpha = 1.8$ and $n = 5$ for different β, and the Galerkin method for $t = 10$.

collocation method acting better than the method in [26]. Figures 9-11 show the absolute error at $t = 1$ for $\alpha = 1.8$, $\beta = 1$ and $n = 5$ in the interval $x \in [0,1]$ in the collocation, Chebyshev spectral-Tau and Galerkin methods, respectively.

Example 9.4 We consider the fractional diffusion equation (1) with $\alpha = 1.5$, $\beta = 0.6$, $\gamma = 0.8$ and the coefficient functions $d(x,t) = t$, $b(x,t) = x$ and the source term $s(x,t)$ such that the exact solution is $u(x,t) = x(2-x)\exp(-t) + t\exp(-t)$ with the initial boundary conditions

$$u(x,0) = x(2-x), \qquad u(0,t) = t\exp(-t), \qquad u(1,t) = \exp(-t) + t\exp(-t).$$

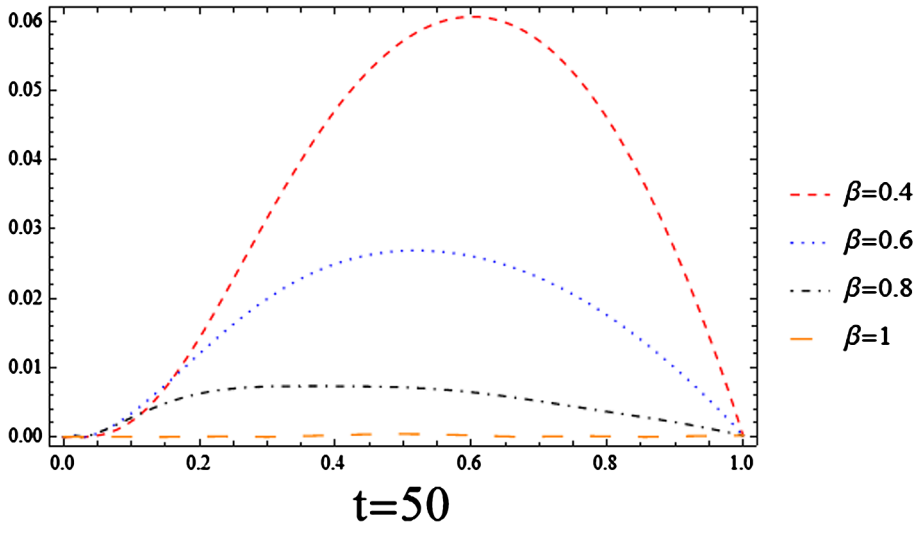


Figure 7 The residual error for Example 9.3 with $\alpha = 1.8$ and $n = 5$ for different β, and the Galerkin method for $t = 50$.

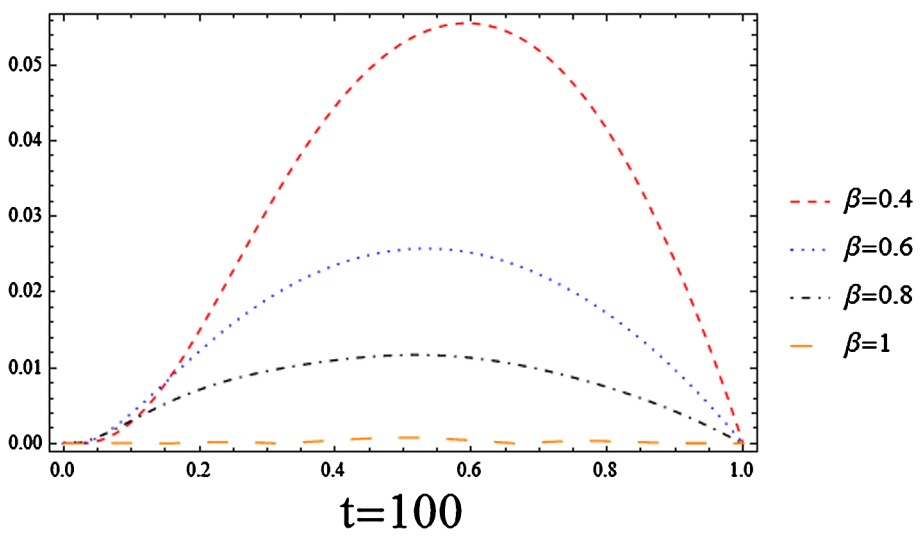


Figure 8 The residual error for Example 9.3 with $\alpha = 1.8$ and $n = 5$ for different β, and the Galerkin method for $t = 100$.

Table 10 The absolute error at $t = 1$ with $\alpha = 1.8$, $\beta = 1$ and $n = 5$ for Example 9.3

x	Proposed method (collocation)	Proposed method (Galerkin)	Method in [26]
0.1	2.20576×10^{-9}	2.35702×10^{-8}	1.930×10^{-6}
0.2	3.09058×10^{-8}	6.54584×10^{-7}	1.434×10^{-7}
0.3	8.8553×10^{-8}	5.16048×10^{-7}	2.644×10^{-6}
0.4	1.60935×10^{-7}	3.24496×10^{-6}	3.862×10^{-6}
0.5	2.25077×10^{-7}	5.80895×10^{-6}	3.466×10^{-6}
0.6	2.57137×10^{-7}	6.17752×10^{-6}	2.012×10^{-6}
0.7	2.40314×10^{-7}	3.18557×10^{-6}	4.571×10^{-7}
0.8	1.72743×10^{-7}	2.29409×10^{-6}	3.391×10^{-7}
0.9	7.54001×10^{-8}	6.17811×10^{-6}	9.767×10^{-8}

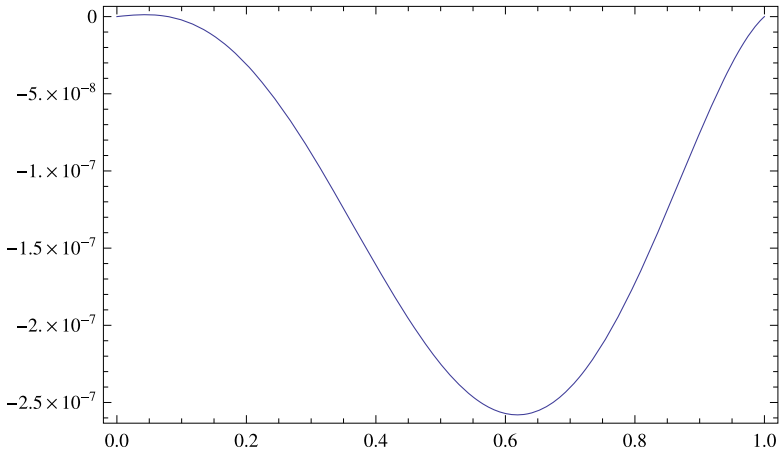
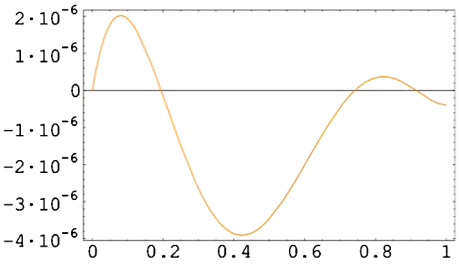


Figure 9 The absolute error at $t = 1$ for $\alpha = 1.8$, $\beta = 1$ and $n = 5$ in the interval $0 < x < 1$ for Example 9.3 by the collocation method.

Figure 10 The absolute error at $t = 1$ for $\alpha = 1.8$, $\beta = 1$ and $n = 5$ in the interval $0 < x < 1$ for Example 9.3 by the method of [26].



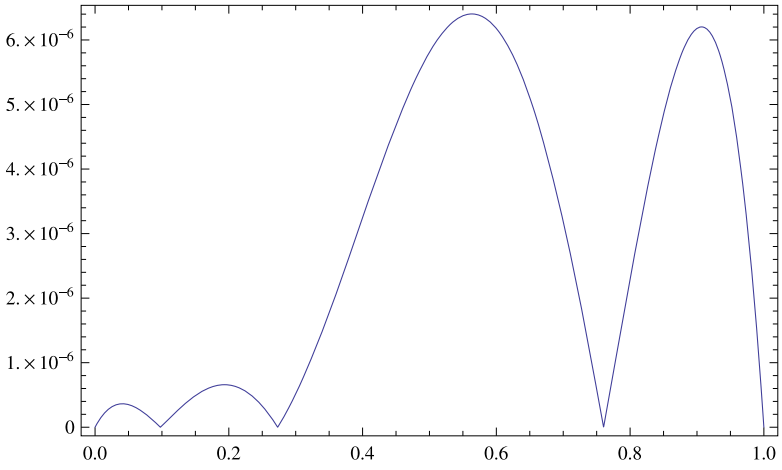


Figure 11 The absolute error at $t = 1$ for $\alpha = 1.8$, $\beta = 1$ and $n = 5$ in the interval $0 < x < 1$ for Example 9.3 by the Galerkin method.

Table 11 The L_∞ error for Example 9.4, the collocation method

n	$T = 2$	$T = 10$	$T = 100$	CPU time (s)	R_∞
5	4.0×10^{-3}	3.0×10^{-3}	2.0×10^{-7}	15.257	7.35512×10^{-3}
8	7.0×10^{-4}	9.0×10^{-3}	7.0×10^{-8}	41.432	1.26371×10^{-6}
10	1.0×10^{-3}	3.4×10^{-4}	4.0×10^{-9}	58.343	2.17796×10^{-9}
16	1.0×10^{-4}	9.5×10^{-4}	8.0×10^{-10}	217.746	1.30039×10^{-16}

Table 12 The absolute error at $t = 1$ and $n = 8$ for Example 9.4

x	Galerkin method	Collocation method
0.1	2.42738×10^{-11}	1.60766×10^{-12}
0.3	1.20381×10^{-11}	1.79642×10^{-11}
0.5	1.69176×10^{-11}	3.06019×10^{-11}
0.7	6.62043×10^{-12}	3.04411×10^{-11}
0.9	3.01258×10^{-11}	1.63679×10^{-11}

The maximum errors L_∞ for different values of T and n are listed in Table 11 for the collocation method, and the results in $x \in (0,1)$ for $n = 8$ are presented in Table 12 at $t = 1$ for both methods.

10 Conclusion

In this article, we presented effective numerical methods for solving a space-time fractional diffusion equation with initial boundary conditions. For these problems defined in the unbounded time domain, we use the rational normalized Bernstein functions as basis functions to approximate the exact solution. We compared the execution of the collocation and Galerkin methods using normalized Bernstein basis for solving a given problem. We have presented some numerical experiments to confirm the theoretical analysis. Precision increases with the increase in the number of terms in the normalized Bernstein expansion. However, for the same number of terms, the collocation method yields relatively more accurate results in a comparatively shorter time compared with the Galerkin method. On the other hand, the collocation method is very sensitive to the collocation points. Generally, the most significant property of the collocation method is its fluency in the application; e.g., matrix elements of the given equation are evaluated directly rather than by numerical integration as in the Galerkin procedure. Generally, the results show that the proposed methods achieve better approximation accuracy than other methods, especially for the long time domain.

Acknowledgements
The authors would like to thank the referee for his valuable comments and suggestions which improved the paper into its present form.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Effective numerical methods for solving space-time fractional diffusion equation with initial boundary conditions are proposed. The rational normalized Bernstein functions as basis functions to approximate the exact solution are used in the unbounded time domain. Some numerical experiments to confirm the theoretical analysis are provided. In our examples we found that the collocation method yields relatively more accurate results in a comparatively shorter time compared with the Galerkin method; on the other hand, the collocation method is very sensitive to the collocation points. All authors read and approved the final manuscript.

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Received: 26 June 2017 Accepted: 15 October 2017 Published online: 25 October 2017

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