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A study on time-delay rumor propagation model with saturated control function

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Abstract

In this paper, a time-delay rumor propagation model with a saturated control function is established. Regarding time delay as a bifurcation parameter, Hopf bifurcation is studied. By means of the normal form and the center manifold theorem, a formula is put forward to determine both Hopf bifurcation direction and bifurcating periodic solution stability, together with some numerical simulations to illustrate the relevant theoretical results. Simulation results indicate that an appropriate government control could make periodic oscillation behaviors of the system become steady so as to improve the balance of a social system.

MSC: 34C23; 34D23

Keywords: rumor; spread; delay; Hopf bifurcation

1 Introduction

After an emergency takes place, the government sector serves as the subject of emergency treatment. As far as the government is concerned, the associated contingency plan should be initiated promptly, including emergency supplies distribution, rescue and information release, etc. During emergency processing, a variety of rumors may be spread along with it to affect government image, course of emergency processing and official contingency strategies. The ultimate aim for studies on rumor propagation law and influencing factors is to effectively control and prevent rumors, guide public opinions, and bring down damages caused by rumor propagation. Research on rumor propagation control strategy is exhibited qualitatively in most cases. The exploration into controls over rumor propagation by qualification has become a hot spot in recent years [1–10]. As for the corresponding theoretical accomplishments, most of them concentrate on rumor propagation regulation by the government that takes advantage of media, etc. to carry out external environment intervention strategy and individual immunity method.

Through media, etc., the government makes use of external environment intervention strategy to control rumor propagation, and such a method has attracted increasingly more attention from scholars and managers. Study on rumor propagation control is required to sufficiently seize rumor propagation laws, psychological features of receivers, and interventions from the external environment, etc., so as to effectively deal with relevant problems. In recent years, scholars have achieved certain research accomplishments from different perspectives and focuses. Reinforcement of media coverage and timely information release by the government is an effective widely recognized measure for rumor propagation control at present. According to Huo et al. [11], factors such as media coverage and governmental information transparency during propagation and diffusion are taken into account to extend the D-K rumor propagation model; moreover, they also utilize the optimal control theory to discuss an optimal control strategy for rumor propagation. Recently, Zhang and Huang [12] have presented an 8-state ICSAR rumor propagation model with government regulation and control. In their opinion, both high and low propagation rates of rumors are able to lead to repeated propagations.

As rumor propagation is similar to infectious disease transmission, the individual immunity method for rumor propagation control is derived from the method of immunization against infectious diseases. Many scholars have carried out relevant studies on immunization strategies of infectious diseases, and rather plentiful research accomplishments have also been obtained [13–16]. Without any doubt, theoretical guidance and methods for references are provided for effective controls over rumor propagation by means of individual immunity. In the process of control over infectious diseases, both treatment and immunization play critical roles. In order to prevent and control transmissions of infectious diseases such as measles, tuberculosis and flu, treatment is an important and valid approach. As for the classic epidemic model, the cure rate of the infected is deemed to be in direct proportion to the number of such infected people [17]. Therefore, dynamic properties according to which application of treatment affects these diseases should be studied. The establishment of an appropriate cure function is a problem rather concerned about by scholars at the time of performing theoretical analysis on epidemicity of diseases. In literature [18–25], they adopt diverse cure functions to probe into controls over epidemic diseases and set up the corresponding epidemic models provided that different medical resources are limited.

Among studies on rumor propagation, regardless of external intervention strategies that are adopted to control it, the control capability of government cannot be enhanced unlimitedly as far as the finiteness of various resources is taken into consideration. Likewise, beneficial from research accomplishments of infectious disease studies, we can use the study on a saturated cure rate of infectious diseases with treatment capacity constraints for references.

Enlightened by research accomplishments above, in this paper, a rumor propagation model that considers government control limits is constructed, and the relevant dynamic properties are also analyzed.

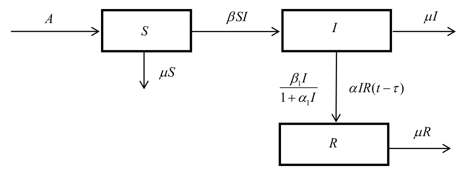
The structure of this paper is arranged as follows. In Section 2, the model is constructed. In Section 3, I study the local stability and the existence of Hopf bifurcation through the study of associated characteristic equations. In Section 4, I study the direction and stability of Hopf bifurcation. In Section 5, some numerical simulations are given to support our theoretical predictions. Finally, this paper ends with a brief conclusion.

2 The model

This section describes a delayed rumor propagation model. Our goal is trying to create a realistic model which can provide wide insight into predicting and controlling rumor prevalence.

Based on the classical SIR epidemic model, in this work, the people can be divided into three classes depending on their different states: ignorants (those not aware of the rumor),

Figure 1 Node state transition relationship.



spreaders (those who are spreading it), and stiflers (those who know the rumor but have ceased communicating it after meeting somebody already informed). For simplicity, we use $S(t)$, $I(t)$ and R to represent the densities of ignorant users, spreading users and stifle users, respectively. To model the propagation of rumor, the following assumptions are imposed:

(i) We consider the recruitment rate of the ignorants is a constant.

(ii) When an ignorant user is infected by spreading users, there is a spreading incubation period during which the infectious agents develop on networks, and it is only after that time that the infected user becomes himself infectious. Therefore, defining a delay for the spreading incubation period is more appropriate.

(iii) Usually, when a rumor is spreading, the government will take effective actions to control and remove the spreading users.

Our assumption on the dynamical transfer of the nodes is depicted in Figure 1. As a result, our model can be represented as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta SI - \mu S, \\ \frac{dI}{dt} = \beta SI - \alpha IR(t-\tau) - \mu I - \frac{\beta_1 I}{1+\alpha_1 I}, \\ \frac{dR}{dt} = \alpha IR(t-\tau) + \frac{\beta_1 I}{1+\alpha_1 I} - \mu R, \end{cases} \tag{1}$$

where S is ignorant, I is spreader, and R is stifler. Λ is the recruitment rate of the ignorants, β is the contact rate of ignorant and spreader, μ is the death rate of nodes, α is the contact rate of spreader and stifler, τ is a non-negative constant representing the spreading incubation period. $\frac{\beta_1 I}{1+\alpha_1 I}$ is a government control function which tends to a saturation level when I gets large.

In the following, we find all possible non-negative equilibria. Clearly, the system has two feasible non-negative equilibria, namely,

(1) The boundary equilibrium $E_1(\frac{\Lambda}{\mu}, 0, 0)$ representing the state corresponding to the extinction of spreaders and stiflers;

(2) The interior equilibrium $E^*(S^*, I^*, R^*)$.

At the interior equilibrium point, we must have

$$\begin{cases} \Lambda - \beta SI - \mu S = 0, \\ \beta SI - \alpha IR - \mu I - \frac{\beta_1 I}{1+\alpha_1 I} = 0, \\ \alpha IR + \frac{\beta_1 I}{1+\alpha_1 I} - \mu R = 0. \end{cases} \tag{2}$$

Solving the first and the third equation of (2), we have $S = \frac{\Lambda}{\beta I+\mu}$ and $R = \frac{\beta_1 I}{(1+\alpha_1 I)(\mu-\alpha I)}$. Substituting S and R into the second equation of (2), we have

$$A_1 I^3 + A_2 I^2 + A_3 I + A_4 = 0, \tag{3}$$

where

$$\begin{aligned} A_1 &= \alpha\alpha_1\beta\mu, \\ A_2 &= \alpha\beta\mu + \alpha\alpha_1\mu^2 - \alpha_1\beta\mu^2 - \Lambda\alpha\alpha_1\beta, \\ A_3 &= \alpha\mu^2 - \alpha_1\mu^3 - \beta\mu^2 - \Lambda\alpha\beta - \beta_1\beta\mu + \Lambda\alpha_1\beta\mu, \\ A_4 &= \Lambda\beta\mu - \mu^3 - \beta_2\mu^2. \end{aligned} \tag{4}$$

We make the following assumption:

(H_1) $\Lambda\beta - \mu^2 - \beta_2\mu < 0.$

The following results are obvious.

Lemma 2.1 *If* (H_1) *holds, then system* (1) *has at least one positive equilibrium* $E^*(S^*, I^*, R^*)$, *where* $S^* = \frac{\Lambda}{\beta I^* + \mu}$ *and* $R^* = \frac{\beta_1 I^*}{(1+\alpha_1 I^*)(\mu - \alpha I^*)}$.

3 Local stability and Hopf bifurcation

In this section, we discuss the local stability and Hopf bifurcation of system (1) by analyzing the corresponding characteristic equations.

Theorem 3.1 *If* $\beta S^* - \mu + \beta_1 < 0$ *holds, then the equilibrium* E_1 *is locally asymptotically stable.*

Proof It is easily obtained that the characteristic equation corresponding to the equilibrium E_1 is as follows:

$$(\lambda + \mu)^2(\lambda - \beta S^* + \mu - \beta_1) = 0. \tag{5}$$

So, we obtain $\lambda_1 = -\mu < 0$ and $\lambda_2 = \beta S^* - \mu + \beta_1$. Therefore, if (H_2) holds, then $\lambda_2 < 0$. It means that the equilibrium E_1 is locally asymptotically stable. □

In the following, we consider the stability of the positive equilibrium E^*. At the positive equilibrium E^*, the corresponding characteristic equation is

$$D(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + (p_4\lambda^2 + p_5\lambda + p_6)e^{-\lambda\tau}, \tag{6}$$

where

$$\begin{aligned} p_1 &= 2\mu + \beta I^* - \alpha_3, \\ p_2 &= -\alpha_3(\mu + \beta I^*) + \mu(\mu + \beta I^* - \alpha_3) + I\beta^2 S^*, \\ p_3 &= \beta^2 S^* I^* \mu - (\mu + \beta I^*)\mu\alpha_3, \\ p_4 &= -\alpha I^*, \\ p_5 &= \alpha^2 I^* R^* + \alpha_2\alpha I^* - (\mu + \beta I^*)\alpha I^* + \alpha I^*\alpha_3, \\ p_6 &= (\mu + \beta I^*)(\alpha^2 I^* R^* + \alpha_2\alpha I^* + I^*\alpha\alpha_3) - \beta^2 S^* I^{*2}\alpha, \\ \alpha_2 &= \frac{\beta_1}{(1 + \alpha_1 I^*)^2}, \qquad \alpha_3 = \frac{\beta_1\alpha_1 I^*}{(1 + \alpha_1 I^*)^2}. \end{aligned} \tag{7}$$

When $\tau = 0$, Eq. (6) becomes

$$\lambda^3 + (p_1 + p_4)\lambda^2 + (p_2 + p_5)\lambda + p_3 + p_6 = 0. \tag{8}$$

By the Routh-Hurwitz criteria, we have the following results.

Lemma 3.1 *If $p_1 + p_4 > 0, p_3 + p_6 > 0, (p_1 + p_4)(p_2 + p_5) - p_3 + p_6 > 0$ hold, then the positive equilibrium of system* (1) *is locally asymptotically stable with $\tau = 0$.*

Now the effect of the delay on the stability of the positive equilibrium of system (1) will be discussed. Providing that there is a root of Eq. (6), it should satisfy the following equation:

$$\begin{cases} -\omega^3 + p_2\omega = (-p_4\omega^2 + p_6)\sin\omega\tau - p_5\omega\cos\omega\tau, \\ \omega(p_1 + p_2 + p_4) = -p_5\omega\sin\omega\tau - (-p_4\omega^2 + p_6)\cos\omega\tau. \end{cases} \tag{9}$$

From Eq. (9), adding the squared terms for both equations yields

$$\omega^6 + ({p_1}^2 - 2p_2 - p_4^2)\omega^4 + (p_2^2 - 2p_1p_3 + 2p_4p_6 - p_5^2)\omega^2 + p_3^2 - p_6^2 = 0. \tag{10}$$

Let $z = \omega^2$, then Eq. (10) becomes

$$z^3 + ({p_1}^2 - 2p_2 - p_4^2)z^2 + (p_2^2 - 2p_1p_3 + 2p_4p_6 - p_5^2)z + p_3^2 - p_6^2 = 0. \tag{11}$$

Denote

$$h(z) = z^3 + ({p_1}^2 - 2p_2 - p_4^2)z^2 + (p_2^2 - 2p_1p_3 + 2p_4p_6 - p_5^2)z + p_3^2 - p_6^2 = 0. \tag{12}$$

Lemma 3.2 *If the following conditions*

$$\alpha^2 I^* R^* + \beta S^*\mu - \alpha\beta S^* I^* > 0, \qquad \beta S^*\mu + \alpha\beta S^* I^* - \mu\alpha_1 - \alpha^2 I^* R^* - \beta\alpha_1\alpha_1 < 0 \tag{13}$$

hold, then Eq. (11) *has at least a positive root.*

Proof From (7), we have

$$\begin{aligned} p_3 + p_6 &= \beta^2 S^* I^*\mu - (\mu + \beta I^*)\mu\alpha_1 + (\mu + \beta I^*)\alpha^2 I^* R^* \\ &\quad + (\mu + \beta I^*)\alpha_1\alpha I^* - \beta^2 S^* {I^*}^2\alpha, \\ p_3 - p_6 &= \beta^2 S^* I^*\mu - (\mu + \beta I^*)\mu\alpha_1 - (\mu + \beta I^*)\alpha^2 I^* R^* \\ &\quad - (\mu + \beta I^*)\alpha_1\alpha I^* + \beta^2 S^* {I^*}^2\alpha. \end{aligned} \tag{14}$$

If conditions (13) hold, then $p_3 + p_6 > 0$ and $p_3 - p_6 < 0$. Obviously, $\lim_{z\to\infty} h(z) = +\infty$. Hence, there is at least $z \in (0, \infty)$, so that $h(z) = 0$. That is to say, Eq. (11) has at least a positive root. $\square$

According to Lemma 3.2, Eq. (11) has a unique positive root, denoted by z_0, and thus Eq. (10) has a unique positive root $\omega_0 = \sqrt{z_0}$. By (9), we have

$$\cos(\omega_0\tau) = \frac{(-p_4\omega^2 + p_6)(p_1 + p_2 + p_4)\omega + p_5\omega(-\omega^2 + p_2\omega)}{(p_4\omega^2 - p_6)^2 + \omega^2 p_5^2}. \tag{15}$$

Thus, if we denote

$$\begin{aligned}\tau_j = {} & \frac{1}{\omega}\left(\arccos\frac{(-p_4\omega^2 + p_6)(p_1 + p_2 + p_4)\omega + p_5\omega(-\omega^2 + p_2\omega)}{(p_4\omega^2 - p_6)^2 + \omega^2 p_5^2} + 2j\pi\right), \\ & j = 0, 1, 2, \ldots, \end{aligned} \tag{16}$$

then $\pm i\omega_0$ is a pair of purely imaginary roots of (10) with $\tau = \tau^j$. Clearly, the sequence $\{\tau^j\}_{j=0}^{\infty}$ is increasing and

$$\lim_{j\to+\infty} \tau^j = +\infty. \tag{17}$$

Thus, we can define

$$\tau_0 = \tau^0 = \min\{\tau^j\}. \tag{18}$$

Lemma 3.3 *Let* $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ *be the root of* (6) *near* $\tau = \tau_0^j$ *satisfying* $\alpha(\tau_0^j) = 0$, $\omega(\tau_0^j) = \omega_0$. *Suppose that, where is defined by* (12), *the following transversality condition holds*:

$$\left.\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}\right|_{\tau=\tau_0^j} \neq 0, \tag{19}$$

and the sign of $\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}|_{\tau=\tau_0^j}$ *is consistent with that of* $h'(z_0)$.

Proof Denote

$$R(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3, \tag{20}$$

$$Q(\lambda) = p_4\lambda^2 + p_5\lambda + p_6. \tag{21}$$

Then Eq. (6) can be written as

$$R(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{22}$$

and (10) can be transformed into the following form:

$$R(i\omega)\bar{R}(i\omega) + Q(i\omega)\bar{Q}(i\omega) = 0. \tag{23}$$

Thus, together with (11) and (12), we have

$$h(\omega^2) = R(i\omega)\bar{R}(i\omega) - Q(i\omega)\bar{Q}(i\omega). \tag{24}$$

Differentiating both sides of Eq. (24) with respect to ω, we obtain

$$2\omega h'(\omega^2) = i\big[R'(i\omega)\bar{R}(i\omega) - R(i\omega)\bar{R}'(i\omega) - Q'(i\omega)\bar{Q}(i\omega) + Q(i\omega)\bar{Q}'(i\omega)\big]. \tag{25}$$

If $i\omega_0$ is not simple, then ω_0 must satisfy

$$\frac{d}{d\lambda}\big[R(\lambda) + Q(\lambda)e^{-2\lambda\tau_0}\big]\bigg|_{\lambda=i\omega_0} = 0, \tag{26}$$

that is, ω_0 must satisfy

$$R'(i\omega) + Q'(i\omega)e^{-i\omega_0\tau_0} - \tau_0 Q(i\omega_0)e^{-i\omega_0\tau_0} = 0. \tag{27}$$

With Eq. (22), we have

$$\tau_0 = \frac{Q'(i\omega_0)}{Q(i\omega_0)} - \frac{R'(i\omega_0)}{R(i\omega_0)}.$$

Thus, by (23) and (24) we obtain

$$\begin{aligned}
\mathrm{Im}(\tau_0) &= \mathrm{Im}\biggl(\frac{Q'(i\omega_0)}{Q(i\omega_0)} - \frac{R'(i\omega_0)}{R(i\omega_0)}\biggr) = \mathrm{Im}\biggl(\frac{Q'(i\omega_0)\bar{Q}(i\omega_0)}{Q(i\omega_0)\bar{Q}(i\omega_0)} - \frac{R'(i\omega_0)\bar{R}(i\omega_0)}{R(i\omega_0)\bar{R}(i\omega_0)}\biggr) \\
&= \mathrm{Im}\biggl(\frac{Q'(i\omega_0)\bar{Q}(i\omega_0) - R'(i\omega_0)\bar{R}(i\omega_0)}{R(i\omega_0)\bar{R}(i\omega_0)}\biggr) \\
&= \frac{-i[Q'(i\omega_0)\bar{Q}(i\omega_0) - R'(i\omega_0)\bar{R}(i\omega_0) - \bar{Q}'(i\omega_0)Q(i\omega_0) + \bar{R}'(i\omega_0)R(i\omega_0)]}{2R(i\omega_0)\bar{R}(i\omega_0)} \\
&= \frac{\omega_0 h'(\omega_0^2)}{|R(i\omega_0)|^2}.
\end{aligned}$$

Since τ_0 is real, i.e., $\mathrm{Im}(\tau_0) = 0$, we have $h'(\omega_0^2) = 0$.

We get a contradiction to the condition $h'(\omega_0^2) \neq 0$. This proves the first conclusion. Differentiating both sides of Eq. (22) with respect to τ, we obtain

$$\big[R'(\lambda) + Q'(\lambda)e^{-\lambda\tau} - \tau Q(\lambda)e^{-\lambda\tau}\big]\frac{d\lambda}{d\tau} - \lambda Q(\lambda)e^{-\lambda\tau} = 0, \tag{28}$$

which implies

$$\begin{aligned}
\frac{d\lambda}{d\tau} &= \frac{\lambda Q(\lambda)}{R'(\lambda)e^{2\lambda\tau} + Q'(\lambda) - \tau Q(\lambda)} = \frac{\lambda Q(\lambda)[\bar{R}'(\lambda)e^{\lambda\tau} + \bar{Q}'(\lambda) - \tau\bar{Q}(\lambda)]}{|R'(\lambda)e^{\lambda\tau} + Q'(\lambda) - \tau Q(\lambda)|^2} \\
&= \frac{\lambda[-R(\lambda)\bar{R}'(\lambda)e^{\lambda\tau} + Q(\lambda)\bar{Q}'(\lambda) - \tau|Q(\lambda)|^2]}{|R'(\lambda)e^{\lambda\tau} + Q'(\lambda) - \tau Q(\lambda)|^2}.
\end{aligned}$$

It follows together with (25) that

$$\begin{aligned}
&\frac{d(\mathrm{Re}\,\lambda(\tau))}{d\tau}\bigg|_{\tau=\tau_0,\lambda=i\omega_0} \\
&\quad= \frac{\mathrm{Re}\{2\lambda[-R(\lambda)\bar{R}'(\lambda)e^{2\lambda\tau} + Q(\lambda)\bar{Q}'(\lambda) - 2\tau|Q(\lambda)|^2]\}_{\tau=\tau_0,\lambda=i\omega_0}}{|R'(\lambda)e^{2\lambda\tau} + Q'(\lambda) - 2\tau Q(\lambda)|^2}
\end{aligned}$$

$$= \frac{i\omega_n[-R(i\omega_n)\bar{R}'(i\omega_n) + Q(i\omega_n)\bar{Q}'(i\omega_n) + R'(i\omega_n)\bar{R}(i\omega_n) - Q'(i\omega_n)\bar{Q}(i\omega_n)]}{|R'(\lambda)e^{2\lambda\tau} + Q'(\lambda) - \tau Q(\lambda)|^2}$$
$$= \frac{\omega_0^2 h'(\omega_0^2)}{|R'(\lambda)e^{2\lambda\tau} + Q'(\lambda) - 2\tau Q(\lambda)|^2} = \frac{\omega_0^2 h'(z_0)}{|R'(\lambda)e^{2\lambda\tau} + Q'(\lambda) - 2\tau Q(\lambda)|^2} \neq 0.$$

Clearly, the sign of $\frac{d(\operatorname{Re}\lambda(\tau))}{d\tau}|_{\tau=\tau_0}$ is determined by that of $h'(z_0)$. □

From the above analysis, we have the following theorem.

Theorem 3.2 *From Lemmas* 3.1-3.3, *the following statements are true*:

(i) *When* $\tau \in [0, \tau_0)$, *the positive equilibrium point of* (1) *is asymptotically stable*;

(ii) *The Hopf bifurcation occurs at* $\tau = \tau_0$. *That is, system* (1) *has a branch of periodic solutions bifurcating from the positive equilibrium near* $\tau = \tau_0$.

4 Direction and stability of Hopf bifurcation

In this section, we derive explicit formulae to determine the properties of the Hopf bifurcation at critical value τ^j by using the normal form theory and the center manifold reduction developed by [26].

First, we let

$$\begin{aligned} &f^{(1)} = \Lambda - \beta SI - \mu S, \qquad f^{(2)} = \beta SI - \alpha IR(t-\tau) - \mu I - \frac{\beta_1 I}{1+\alpha_1 I}, \\ &f^{(3)} = \alpha IR(t-\tau) + \frac{\beta_1 I}{1+\alpha_1 I} - \mu R, \\ &f^{(1)}_{ij} = \frac{\partial^{i+j} f^{(1)}}{\partial S^i \partial I^j}, \qquad f^{(2)}_{ijlk} = \frac{\partial^{i+j+l+k} f^{(2)}}{\partial S^i \partial I^j \partial R^l \partial R^k}, \qquad f^{(3)}_{ijl} = \frac{\partial^{i+j+l} f^{(3)}}{\partial I^i \partial R^j \partial R^l}. \end{aligned} \tag{29}$$

Denote τ^j by τ^* and introduce the new parameter $\mu = \tau - \tau^*$. Normalize the delay τ by the time-scaling $t \to t/\tau$. Denote

$$U(t) = \bigl(S(t), I(t), R(t)\bigr)^T,$$

then system (1) can be rewritten as an abstract differential equation in the phase space $\mathcal{C} = C([-\tau, 0], \mathcal{R}^n)$ of the form

$$\frac{dU(t)}{dt} = L\bigl(\tau^*\bigr)(U_t) + F(U_t, \mu), \tag{30}$$

where

$$U_t(\theta) = U(t+\theta), \quad -\tau \le \theta \le 0,$$
$$L(\gamma)(\phi) = \mu \begin{pmatrix} -(\beta I^* + \mu)\phi_1(0) - \beta S^* \phi_2(0) \\ \beta I^* \phi_1(0) + \alpha_2 \phi_2(0) - \alpha I^* \phi_3(-\tau) \\ (\alpha R^* + \alpha_3)\phi_2(0) - \mu \phi_3(0) + \alpha I^* \phi_3(-\tau) \end{pmatrix}, \tag{31}$$

$$F(\phi, \gamma) = L(\gamma)\phi + q(\phi, \gamma) \tag{32}$$

and

$$q(\varphi,\mu)=\left(\tau^*+\mu\right)\begin{pmatrix}\sum_{i+j=2}\frac{1}{i!j!}f_{ij}^{(1)}\varphi_1^i(0)\varphi_3^j(0)\\ \sum_{i+j+l+k=2}\frac{1}{i!j!l!k!}f_{ijlk}^{(2)}\varphi_1^i(0)\varphi_2^j(0)\varphi_3^l(0)\varphi_3^k(-1)\\ \sum_{i+j+l=2}\frac{1}{i!j!l!}f_{ijl}^{(3)}\varphi_2^i(0)\varphi_3^j(0)\varphi_3^l(-1)\end{pmatrix}+h.o.t., \tag{33}$$

for $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T\in\mathcal{C}$.

Then the linearized system of (30) at the positive equilibrium is

$$\frac{dU(t)}{dt}=L\left(\tau^*\right)(U_t). \tag{34}$$

Based on the discussion in Section 3, we can easily know that for $\tau=\tau^*$, the characteristic equation of (6) has a pair of simple purely imaginary eigenvalues $\Lambda_0=\{i\omega_0\tau^*,-i\omega_0\tau^*\}$.

Let $\mathcal{C}:=C([-1,0],\mathbb{R}^3)$, consider the following FDE on $\mathcal{C}$:

$$\dot{z}=L\left(\tau^*\right)(z_t). \tag{35}$$

Obviously, $L(\tau^*)$ is a continuous linear function mapping $C([-1,0],\mathbb{R}^3)$ into $\mathbb{R}^3$. By the Riesz representation theorem, there exists a 3×3 matrix function η $(-1\le\theta\le 0)$, whose elements are of bounded variation such that

$$L\left(\tau^*\right)(\varphi)=\int_{-1}^{0}\left[d\eta\left(\theta,\tau^*\right)\right]\varphi(\theta)\quad\text{for }\varphi\in C. \tag{36}$$

In fact, we can choose

$$\begin{aligned}\eta\left(\theta,\tau^*\right)={}&\tau^*\begin{pmatrix}-(\beta I^*+\mu) & -\beta S^* & 0\\ \beta I^* & \alpha_2 & 0\\ 0 & \alpha R^*+\alpha_3 & -\mu\end{pmatrix}\delta(\theta)\\ &-\tau^*\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\alpha I^*\\ 0 & 0 & \alpha I^*\end{pmatrix}\delta(\theta+1),\end{aligned} \tag{37}$$

where δ is the Dirac delta function.

Let $A(\tau^*)$ denote the infinitesimal generator of the semigroup induced by the solutions of (35), and let A^* be the formal adjoint of $A(\tau^*)$ under the bilinear pairing

$$\begin{aligned}(\psi,\phi)&=\left(\psi(0),\phi(0)\right)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi\\ &=\left(\psi(0),\phi(0)\right)+\tau^*\int_{-1}^{0}\psi(\theta+1)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\alpha I^*\\ 0 & 0 & \alpha I^*\end{pmatrix}\phi(\theta)\,d\theta\end{aligned} \tag{38}$$

for $\phi\in C,\psi\in C^*=C([0,1],R^3)$. Then $A(\tau^*)$ and A^* are a pair of adjoint operators. From the discussion in Section 3, we know that $A(\tau^*)$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_0\tau^*$, and they are also eigenvalues of A^* since $A(\tau^*)$ and A^* are a pair of adjoint operators. Let P and P^* be the center spaces, that is, the generalized eigenspaces of $A(\tau^*)$ and A^* associated with Λ_0, respectively. Then P^* is the adjoint space of P and $\dim P = \dim P^* = 2$. Direct computations give the following results.

Lemma 4.1 *Let*

$$\begin{cases} \sigma_1 = -\frac{i\omega+\beta I^*+\mu}{\beta S^*}, & \sigma_2 = \frac{i\omega(i\omega+\beta I^*+\mu)+\beta^2 S^* I^*}{\beta S^*(\alpha I^* e^{-i\omega\tau^*}-\alpha_2)}, \\ \sigma_1^* = \frac{\beta I^*+\mu+i\omega}{\beta I^*}, & \sigma_2^* = \frac{(i\omega-\alpha_2)(i\omega+\beta I^*+\mu)+\beta S^*}{\alpha R^*+\alpha_3}. \end{cases} \tag{39}$$

Then

$$p_1(\theta) = e^{i\omega_0\tau^*\theta}(1,\sigma_1,\sigma_2)^T, \qquad p_2(\theta) = \overline{p_1(\theta)}, \quad -1 \le \theta \le 0, \tag{40}$$

is a basis of P associated with Λ_0 and

$$q_1(s) = \left(1,\sigma_1^*,\sigma_2^*\right)e^{-i\omega_0\tau^* s}, \qquad q_2(s) = \overline{q_1(s)}, \quad 0 \le s \le 1, \tag{41}$$

is a basis of Q associated with Λ_0.

Let $\Phi = (\Phi_1,\Phi_2)$ and $\Psi^* = (\Psi_1^*,\Psi_2^*)^T$ with

$$\Phi_1(\theta) = \frac{p_1(\theta)+p_2(\theta)}{2} = \begin{pmatrix} \operatorname{Re}\{e^{i\omega_0\tau^*\theta}\} \\ \operatorname{Re}\{\sigma_1 e^{i\omega_0\tau^*\theta}\} \\ \operatorname{Re}\{\sigma_2 e^{i\omega_0\tau^*\theta}\} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega_0\tau^*\theta \\ \frac{\omega}{\beta S^*}\sin\omega\tau^*\theta - \frac{(\beta I^*+\mu)}{\beta S^*}\cos\omega\tau^*\theta \\ \frac{1}{\beta S^*((\alpha I^*\cos(\omega\tau^*)-\alpha_2)^2+\sin^2(\omega\tau^*))}(((-\omega^2+\beta^2 S^* I^*)(\alpha I^*\cos(\omega\tau^*)-\alpha_2) \\ -(\beta I^*+\mu)\omega\sin\omega\tau^*)\cos\omega\tau^*\theta - (\sin\omega\tau^*(-\omega^2+\beta^2 S^* I^*) \\ +\omega(\beta I^*+\mu)(\alpha I^*\cos(\omega\tau^*)-\alpha_2))\sin\omega\tau^*\theta) \end{pmatrix},$$

and

$$\Phi_2(\theta) = \frac{p_1(\theta)-p_2(\theta)}{2i} = \begin{pmatrix} \operatorname{Im}\{e^{i\omega_0\tau^*\theta}\} \\ \operatorname{Im}\{\sigma_1 e^{i\omega_0\tau^*\theta}\} \\ \operatorname{Im}\{\sigma_2 e^{i\omega_0\tau^*\theta}\} \end{pmatrix} = \begin{pmatrix} \sin\omega_0\tau^*\theta \\ -\frac{\beta\sin\omega_0\tau^*\theta+\omega\cos\omega_0\tau^*\theta+\mu\sin\omega_0\tau^*\theta}{\beta S^*} \\ \frac{B_1\sin\omega\tau^*\theta+B_2\cos\omega\tau^*\theta}{\beta S^*((\alpha I^*\cos(\omega\tau^*)-\alpha_2)^2+\sin^2(\omega\tau^*))} \end{pmatrix},$$

for $\theta \in [-1,0]$, where

$$B_1 = \left(-\omega^2+\beta^2 S^* I^*\right)\left(\alpha I^*\cos\left(\omega\tau^*\right)-\alpha_2\right) - \left(\beta I^*+\mu\right)\omega\sin\omega\tau^*,$$

$$B_2 = \sin\omega\tau^*\left(-\omega^2+\beta^2 S^* I^*\right) + \omega\left(\beta I^*+\mu\right)\left(\alpha I^*\cos\left(\omega\tau^*\right)-\alpha_2\right),$$

$$\Psi_1^*(s) = \frac{q_1(s)+q_2(s)}{2} = \begin{pmatrix} \operatorname{Re}\{e^{-i\omega_0\tau^* s}\} \\ \operatorname{Re}\{\sigma_1^* e^{-i\omega_0\tau^* s}\} \\ \operatorname{Re}\{\sigma_2^* e^{-i\omega_0\tau^* s}\} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega_0\tau^* s \\ \frac{(\beta I^*+\mu)\cos\omega_0\tau^* s+\omega\sin\omega_0\tau^* s}{\beta I^*} \\ \frac{(\beta^2 S^* I^*-\omega^2-\alpha_2(\beta I^*+\mu))\cos\omega_0\tau^* s+\omega_0(\beta I^*+\mu-\alpha_2)\sin\omega_0\tau^* s}{\beta I^*(\alpha R^*+\alpha_3)} \end{pmatrix},$$

$$\Psi_2^*(s) = \frac{q_1(s) - q_2(s)}{2i} = \begin{pmatrix} \mathrm{Im}\{e^{-i\omega_0\tau^*s}\} \\ \mathrm{Im}\{\sigma_1^* e^{-i\omega_0\tau^*s}\} \\ \mathrm{Im}\{\sigma_2^* e^{-i\omega_0\tau^*s}\} \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\omega_0\tau^*s \\ \frac{\omega\cos\omega_0\tau^*s-(\beta I^*+\mu)\sin\omega_0\tau^*s}{\beta I^*} \\ \frac{-(\beta^2 S^* I^*-\omega^2-\alpha_2(\beta I^*+\mu))\sin\omega_0\tau^*s+\omega_0(\beta I^*+\mu-\alpha_2)\cos\omega_0\tau^*s}{\beta I^*(\alpha R^*+\alpha_3)} \end{pmatrix}$$

for $s \in [0,1]$. From (38), we can obtain (Ψ_1^*, Φ_1) and (Ψ_1^*, Φ_2). Note that

$$(q_1, p_1) = (\Psi_1^*, \Phi_1) - (\Psi_2^*, \Phi_2) + i\big[(\Psi_1^*, \Phi_2) + (\Psi_2^*, \Phi_1)\big]$$

and

$$(q_1, p_1) = 1 + \sigma_1\sigma_1{}^* + \sigma_2\sigma_2^* - \tau^*\alpha I^*\big(\sigma_2\sigma_2^* - \sigma_2\sigma_1^*\big)e^{-i\omega_0\tau^*} := D^*.$$

Therefore, we have

$$\begin{aligned} &(\Psi_1^*, \Phi_1) - (\Psi_2^*, \Phi_2) = \mathrm{Re}\{D^*\}, \\ &(\Psi_1^*, \Phi_2) + (\Psi_2^*, \Phi_1) = \mathrm{Im}\{D^*\}. \end{aligned}$$

Now, we define $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k)$ $(j,k = 1,2)$ and construct a new basis ψ for Q by

$$\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1}\Psi^*.$$

Obviously, $(\Psi, \Phi) = I_{2\times 2}$, the second order identity matrix. In addition, define $f_0 = (\xi_0^1, \xi_0^2, \xi_0^3)$, where

$$\xi_0^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_0^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi_0^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $c \cdot f_0$ be defined by

$$c \cdot f_0 = c_1\xi_0^1 + c_2\xi_0^2 + c_3\xi_0^3$$

for $c = (c_1, c_2, c_3)^T$, $c_j \in R$ $(j = 1,2,3)$.

Then the center space of linear equation (34) is given by $P_{CN}\mathcal{C}$, where

$$P_{CN}\varphi = \Phi\big(\Psi, \langle\varphi, f_0\rangle\big) \cdot f_0, \quad \varphi \in c, \tag{42}$$

and $\mathcal{C} = P_{CN}\mathcal{C} \oplus P_S\mathcal{C}$, here $P_S\mathcal{C}$ denotes the complementary subspace of $P_{CN}\mathcal{C}$ and $\langle\cdot,\cdot\rangle$ is the Euclidean product.

Let A_{τ^*} be defined by

$$A_{\tau^*}\varphi(\theta) = \dot{\varphi}(\theta) + X_0(\theta)\big[L(\tau^*)\big(\varphi(\theta)\big) - \dot{\varphi}(0)\big], \quad \varphi \in BC, \tag{43}$$

where $X_0 : [-1, 0] \to B(X, X)$ is given by

$$X_0(\theta) = \begin{cases} 0, & -1 \leq \theta < 0, \\ I, & \theta = 0. \end{cases} \tag{44}$$

Then A_{τ^*} is the infinitesimal generator induced by the solution of (34), and (30) can be rewritten as the following operator differential equation:

$$\dot{U}_t = A_{\tau^*} U_t + X_0 F(U_t, \mu). \tag{45}$$

Using the decomposition $\mathcal{C} = P_{CN}\mathcal{C} \oplus P_S\mathcal{C}$ and (42), the solution of (30) can be rewritten as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_0 + h(x_1, x_2, \mu), \tag{46}$$

where

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \bigl(\Psi, \langle U_t, f_0\rangle\bigr), \tag{47}$$

and $h(x_1, x_2, \mu) \in P_s c$ with $h(0, 0, 0) = Dh(0, 0, 0) = 0$. In particular, the solution of (30) on the center manifold is given by

$$U_t^* = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_0 + h(x_1, x_2, 0). \tag{48}$$

Setting $z = x_1 - ix_2$ and noticing that $p_1 = \Phi_1 + i\Phi_2$, then (48) can be rewritten as

$$U_t^* = \frac{1}{2}\Phi \begin{pmatrix} z + \bar{z} \\ i(z - \bar{z}) \end{pmatrix} \cdot f_0 + w(z, \bar{z}) = \frac{1}{2}(p_1 z + \bar{p_1}\bar{z}) \cdot f_0 + W(z, \bar{z}), \tag{49}$$

where $W(z, \bar{z}) = h(\frac{z+\bar{z}}{2}, -\frac{z-\bar{z}}{2i}, 0)$. Moreover, by [26], z satisfies

$$\dot{z} = i\omega_0\tau^* z + g(z, \bar{z}), \tag{50}$$

where

$$g(z, \bar{z}) = \bigl(\Psi_1(0) - i\Psi_2(0)\bigr)\bigl\langle F\bigl(U_t^*, 0\bigr), f_0\bigr\rangle. \tag{51}$$

Let

$$W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{52}$$

and

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots . \tag{53}$$

From (49), we have

$$
\begin{aligned}
&\langle F(U_t^*,0),f_0\rangle \\
&= \frac{\tau^* z^2}{4}\begin{pmatrix}
\frac{1}{2}f_{20}^{(1)} + f_{11}^{(1)}\sigma_2 + \frac{1}{2}f_{02}^{(1)}\sigma_2^2 \\
\frac{1}{2}f_{2000}^{(2)} + \frac{1}{2}f_{0200}^{(2)}\sigma_1^2 + \frac{1}{2}f_{0020}^{(2)}\sigma_2^2 + \frac{1}{2}f_{0002}^{(2)}\sigma_2^2 e^{-2i\omega\tau^*} + f_{1100}^{(2)}\sigma_1 + f_{1010}^{(2)}\sigma_2 \\
+ f_{1001}^{(2)}\sigma_2 e^{-i\omega\tau^*} + f_{0110}^{(2)}\sigma_1\sigma_2 + f_{0101}^{(2)}\sigma_1\sigma_2 e^{-i\omega\tau^*} + f_{0011}^{(2)}\sigma_2^2 e^{-i\omega\tau^*} \\
\frac{1}{2}f_{200}^{(3)}\sigma_1^2 + \frac{1}{2}\alpha^2 f_{020}^{(3)}\sigma_2^2 + \frac{1}{2}f_{002}^{(3)}\sigma_2^2 e^{-2i\omega\tau^*} + f_{110}^{(3)}\sigma_1\sigma_2 + f_{101}^{(3)}\sigma_1\sigma_2 e^{-i\omega\tau^*} \\
+ f_{011}^{(3)}\sigma_2^2 e^{-i\omega\tau^*}
\end{pmatrix} \\
&\quad + \frac{\tau^* z\bar{z}}{4}\begin{pmatrix}
f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2 + \bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\
f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1 + \bar{\sigma}_1) + f_{1010}^{(2)}(\sigma_2 \\
+ \bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\
+ f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\
f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\
+ f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*})
\end{pmatrix} \\
&\quad + \frac{\tau^* \bar{z}^2}{4}\begin{pmatrix}
\frac{1}{2}f_{20}^{(1)} + f_{11}^{(1)}\bar{\sigma}_2 e^{i\omega_0\tau^*} + \frac{1}{2}f_{02}^{(1)}\bar{\sigma}_2^2 \\
\frac{1}{2}f_{2000}^{(2)} + \frac{1}{2}f_{0200}^{(2)}\bar{\sigma}_1^2 + \frac{1}{2}f_{0020}^{(2)}\bar{\sigma}_2^2 + \frac{1}{2}f_{0002}^{(2)}\bar{\sigma}_2^2 e^{2i\omega\tau^*} + f_{1100}^{(2)}\sigma_1 + f_{1010}^{(2)}\bar{\sigma}_2 \\
+ f_{1001}^{(2)}\bar{\sigma}_2 e^{i\omega\tau^*} + f_{0110}^{(2)}\bar{\sigma}_1\bar{\sigma}_2 + f_{0101}^{(2)}\bar{\sigma}_1\bar{\sigma}_2 e^{i\omega_0\tau^*} + f_{0011}^{(2)}\bar{\sigma}_2^2 e^{i\omega_0\tau^*} \\
\frac{1}{2}f_{200}^{(3)}\bar{\sigma}_1^2 + \frac{1}{2}f_{020}^{(3)}\bar{\sigma}_2^2 + \frac{1}{4}f_{002}^{(3)}\bar{\sigma}_2^2 e^{2i\omega_0\tau^*} + f_{110}^{(3)}\bar{\sigma}_1\bar{\sigma}_2 + f_{101}^{(3)}\bar{\sigma}_1\bar{\sigma}_2 e^{i\omega_0\tau^*} \\
+ f_{011}^{(3)}\bar{\sigma}_2^2 e^{i\omega_0\tau^*}
\end{pmatrix} \\
&\quad + \frac{\tau^*}{4}\begin{pmatrix}
\langle 2f_{20}^{(1)}(w_{20}^{(1)}(0) + 2w_{11}^{(1)}(0)) + f_{11}^{(1)}(w_{20}^{(3)}(0) + 2w_{11}^{(3)}(0) + 2\sigma_2 w_{11}^{(2)}(0) \\
+ \bar{\sigma}_2 w_{20}^{(1)}(0)) + f_{02}^{(1)}(\bar{\sigma}_2 e^{i\omega_0\tau^*} w_{20}^{(3)}(-1) + 4\sigma_2 e^{-i\omega_0\tau^*} w_{11}^{(3)}(-1)), 1\rangle \\
\langle f_{2000}^{(2)}(2w_{11}^{(1)}(0) + w_{20}^{(1)}(0)) + f_{0200}^{(2)}(w_{20}^{(2)}(0)\bar{\sigma}_1 + 2\sigma_1 w_{11}^{(2)}(0)) \\
+ f_{0020}^{(2)}(w_{20}^{(3)}(0)\bar{\sigma}_2 + 2\sigma_2 w_{11}^{(3)}(0)) + f_{0002}^{(2)}(w_{20}^{(3)}(-1)\bar{\sigma}_2 e^{i\omega_0\tau^*} \\
+ 2\sigma_2 w_{11}^{(3)}(-1)e^{-i\omega_0\tau^*}) + f_{1100}^{(2)}(2w_{11}^{(2)}(0) + w_{20}^{(2)}(0) + 2\sigma_1 w_{11}^{(1)}(0) + \bar{\sigma}_1 w_{20}^{(1)}(0)) \\
+ f_{1010}^{(2)}(w_{20}^{(3)}(0) + 2w_{11}^{(3)}(0) + 2\sigma_2 w_{11}^{(2)}(0) + \bar{\sigma}_2 w_{20}^{(1)}(0)) + f_{1001}^{(2)}(w_{20}^{(3)}(-1) \\
+ 2w_{11}^{(3)}(-1) + \bar{\sigma}_2 e^{i\omega_0\tau^*} w_{20}^{(1)}(0) + 2\sigma_2 e^{-i\omega_0\tau^*} w_{11}^{(1)}(0)) + f_{0110}^{(2)}(2\sigma_1 w_{11}^{(3)}(0) \\
+ \bar{\sigma}_1 w_{20}^{(3)}(0) + 2\sigma_2 w_{11}^{(2)}(0) + \bar{\sigma}_2 w_{20}^{(3)}(0)) + f_{0101}^{(2)}(\bar{\sigma}_1 w_{20}^{(1)}(-1) + 2\sigma_1 w_{11}^{(3)}(-1) \\
+ 2\sigma_2 e^{-i\omega_0\tau^*} w_{11}^{(2)}(0) + \bar{\sigma}_2 e^{i\omega_0\tau^*} w_{20}^{(2)}(0)) + f_{0011}^{(2)}(2\sigma_2 w_{11}^{(3)}(-1) + \bar{\sigma}_2 w_{20}^{(3)}(-1) \\
+ 2\sigma_2 w_{11}^{(3)}(0)e^{-i\omega_0\tau^*} + \bar{\sigma}_2 e^{i\omega_0\tau^*} w_{20}^{(3)}(0)), 1\rangle \\
\langle f_{200}^{(3)}(\bar{\sigma}_1 w_{20}^{(2)}(0) + 2\sigma_1 w_{11}^{(2)}(0)) + f_{020}^{(3)}(\bar{\sigma}_2 w_{20}^{(3)}(0) + 2\sigma_2 w_{11}^{(3)}(0)) \\
+ f_{002}^{(3)}(\bar{\sigma}_2 e^{i\omega_0\tau^*} w_{20}^{(3)}(-1) + 2\sigma_2 e^{-i\omega_0\tau^*} w_{11}^{(3)}(-1)) + f_{110}^{(3)}(2\sigma_1 w_{11}^{(3)}(0) \\
+ \bar{\sigma}_1 w_{20}^{(3)}(0) + 2\sigma_2 w_{11}^{(2)}(0) + \bar{\sigma}_2 w_{20}^{(2)}(0)) + f_{101}^{(3)}(\bar{\sigma}_1 w_{20}^{(3)}(-1) + 2\bar{\sigma}_1 w_{11}^{(3)}(-1) \\
+ 2\sigma_2 e^{-i\omega_0\tau^*} w_{11}^{(2)}(0) + \bar{\sigma}_2 w_{20}^{(2)}(0)e^{i\omega_0\tau^*}) + f_{011}^{(3)}(2\sigma_2 w_{11}^{(3)}(-1) \\
+ \bar{\sigma}_2 w_{20}^{(3)}(-1) + 2\sigma_2 w_{11}^{(3)}(0)e^{-i\omega_0\tau^*} + \bar{\sigma}_2 w_{20}^{(3)}(0)e^{i\omega_0\tau^*}), 1\rangle
\end{pmatrix} \\
&\quad \times z^2\bar{z} + \cdots,
\end{aligned}
$$

where

$$\langle W_{ij}^n(\theta), 1\rangle = \frac{1}{\pi}\int_0^\pi W_{ij}^n(\theta)(x)\,dx, \quad i + j = 2, n \in \mathbb{N}.$$

Let $(\psi_1,\psi_2,\psi_3)=\Psi_1(0)-i\Psi_2(0)$. Then, by (51), (52) and (53), we can obtain the following quantities:

$$\begin{aligned}
g_{20} ={}& \frac{\tau^*}{2}\left\{\left[\frac{1}{2}f_{20}^{(1)}+f_{11}^{(1)}\sigma_2+\frac{1}{2}f_{02}^{(1)}\sigma_2^2\right]\psi_1+\left[\frac{1}{2}f_{2000}^{(2)}+\frac{1}{2}f_{0200}^{(2)}\sigma_1^2+\frac{1}{2}f_{0020}^{(2)}\sigma_2^2\right.\right.\\
&+\frac{1}{2}f_{0002}^{(2)}\sigma_2^2e^{-2i\omega\tau^*}+f_{1100}^{(2)}\sigma_1+f_{1010}^{(2)}\sigma_2+f_{1001}^{(2)}\sigma_2e^{-i\omega\tau^*}+f_{0110}^{(2)}\sigma_1\sigma_2\\
&\left.+f_{0101}^{(2)}\sigma_1\sigma_2e^{-i\omega\tau^*}+f_{0011}^{(2)}\sigma_2^2e^{-i\omega\tau^*}\right]\psi_2+\left[\frac{1}{2}f_{200}^{(3)}\sigma_1^2+\frac{1}{2}\alpha^2f_{020}^{(3)}\sigma_2^2\right.\\
&\left.\left.+\frac{1}{2}f_{002}^{(3)}\sigma_2^2e^{-2i\omega\tau^*}+f_{110}^{(3)}\sigma_1\sigma_2+f_{101}^{(3)}\sigma_1\sigma_2e^{-i\omega\tau^*}+f_{011}^{(3)}\sigma_2^2e^{-i\omega\tau^*}\right]\psi_3\right\},\\
g_{11} ={}& \frac{\tau^*}{4}\left\{\left[f_{20}^{(1)}+f_{11}^{(1)}(\sigma_2+\bar\sigma_2)+f_{02}^{(1)}\sigma_2\bar\sigma_2\right]\psi_1+\left[f_{2000}^{(2)}+f_{0200}^{(2)}\sigma_1\bar\sigma_1+f_{0020}^{(2)}\sigma_2\bar\sigma_2\right.\right.\\
&+f_{0002}^{(2)}\sigma_2\bar\sigma_2+f_{1100}^{(2)}(\sigma_1+\bar\sigma_1)+f_{1010}^{(2)}(\sigma_2+\bar\sigma_2)+f_{1001}^{(2)}\left(\sigma_2e^{-i\omega\tau^*}+\bar\sigma_2e^{i\omega\tau^*}\right)\\
&+f_{0110}^{(2)}(\sigma_1\bar\sigma_2+\bar\sigma_1\sigma_2)+f_{0101}^{(2)}\left(\sigma_1\bar\sigma_2e^{i\omega\tau^*}+\bar\sigma_1\sigma_2e^{-i\omega\tau^*}\right)+f_{0011}^{(2)}\left(\sigma_2\bar\sigma_2e^{i\omega\tau^*}\right.\\
&\left.\left.+\bar\sigma_2\sigma_2e^{-i\omega\tau^*}\right)\right]\psi_2+\left[f_{200}^{(3)}\sigma_1\bar\sigma_1+f_{020}^{(3)}\sigma_2\bar\sigma_2+f_{002}^{(3)}\sigma_2\bar\sigma_2+f_{110}^{(3)}(\sigma_1\bar\sigma_2+\bar\sigma_1\sigma_2)\right.\\
&\left.\left.+f_{101}^{(3)}\left(\sigma_1\bar\sigma_2e^{i\omega\tau^*}+\bar\sigma_1\sigma_2e^{-i\omega\tau^*}\right)+f_{011}^{(3)}\left(\sigma_2\bar\sigma_2e^{i\omega\tau^*}+\sigma_2\bar\sigma_2e^{-i\omega\tau^*}\right)\right]\psi_3\right\},\\
g_{02} ={}& \frac{\tau^*}{2}\left\{\left[\frac{1}{2}f_{20}^{(1)}+f_{11}^{(1)}\bar\sigma_2e^{i\omega_0\tau^*}+\frac{1}{2}f_{02}^{(1)}\bar\sigma_2^2\right]\psi_1+\left[\frac{1}{2}f_{2000}^{(2)}+\frac{1}{2}f_{0200}^{(2)}\bar\sigma_1^2+\frac{1}{2}f_{0020}^{(2)}\bar\sigma_2^2\right.\right.\\
&+\frac{1}{2}f_{0002}^{(2)}\bar\sigma_2^2e^{2i\omega\tau^*}+f_{1100}^{(2)}\sigma_1+f_{1010}^{(2)}\bar\sigma_2+f_{1001}^{(2)}\bar\sigma_2e^{i\omega\tau^*}+f_{0110}^{(2)}\bar\sigma_1\bar\sigma_2+f_{0101}^{(2)}\bar\sigma_1\bar\sigma_2e^{i\omega_0\tau^*}\\
&\left.+f_{0011}^{(2)}\bar\sigma_2^2e^{i\omega_0\tau^*}\right]\psi_2+\left[\frac{1}{2}f_{200}^{(3)}\bar\sigma_1^2+\frac{1}{2}f_{020}^{(3)}\bar\sigma_2^2+\frac{1}{4}f_{002}^{(3)}\bar\sigma_2^2e^{2i\omega_0\tau^*}+f_{110}^{(3)}\bar\sigma_1\bar\sigma_2\right.\\
&\left.\left.+f_{101}^{(3)}\bar\sigma_1\bar\sigma_2e^{i\omega_0\tau^*}+f_{011}^{(3)}\bar\sigma_2^2e^{i\omega_0\tau^*}\right]\psi_3\right\},\\
g_{21} ={}& \frac{\tau^*}{2}\Big[\Big\langle 2f_{20}^{(1)}\left(w_{20}^{(1)}(0)+2w_{11}^{(1)}(0)\right)+f_{11}^{(1)}\left(w_{20}^{(3)}(0)+2w_{11}^{(3)}(0)+2\sigma_2w_{11}^{(2)}(0)+\bar\sigma_2w_{20}^{(1)}(0)\right)\\
&+f_{02}^{(1)}\left(\bar\sigma_2e^{i\omega_0\tau^*}w_{20}^{(3)}(-1)+4\sigma_2e^{-i\omega_0\tau^*}w_{11}^{(3)}(-1)\right),1\Big\rangle\psi_1+\Big\langle f_{2000}^{(2)}\left(2w_{11}^{(1)}(0)+w_{20}^{(1)}(0)\right)\\
&+f_{0200}^{(2)}\left(w_{20}^{(2)}(0)\bar\sigma_1+2\sigma_1w_{11}^{(2)}(0)\right)+f_{0020}^{(2)}\left(w_{20}^{(3)}(0)\bar\sigma_2+2\sigma_2w_{11}^{(3)}(0)\right)\\
&+f_{0002}^{(2)}\left(w_{20}^{(3)}(-1)\bar\sigma_2e^{i\omega_0\tau^*}+2\sigma_2w_{11}^{(3)}(-1)e^{-i\omega_0\tau^*}\right)+f_{1100}^{(2)}\left(2w_{11}^{(2)}(0)+w_{20}^{(2)}(0)\right.\\
&\left.+2\sigma_1w_{11}^{(1)}(0)+\bar\sigma_1w_{20}^{(1)}(0)\right)+f_{1010}^{(2)}\left(w_{20}^{(3)}(0)+2w_{11}^{(3)}(0)+2\sigma_2w_{11}^{(2)}(0)+\bar\sigma_2w_{20}^{(1)}(0)\right)\\
&+f_{1001}^{(2)}\left(w_{20}^{(3)}(-1)+2w_{11}^{(3)}(-1)+\bar\sigma_2e^{i\omega_0\tau^*}w_{20}^{(1)}(0)+2\sigma_2e^{-i\omega_0\tau^*}w_{11}^{(1)}(0)\right)\\
&+f_{0110}^{(2)}\left(2\sigma_1w_{11}^{(3)}(0)+\bar\sigma_1w_{20}^{(3)}(0)+2\sigma_2w_{11}^{(2)}(0)+\bar\sigma_2w_{20}^{(3)}(0)\right)+f_{0101}^{(2)}\left(\bar\sigma_1w_{20}^{(1)}(-1)\right.\\
&\left.+2\sigma_1w_{11}^{(3)}(-1)+2\sigma_2e^{-i\omega_0\tau^*}w_{11}^{(2)}(0)+\bar\sigma_2e^{i\omega_0\tau^*}w_{20}^{(2)}(0)\right)+f_{0011}^{(2)}(2\sigma_2w_{11}^{(3)}(-1)\\
&+\bar\sigma_2w_{20}^{(3)}(-1)+2\sigma_2w_{11}^{(3)}(0)e^{-i\omega_0\tau^*}+\bar\sigma_2e^{i\omega_0\tau^*}w_{20}^{(3)}(0)),1\Big\rangle\psi_2+\Big\langle f_{200}^{(3)}\left(\bar\sigma_1w_{20}^{(2)}(0)\right.\\
&\left.+2\sigma_1w_{11}^{(2)}(0)\right)+f_{020}^{(3)}\left(\bar\sigma_2w_{20}^{(3)}(0)+2\sigma_2w_{11}^{(3)}(0)\right)+f_{002}^{(3)}\left(\bar\sigma_2e^{i\omega_0\tau^*}w_{20}^{(3)}(-1)\right.\\
&\left.+2\sigma_2e^{-i\omega_0\tau^*}w_{11}^{(3)}(-1)\right)+f_{110}^{(3)}\left(2\sigma_1w_{11}^{(3)}(0)+\bar\sigma_1w_{20}^{(3)}(0)+2\sigma_2w_{11}^{(2)}(0)+\bar\sigma_2w_{20}^{(2)}(0)\right)\\
&+f_{101}^{(3)}\left(\bar\sigma_1w_{20}^{(3)}(-1)+2\bar\sigma_1w_{11}^{(3)}(-1)+2\sigma_2e^{-i\omega_0\tau^*}w_{11}^{(2)}(0)+\bar\sigma_2w_{20}^{(2)}(0)e^{i\omega_0\tau^*}\right)\\
&+f_{011}^{(3)}\left(2\sigma_2w_{11}^{(3)}(-1)+\bar\sigma_2w_{20}^{(3)}(-1)+2\sigma_2w_{11}^{(3)}(0)e^{-i\omega_0\tau^*}+\bar\sigma_2w_{20}^{(3)}(0)e^{i\omega_0\tau^*}\right),1\Big\rangle\psi_3\Big].
\end{aligned}$$

Since $W_{20}(\theta), W_{11}(\theta)$ for $\theta \in [-1,0]$ appear in g_{21}, we still need to compute them. It follows easily from (52) that

$$\dot{W}(z,\bar{z}) = W_{20}z\dot{z} + W_{11}(\dot{z}\bar{z} + z\dot{\bar{z}}) + W_{02}\bar{z}\dot{\bar{z}} + \cdots \tag{54}$$

and

$$A_{\tau^*}W = A_{\tau^*}W_{20}\frac{z^2}{2} + A_{\tau^*}W_{11}z\bar{z} + A_{\tau^*}W_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{55}$$

In addition, by [26], $W(z(t),\bar{z}(t))$ satisfy

$$\dot{W} = A_{\tau^*}W + H(z,\bar{z}), \tag{56}$$

where

$$\begin{aligned} H(z,\bar{z}) &= H_{20}\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \\ &= X_0F(U_t^*,0) - \Phi\big(\Psi,\langle X_0F(U_t^*,0),f_0\rangle\big)\cdot f_0, \end{aligned} \tag{57}$$

with $H_{ij} \in P_S\mathcal{C}$, $i+j=2$. Thus, from (49) and (54)-(56), we can obtain that

$$\begin{cases} (2i\omega_0\tau^* - A_{\tau^*})W_{20} = H_{20}, \\ -A_{\tau^*}W_{11} = H_{11}. \end{cases} \tag{58}$$

Notice that A_{τ^*} has only two eigenvalues $\pm i\omega_0\tau^*$ with zero real parts, therefore, (56) has a unique solution W_{ij} $(i+j=2)$ in $P_S\mathcal{C}$ given by

$$\begin{cases} W_{20} = (2i\omega_0\tau^* - A_{\tau^*})^{-1}H_{20}, \\ W_{11} = -A_{\tau^*}^{-1}H_{11}. \end{cases} \tag{59}$$

From (57), we know that for $-1 \leq \theta < 0$,

$$\begin{aligned} H(z,\bar{z}) &= -\Phi(\theta)\Psi(0)\langle F(U_t^*,0),f_0\rangle \cdot f_0 \\ &= -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2i}\right)\big(\Psi_1(0)\Psi_2(0)\big) \times \langle F(U_t^*,0),f_0\rangle \cdot f_0 \\ &= -\frac{1}{2}\big[p_1(\theta)\big(\Psi_1(0) - i\Psi_2(0)\big) + p_2(\theta)\big(\Psi_1(0) + i\Psi_2(0)\big)\big] \times \langle F(U_t^*,0),f_0\rangle \cdot f_0 \\ &= -\frac{1}{4}\big[g_{20}p_1(\theta) + \bar{g}_{02}p_2(\theta)\big]z^2 \cdot f_0 - \frac{1}{2}\big[g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta)\big]z\bar{z}\cdot f_0 + \cdots. \end{aligned}$$

Therefore, for $-1 \leq \theta < 0$,

$$H_{20}(\theta) = -\frac{1}{2}\big[g_{20}p_1(\theta) + \bar{g}_{02}p_2(\theta)\big]\cdot f_0, \tag{60}$$

$$H_{11}(\theta) = -\frac{1}{2}\big[g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta)\big]\cdot f_0 \tag{61}$$

and

$$H(z,\bar{z})(0) = F\big(U_t^*,0\big) - \Phi\big(\Psi,\big\langle F\big(U_t^*,0\big),f_0\big\rangle\big)\cdot f_0,$$

$$H_{20}(0) = \frac{\tau^*}{2}\begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2+\bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1+\bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2+\bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 \\ + \bar{\sigma}_1\sigma_2) + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} \\ + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 \\ + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) \\ + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix} - \frac{1}{2}\big[g_{20}p_1(0) + \bar{g}_{02}p_2(0)\big]\cdot f, \tag{62}$$

$$H_{11}(0) = \frac{\tau^*}{4}\begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2+\bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1+\bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2+\bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\ f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix} - \frac{1}{2}\big[g_{11}p_1(0) + \underset{11}{\bar{g}}\,p_2(0)\big]\cdot f. \tag{63}$$

By the definition of A_{τ^*}, we get from (59) that

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau^* W_{20}(\theta) + \frac{1}{2}\big[g_{20}p_1(\theta) + \bar{g}_{02}p_2(\theta)\big]\cdot f_0, \quad -1 \le \theta < 0.$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_0\tau^*}$, $-1 \le \theta \le 0$. Hence

$$W_{20}(\theta) = \frac{i}{2}\left[\frac{g_{20}}{\omega_0\tau^*}p_1(\theta) + \frac{\bar{g}_{02}}{3\omega_0\tau^*}p_2(\theta)\right]\cdot f_0 + Ee^{2i\omega_0\tau^*\theta}, \tag{64}$$

and

$$E = W_{20}(0) - \frac{i}{2}\left[\frac{g_{20}}{\omega_0\tau^*}p_1(0) + \frac{\bar{g}_{02}}{3\omega_0\tau^*}p_2(0)\right]\cdot f_0. \tag{65}$$

Using the definition of A_{τ^*} and combining (59) and (65), we get

$$\begin{aligned} &2i\omega_0\tau^*\left[\frac{ig_{20}}{2\omega_0\tau^*}p_1(0)\cdot f + \frac{i\bar{g}_{02}}{6\omega_0\tau^*}p_2(0)\cdot f + E\right] \\ &\quad - L(\tau^*)\left[\frac{ig_{20}}{2\omega_0\tau^*}p_1(\theta)\cdot f + \frac{i\bar{g}_{02}}{6\omega_0\tau^*}p_2(\theta)\cdot f + Ee^{2i\omega_0\tau^*\theta}\right] \\ &= \frac{\tau^*}{2}\begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2+\bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1+\bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2+\bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\ f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix} \end{aligned}$$

$$-\frac{1}{2}\big[g_{20}p_1(0)+\underset{02}{\bar{g}}\,p_2(0)\big]\cdot f.$$

Notice that

$$\begin{cases} L(\tau^*)[p_1(\theta)\cdot f_0] = i\omega_0\tau^* p_1(0)\cdot f_0, \\ L(\tau^*)[p_2(\theta)\cdot f_0] = -i\omega_0\tau^* p_2(0)\cdot f_0. \end{cases}$$

Then we have

$$\begin{aligned} &2i\omega_0\tau^*E - L(\tau^*)\big(Ee^{2i\omega_0\tau^*\theta}\big) \\ &\quad= \frac{\tau^*}{2}\begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2+\bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1+\bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2+\bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\ f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix}. \end{aligned}$$

From the above expression, we can see easily that

$$\begin{aligned} E &= \frac{1}{2}\begin{pmatrix} 2i\omega_0 + \beta I^* + \mu & \beta S^* & 0 \\ -\beta I^* & 2i\omega_0 - \alpha_2 & \alpha I^* e^{-2i\omega\tau^*} \\ 0 & \alpha R^* + \alpha_3 & 2i\omega_0 + \mu - \alpha I^* e^{-2i\omega\tau^*} \end{pmatrix}^{-1} \\ &\quad\times \begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2+\bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1+\bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2+\bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\ f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix}. \end{aligned}$$

By a similar way, we have

$$\dot{W}_{11}(\theta) = \frac{1}{2}\big[g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta)\big]\cdot f_0, \quad -1 \le \theta < 0,$$

and

$$W_{11}(\theta) = \frac{i}{2\omega_0\tau^*}\big[-g_{11}p_1(\theta) + \bar{g}_{11}p_2(\theta)\big]\cdot f_0 + F.$$

Similar to the above, we can obtain that

$$F = \frac{1}{4}\begin{pmatrix} \beta I^* + \mu & \beta S^* & 0 \\ -\beta I^* & -\alpha_2 & 0 \\ 0 & \alpha R^* + \alpha_3 & 2i\omega_0 + \mu \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} f_{20}^{(1)} + f_{11}^{(1)}(\sigma_2 + \bar{\sigma}_2) + f_{02}^{(1)}\sigma_2\bar{\sigma}_2 \\ f_{2000}^{(2)} + f_{0200}^{(2)}\sigma_1\bar{\sigma}_1 + f_{0020}^{(2)}\sigma_2\bar{\sigma}_2 + f_{0002}^{(2)}\sigma_2\bar{\sigma}_2 + f_{1100}^{(2)}(\sigma_1 + \bar{\sigma}_1) \\ + f_{1010}^{(2)}(\sigma_2 + \bar{\sigma}_2) + f_{1001}^{(2)}(\sigma_2 e^{-i\omega\tau^*} + \bar{\sigma}_2 e^{i\omega\tau^*}) + f_{0110}^{(2)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{0101}^{(2)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{0011}^{(2)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_2\sigma_2 e^{-i\omega\tau^*}) \\ f_{200}^{(3)}\sigma_1\bar{\sigma}_1 + f_{020}^{(3)}\sigma_2\bar{\sigma}_2 + f_{002}^{(3)}\sigma_2\bar{\sigma}_2 + f_{110}^{(3)}(\sigma_1\bar{\sigma}_2 + \bar{\sigma}_1\sigma_2) \\ + f_{101}^{(3)}(\sigma_1\bar{\sigma}_2 e^{i\omega\tau^*} + \bar{\sigma}_1\sigma_2 e^{-i\omega\tau^*}) + f_{011}^{(3)}(\sigma_2\bar{\sigma}_2 e^{i\omega\tau^*} + \sigma_2\bar{\sigma}_2 e^{-i\omega\tau^*}) \end{pmatrix}.$$

So far, $W_{20}(\theta)$ and $W_{11}(\theta)$ have been expressed by the parameters of system (1). Therefore, g_{21} can be expressed explicitly.

Theorem 4.1 *System* (1) *has the following Poincaré normal form*:

$$\dot{\xi} = i\omega_0\tau^*\xi + c_1(0)\xi|\xi|^2 + o\big(|\xi|^5\big),$$

where

$$c_1(0) = \frac{i}{2\omega_0\tau^*}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2},$$

so we can compute the following results:

$$\sigma_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tau^*))},$$
$$\beta_2 = 2\,\mathrm{Re}\big(c_1(0)\big),$$
$$T_2 = -\frac{\mathrm{Im}(c_1(0)) + \sigma_2\,\mathrm{Im}(\lambda'(\tau^*))}{\omega_0\tau^*},$$

which determine the properties of bifurcating periodic solutions at the critical values τ^*, *i.e.*, σ_2 *determines the directions of the Hopf bifurcation*: *if* $\sigma_2 > 0$ ($\sigma_2 < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*) *and the bifurcating periodic solutions exist for* $\tau > \tau^*$; β_2 *determines the stability of the bifurcating periodic solutions*: *the bifurcating periodic solutions on the center manifold are stable* (*unstable*) *if* $\beta_2 < 0$ ($\beta_2 > 0$); *and* T_2 *determines the period of the bifurcating periodic solutions*: *the periodic increase* (*decrease*) *if* $T_2 > 0$ ($T_2 < 0$).

5 Numerical simulation

In this section, numerical simulations of some examples are presented to illustrate the theoretical results.

5.1 Stability of the boundary equilibrium E_1

Let the parameters of system (1) be $\Lambda = 0.8$, $\beta = 0.2$, $\beta_1 = 0.7$, $\alpha_1 = 0.2$, $\mu = 0.6$, and $\alpha = 0.7$. Calculation reveals that the boundary equilibrium of system (1) is $(1.3333, 0, 0)$. Obviously, condition (H_1) holds. According to Theorem 3.1, system (1) is locally asymptotically stable at the boundary equilibrium for all $\tau \geq 0$, as shown in Figure 2. This means that with these parameter values the rumor will not be propagated.

Figure 2 Local asymptotic stability diagram for system (1) for all $\tau \geq 0$.

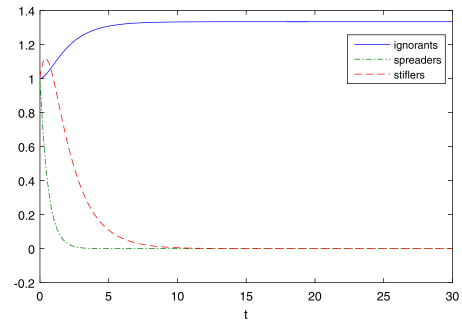


Figure 3 The amplitude of the density of spreaders with delay τ increasing $\in [0, 2.5]$.

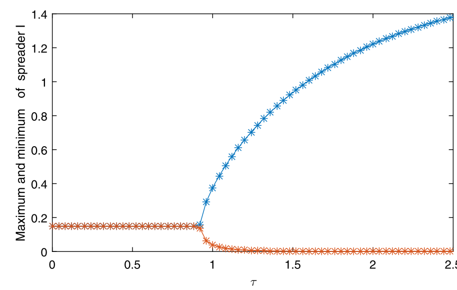
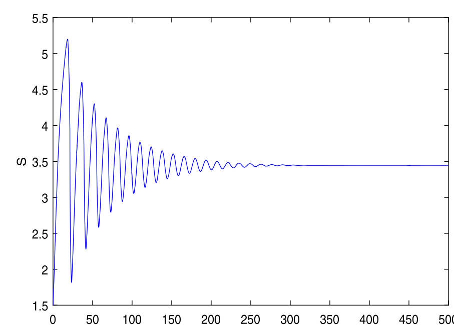


Figure 4 Local asymptotic stability diagram for system (1) when $\tau = 0.5$.



5.2 Stability and Hopf bifurcation of system (1)

Let $\Lambda = 0.6$, $\beta = 0.5$, $\beta_1 = 0.7$, $\alpha_1 = 0.4$, $\mu = 0.1$, $\alpha = 0.4$. Calculation reveals that the positive equilibrium of system (1) is $(3.4459, 0.1482, 2.4059)$ and the critical value is $\tau_0 = 0.9583$. Figure 3 gives the maximum and minimum of the density of spreaders for system (1) when $\tau \in [0, 2.5]$. From it we can find that when $\tau < 0.9583$ the amplitude is zero, which means system (1) is locally asymptotically stable at the positive equilibrium E^*, as observed in Figure 4. That is, the rumor continuously propagates with a fixed density of spreaders. When $0.9583 < \tau < 2.5$, as τ increases, the amplitude will increase. That is, system (1) is unstable and oscillation occurs, see Figure 5. As Figure 5 shows, when $\tau = 1$, the periodic solutions emerge from the positive equilibrium E^*, which implies that the rumor explosively spreads in a short period and may destroy network stability and block regular communications in online social networks, or even cause a panic in the real society.

Figure 5 Instability diagram and periodic solutions for system (1) when $\tau = 1$.

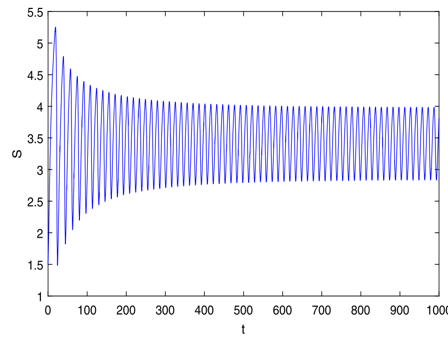
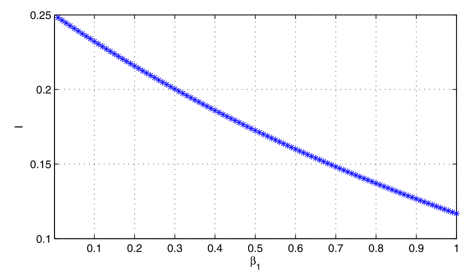


Figure 6 The spreaders vary with β_1 increasing.



In addition, when $\tau = \tau_0 = 0.9583$, we get $c_1(0) = -0.6013 + 6.8425i$, $\sigma_2 = -\frac{-0.6013}{\mathrm{Re}(\lambda'(\tau^*))} = 12.9212 > 0$, $\beta_2 = 2\,\mathrm{Re}(c_1(0)) < 0$. According to Theorem 4.1 in Section 4, the bifurcated periodic solutions of system (1) when $\tau_0 = 0.9583$ in the whole phase space are orbitally asymptotically stable, and the Hopf bifurcation is supercritical for $\sigma_2 > 0$.

5.3 Effect of the government adjustment power

Taking the same parameters as in Section 5.2, but β_1 varies in $[0, 1]$, the corresponding situations of spreaders are shown in Figure 6. Numerical evidence shows that with the increase of the adjustment power β_1, the adjustment power makes the population of the spreaders reduced. This is to say, if the government uses TV (the most popular and believable media in China) to announce the truth, the population of the spreaders will reduce immediately.

In addition, if we let $\tau = 1$ and $\beta_1 = 0.1$, then the positive equilibrium is locally asymptotically stable (see Figure 7). However, if we let $\beta_1 = 0.8$, then the positive equilibrium becomes unstable as is shown in Figure 8. It means that the government adjustment power has great effect on system (1).

6 Conclusions

In this paper, we considered a delayed rumor model with a saturated control function. Through the theoretical analysis and numerical simulation, we found that government adjustment power can affect system's stability. These can be found in Section 5.3.

By using DDE's stability theory, we take delay τ as a bifurcation parameter to study the Hopf bifurcation of system (1). Theoretical analysis and numerical simulations show that the discrete delays are responsible for the stability switch of the model, and a Hopf bifurcation occurs as the delays increase through a certain threshold (see Section 5.2).

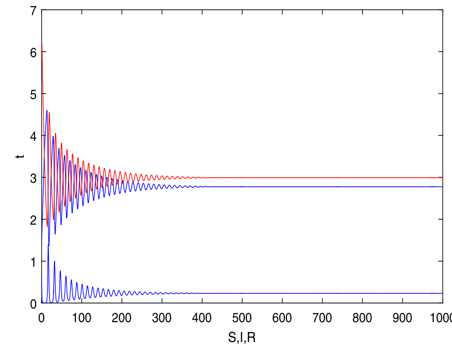


Figure 7 Local asymptotic stability diagram for system (1) with $\tau = 1$ and $\beta_1 = 0.1$.

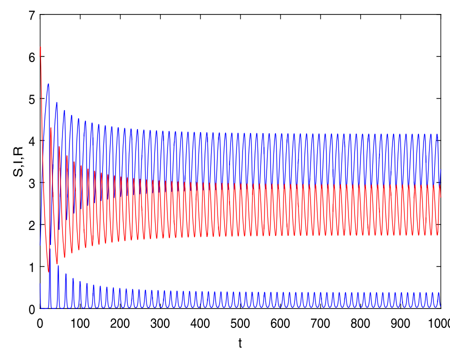


Figure 8 Instability diagram and periodic solutions for system (1) when $\tau = 1$ and $\beta_1 = 0.8$.

In summary, our study contributes to rumor management by offering an interplay model between rumor spreading and government adjustment. According to the transmission of the rumor, the government should use TV (the most popular and believable media in China) to announce the truth, then the population of the spreaders will be reduced immediately.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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