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Study of generalized type K-fractional derivatives

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Abstract

In this paper, the generalized type *k*-fractional derivatives are introduced and their semi-group, commutative and inverse properties are presented. These derivatives can be reduced to other fractional derivatives by substituting the values of the parameters involved. The Mellin transform of generalized Caputo type *k*-fractional derivative is also found.

MSC: 42A38; 26A33

Keywords: generalized *k*-fractional derivative; generalized Caputo type *k*-fractional derivative; Mellin transform

1 Introduction

Fractional calculus is the study of theory and applications of derivatives and integrals of non-integer order. It is a generalized form of calculus, so it retains many properties of calculus. It is worth mentioning that, in recent times, theory of fractional calculus has developed quickly and played many important roles in science and engineering, serving as a powerful and very effective tool for many mathematical problems. It has been extensively investigated in the last two decades.

Fractional derivatives are of vital importance in fractional calculus. These fractional derivatives are used in mathematical physics, astrophysics, control theory, electric conductance of biological systems, statistical mechanics, finance, biophysics, electrochemistry, computed tomography, geological surveying, thermodynamics, hydrology and engineering; moreover, they are also drawn on for the mathematical modelling of viscoelastic material.

Fractional derivatives have also been employed recently in signal and image possessing. They also have a key role in electric conductance of biological systems and fractional order models of neurons. The application of fractional order derivatives to the modelling of diffusion in a specific type of porous medium is in practice as well.

The objective and motivation of this work is to develop new generalized type k-fractional derivatives, which are the generalized form of the existing fractional derivatives, as well as to highlight the importance of their applications in diverse research areas. The generalized k-fractional and generalized Caputo type k-fractional derivatives are the generalized forms of some existing fractional derivatives.

Over the years, a large body of dedicated literature has become available to study fractional derivatives. In particular, we can find the theory and applications related to fractional derivatives in the books or papers in [1–24] and in their references. Diaz and Pariguan [25] defined the k-gamma function as

$$\Gamma_k(u) = \int_0^\infty e^{-\frac{x^k}{k}} x^{u-1}\,dx = k^{\frac{u}{k}-1}\Gamma\left(\frac{u}{k}\right), \quad \operatorname{Re}(u) > 0. \tag{1}$$

And

$$\Gamma(u) = \lim_{k\to 1}\Gamma_k(u), \qquad \Gamma_k(k) = 1, \qquad \Gamma_k(u+k) = u\Gamma_k(u). \tag{2}$$

The k-beta function is defined as

$$B_k(u,v) = \frac{1}{k}\int_0^1 z^{\frac{u}{k}-1}(1-z)^{\frac{v}{k}-1}\,dz, \quad \operatorname{Re}(u) > 0, \operatorname{Re}(v) > 0. \tag{3}$$

And

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)} = \frac{1}{k}B\left(\frac{m}{k},\frac{n}{k}\right). \tag{4}$$

Here, we introduce the generalized k-fractional derivative of order α. Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall 0 < t < x < \infty$

$$\left({}_k^s D_{a^+}^\alpha f\right)(x) = \frac{(s)^{\left(\frac{\alpha-nk+k}{k}\right)}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n \int_a^x \left(x^s - t^s\right)^{\frac{nk-\alpha}{k}-1} t^{s-1} f(t)\,dt, \quad \forall 0 < a < x \tag{5}$$

is called a generalized k-fractional derivative provided it exists. It can be written as

$$\left({}_k^s D_{a^+}^\alpha f\right)(x) = \left(x^{1-s}\frac{d}{dx}\right)^n \left({}_k^s I_{a^+}^{nk-\alpha} f\right)(x), \tag{6}$$

where ${}_k^s I_{a^+}^{nk-\alpha}$ is a (k,s)-Riemann-Liouville fractional integral [23].

Note that: (i) when $k \to 1$, it reduces to a generalized fractional derivative [13].

(ii) For $s = 1$, it reduces to a k-Riemann-Liouville fractional derivative [21]. It can be written as

$$\left({}_k D_{a^+}^\alpha f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_k^{nk-\alpha} f\right)(x), \tag{7}$$

where $I_k^{nk-\alpha}$ is a k-Riemann-Liouville fractional integral (see Mubeen and Habibullah [16]).

This further reduces to a Riemann-Liouville fractional derivative [15] for $k \to 1$.

(iii) For $s = 1$, $a = -\infty$, it reduces to a k-Weyl fractional derivative [20] which further reduces to a Weyl fractional derivative for $k \to 1$.

(iv) It gives a k-Hadamard fractional derivative for $s \to 0^+$, using L'Hospital rule

$$\begin{aligned}
\lim_{s\to 0^+} \left({}^s_k D^\alpha_{a^+} f\right)(x) &= \lim_{s\to 0^+} \frac{(s)^{1-(\frac{nk-\alpha}{k})}}{k\Gamma_k(nk-\alpha)} \left(x^{1-s}\frac{d}{dx}\right)^n \int_a^x \left(x^s - t^s\right)^{\frac{nk-\alpha}{k}-1} t^{s-1} f(t)\,dt \\
&= \frac{1}{k\Gamma_k(nk-\alpha)} \left(x\frac{d}{dx}\right)^n \\
&\quad \times \int_a^x \lim_{s\to 0^+} \left[\frac{x^s \log x - t^s \log t}{1}\right]^{\frac{nk-\alpha}{k}-1} t^{s-1} f(t)\,dt, \\
\left({}^H_k D^\alpha_{a^+} f\right)(x) &= \frac{1}{k\Gamma_k(nk-\alpha)} \left(x\frac{d}{dx}\right)^n \int_a^x \left(\log\frac{x}{t}\right)^{\frac{nk-\alpha}{k}-1} f(t)\frac{dt}{t},
\end{aligned} \tag{8}$$

which can also be written as

$$\left({}^H_k D^{nk-\alpha}_{a^+} f\right)(x) = \left(x\frac{d}{dx}\right)^n \left({}^H_k I^{nk-\alpha}_{a^+} f\right)(x), \tag{9}$$

where ${}^k_H I^{nk-\alpha}_{a^+}$ is a Hadamard k-fractional integral (see Farid and Habibullah [9]).

This further reduces to the usual Hadamard fractional derivative [11] for $k \to 1$.

Here, we also introduce the generalized Caputo type k-fractional derivative of order α. Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$, $k(n-1) < \alpha < nk$. Then $\forall 0 < t < x < \infty$

$$\begin{aligned}
\left({}^C_k D^\alpha_{s;a^+} f\right)(x) &= \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)} \int_a^x \left(x^s - t^s\right)^{\frac{nk-\alpha}{k}-1} t^{s-1} \left(t^{1-s}\frac{d}{dt}\right)^n f(t)\,dt, \\
&\forall 0 < a < x
\end{aligned} \tag{10}$$

is called a generalized Caputo type k-fractional derivative provided it exists. It can be written as

$$\left({}^C_k D^\alpha_{s;a^+} f\right)(x) = {}^s_k I^{nk-\alpha}_{a^+} \left(D^n f\right)(x), \tag{11}$$

where ${}^s_k I^{nk-\alpha}_{a^+}$ is a (k,s)-Riemann-Liouville fractional integral [23].

Note that: (i) When $k \to 1$, it reduces to a generalized Caputo type fractional derivative written as

$$\begin{aligned}
\left({}^C_s D^\alpha_{a^+} f\right)(x) &= \frac{(s)^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^x \left(x^s - t^s\right)^{n-\alpha-1} t^{s-1} \left(t^{1-s}\frac{d}{dt}\right)^n f(t)\,dt, \\
&\forall 0 < a < x.
\end{aligned} \tag{12}$$

(ii) When $s = 1$, it reduces to a k-Caputo fractional derivative [26].

(iii) For $k \to 1$, $s = 1$, it reduces to the well-known Caputo fractional derivative [27].

(iv) It gives a k-Caputo Hadamard fractional derivative for $s \to 0^+$, using L'Hospital rule

$$\begin{aligned}
\lim_{s\to 0^+} \left({}^C_k D^\alpha_{s;a^+} f\right)(x) &= \frac{1}{k\Gamma_k(nk-\alpha)} \int_a^x \lim_{s\to 0^+} \left(\frac{x^s - t^s}{s}\right)^{\frac{nk-\alpha}{k}-1} t^{s-1} \left(t^{1-s}\frac{d}{dt}\right)^n f(t)\,dt, \\
\lim_{s\to 0^+} \left({}^C_k D^\alpha_{s;a^+} f\right)(x) &= \frac{1}{k\Gamma_k(nk-\alpha)} \int_a^x \left(\lim_{s\to 0^+} \frac{x^s \log x - t^s \log t}{1}\right)^{\frac{nk-\alpha}{k}-1} \\
&\quad \times \frac{1}{t}\left(t\frac{d}{dt}\right)^n f(t)\,dt, \\
\left({}^{C-H}_k D^\alpha_{a^+} f\right)(x) &= \frac{1}{k\Gamma_k(nk-\alpha)} \int_a^x \left(\log\frac{x}{t}\right)^{\frac{nk-\alpha}{k}-1} \left(t\frac{d}{dt}\right)^n f(t)\frac{dt}{t},
\end{aligned} \tag{13}$$

which can also be written as

$$\left({}^{C-H}_k D^{nk-\alpha}_{a^+} f\right)(x) = {}^H_k I^{nk-\alpha}_{a^+} \left(x\frac{d}{dx}\right)^n f(x), \tag{14}$$

where ${}^k_H I^{nk-\alpha}_{a^+}$ is the Hadamard k-fractional integral (see Farid and Habibullah [9]).

This further reduces to Caputo modification of the Hadamard fractional derivative [10] for $k \to 1$.

(v) For $k \to 1$, $s = 1$, $a = -\infty$, we can also find the Caputo type Weyl fractional derivative.

The Mellin transform of a real scalar function $g(x)$ is given by

$$g^*(s_1) = M\left[g(x)\right] = \int_0^\infty x^{s_1-1} g(x)\,dx, \quad \operatorname{Re}(s_1) > 0 \tag{15}$$

whenever $g^*(s_1)$ exists. It is a function of the arbitrary parameter s_1.

Proposition 1.1 *For continuous $f(x)$ on $[0,\infty)$ and $\alpha, \beta \in \mathbb{R}$ and $k, s \in (0,\infty)$. Then $\forall 0 < a < x$*

$${}^s_k I^\alpha_{a^+} \left({}^s_k I^\beta_{a^+} f\right)(x) = \left({}^s_k I^{\alpha+\beta}_{a^+} f\right)(x) = {}^s_k I^\beta_{a^+} \left({}^s_k I^\alpha_{a^+} f\right)(x) \quad \left(\textit{see } [23]\right). \tag{16}$$

2 Results and discussion

Theorem 2.1 *Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall 0 < a < x$*

$${}^s_k D^\alpha_{a^+} \left({}^s_k I^\alpha_{a^+} f\right)(x) = \frac{1}{(k)^n} f(x). \quad \textit{(Inverse Property)} \tag{17}$$

Proof Using the result (5) in the LHS of equation (17), we have

$${}^s_k D^\alpha_{a^+} \left({}^s_k I^\alpha_{a^+} f\right)(x) = \frac{(s)^{\left(\frac{\alpha-nk+k}{k}\right)}}{k\Gamma_k(nk-\alpha)} \left(x^{1-s}\frac{d}{dx}\right)^n \int_a^x \left(x^s - y^s\right)^{\frac{nk-\alpha}{k}-1} y^{s-1} \left({}^s_k I^\alpha_{a^+} f\right)(y)\,dy.$$

By using the result of ${}^s_k I^\alpha_{a^+}$ and Fubini's theorem, we get

$$\begin{aligned}
{}^s_k D^\alpha_{a^+} \left({}^s_k I^\alpha_{a^+} f\right)(x) &= \frac{(s)^{2-n}}{k^2\Gamma_k(\alpha)\Gamma_k(nk-\alpha)} \left(x^{1-s}\frac{d}{dx}\right)^n \\
&\quad \times \int_a^x t^{s-1} f(t) \left[\int_t^x \left(y^s - t^s\right)^{\frac{\alpha}{k}-1} \left(x^s - y^s\right)^{\frac{nk-\alpha}{k}-1} y^{s-1}\,dy\right] dt.
\end{aligned}$$

By substituting $z = \frac{y^s - t^s}{x^s - t^s}$,

$$
{}_k^s D_{a^+}^{\alpha}\big({}_k^s I_{a^+}^{\alpha} f\big)(x) = \frac{(s)^{1-n}}{k^2 \Gamma_k(\alpha)\Gamma_k(nk-\alpha)} \left(x^{1-s}\frac{d}{dx}\right)^n \times \int_a^x \big(x^s - t^s\big)^{n-1} t^{s-1} f(t) \left[\int_0^1 (1-z)^{\frac{\alpha}{k}-1}(z)^{\frac{nk-\alpha}{k}-1}\,dz\right] dt.
$$

Using the result (2), we get

$$
{}_k^s D_{a^+}^{\alpha}\big({}_k^s I_{a^+}^{\alpha} f\big)(x) = \frac{1}{k^n}\left(x^{1-s}\frac{d}{dx}\right)^n \frac{(s)^{1-n}}{\Gamma(n)} \int_a^x \big(x^s - t^s\big)^{n-1} t^{s-1} f(t)\,dt,
$$

which gives the required result. □

Corollary 2.2 *Let f be continuous on $[0,\infty)$, and let $\alpha, \beta, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall 0 < a < x$*

$$
{}_k^s D_{a^+}^{\alpha}\big({}_k^s I_{a^+}^{\beta} f\big)(x) = \frac{1}{(k)^n}\big({}_k^s D_{a^+}^{\alpha-\beta} f\big)(x). \tag{18}
$$

Corollary 2.3 *For continuous f on $[0,\infty)$ and $\alpha, \beta, s \in \mathbb{R}^+$, $k, m, n \in \mathbb{N}$, $n = [\alpha] + 1$, $m = [\beta] + 1$. Then $\forall 0 < a < x$ and $\alpha + \beta < nk$,*

$$
{}_k^s D_{a^+}^{\alpha}\big({}_k^s D_{a^+}^{\beta} f\big)(x) = \frac{1}{(k)^n}\big({}_k^s D_{a^+}^{\alpha+\beta} f\big)(x). \quad \textit{(Semi-group Property)} \tag{19}
$$

Proof Using the result (5) in the LHS of equation (19), we have

$$
\begin{aligned}
{}_k^s D_{a^+}^{\alpha}\big({}_k^s D_{a^+}^{\beta} f\big)(x) &= \left(x^{1-s}\frac{d}{dx}\right)^n \big({}_k^s I_{a^+}^{nk-\alpha}\big)\big({}_k^s D_{a^+}^{\beta} f\big)(x) \\
&= \left(x^{1-s}\frac{d}{dx}\right)^n \big({}_k^s I_{a^+}^{nk-\alpha}\big)\big({}_k^s D_{a^+}^{\beta}\big)\big({}_k^s I_{a^+}^{\beta}\big)\big({}_k^s I_{a^+}^{-\beta} f\big)(x).
\end{aligned}
$$

By using the result (17), we get

$$
\begin{aligned}
{}_k^s D_{a^+}^{\alpha}\big({}_k^s D_{a^+}^{\beta} f\big)(x) &= \frac{1}{(k)^n}\left(x^{1-s}\frac{d}{dx}\right)^n \big({}_k^s I_{a^+}^{nk-\alpha}\big)\big({}_k^s I_{a^+}^{-\beta} f\big)(x) \\
&= \frac{1}{(k)^n}\left(x^{1-s}\frac{d}{dx}\right)^n \big({}_k^s I_{a^+}^{nk-(\alpha+\beta)} f\big)(x).
\end{aligned}
$$

Using the result (5), we get the required result. □

Corollary 2.4 *For continuous f on $[0,\infty)$ and $\alpha, \beta, s \in \mathbb{R}^+$, $k, n, m \in \mathbb{N}$ and $n = [\alpha] + 1$, $m = [\beta] + 1$, $\alpha + \beta < nk$. Then $\forall 0 < a < x$*

$$
{}_k^s D_{a^+}^{\alpha}\big({}_k^s D_{a^+}^{\beta} f\big)(x) = {}_k^s D_{a^+}^{\beta}\big({}_k^s D_{a^+}^{\alpha} f\big)(x). \quad \textit{(Commutative Property)} \tag{20}
$$

Corollary 2.5 *Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall 0 < a < x$*

$$
\big({}_k^s D_{a^+}^{\alpha}\big)\big[g(x) + \mu h(x)\big] = \big({}_k^s D_{a^+}^{\alpha}\big)g(x) + \mu\big({}_k^s D_{a^+}^{\alpha}\big)h(x). \quad \textit{(Linearity)} \tag{21}
$$

Example 2.6 Let $\alpha, s, \gamma \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall x > 0$

$$ {}_k^sD_{0^+}^{\alpha}\left[(x^s)^{\frac{\gamma}{k}}\right] = \frac{(s)^{(\frac{\alpha-nk}{k})}\Gamma_k(k+\gamma)}{\Gamma_k(nk+k+\gamma-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n (x^s)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}. \tag{22} $$

Solution Using the result (5) in the LHS of equation (22), we have

$$ {}_k^sD_{0^+}^{\alpha}\left[(x^s)^{\frac{\gamma}{k}}\right] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n \int_0^x \left(x^s-t^s\right)^{\frac{nk-\alpha}{k}-1} t^{s-1}\left[(t^s)^{\frac{\gamma}{k}}\right] dt. $$

By substituting $y = \frac{t^s}{x^s}$,

$$ {}_k^sD_{0^+}^{\alpha}\left[(x^s)^{\frac{\gamma}{k}}\right] = \frac{(s)^{(\frac{\alpha-nk}{k})}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n (x^s)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}\left[\int_0^1 (1-y)^{\frac{nk-\alpha}{k}-1} y^{(1+\frac{\gamma}{k})-1}\,dy\right]. $$

Using the results (3) and (4), we get the required result.

Example 2.7 Let $\alpha, s, \mu \in \mathbb{R}^+$, $k, n \in \mathbb{N}$ and $n = [\alpha] + 1$. Then $\forall x < \infty$

$$ {}_k^sD_{-\infty}^{\alpha}\left[e^{\mu x^s}\right] = (\mu ks)^{(\frac{\alpha-nk}{k})}\left(x^{1-s}\frac{d}{dx}\right)^n e^{\mu x^s}. \tag{23} $$

Solution Using the result (5) in the LHS of equation (23), we have

$$ {}_k^sD_{-\infty}^{\alpha}\left[e^{\mu x^s}\right] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n \int_{-\infty}^x \left(x^s-t^s\right)^{\frac{nk-\alpha}{k}-1} t^{s-1}\left[e^{\mu t^s}\right] dt. $$

By substituting $x^s - t^s = z^s$,

$$ {}_k^sD_{-\infty}^{\alpha}\left[e^{\mu t^s}\right] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n e^{\mu x^s}\int_0^{\infty} \left(z^s\right)^{\frac{nk-\alpha}{k}-1}\left[e^{-\mu z^s}\right] z^{s-1}\,dz. $$

By substituting $u = \mu z^s$,

$$ {}_k^sD_{-\infty}^{\alpha}\left[e^{\mu t^s}\right] = \frac{(\mu s)^{(\frac{\alpha-nk}{k})}}{k\Gamma_k(nk-\alpha)}\left(x^{1-s}\frac{d}{dx}\right)^n e^{\mu x^s}\int_0^{\infty} (u)^{n-\frac{\alpha}{k}-1} e^{-u}\,du. $$

Using the results (1) and (2), we get the required result.

Theorem 2.8 *Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$, $k(n-1) < \alpha < nk$. Then $\forall 0 < a < x$*

$$ {}_k^sI_{a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x) = \frac{1}{k^n}\left[f(x) - f(a)\right]. \tag{24} $$

Proof Using the result of ${}_k^sI_{a^+}^{\alpha}$,

$$ {}_k^sI_{a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x) = \frac{(s)^{\frac{k-\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^s-y^s\right)^{\frac{\alpha}{k}-1} y^{s-1}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(y)\,dy. $$

Using the result (10) and Fubini's theorem, we obtain

$$\begin{aligned} {}_k^sI_{a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x) &= \frac{(s)^{2-n}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)} \\ &\quad\times\int_a^x t^{s-1}\left(t^{1-s}\right)^n f^{(n)}(t)\left[\int_t^x \left(y^s-t^s\right)^{n-\frac{\alpha}{k}-1}\left(x^s-y^s\right)^{\frac{\alpha}{k}-1}y^{s-1}\,dy\right]dt. \end{aligned}$$

By substituting $z=\frac{y^s-t^s}{x^s-t^s}$ and using the results (3) and (4), we get

$${}_k^sI_{a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x)=\frac{(s)^{1-n}}{k\Gamma_k(nk)}\int_a^x \left(x^s-t^s\right)^{n-1}t^{s-1}\left(t^{1-s}\right)^n f^{(n)}(t)\,dt,$$

which gives the required result. □

Corollary 2.9 *Let f be continuous on $[0,\infty)$, and let $\alpha,\beta,s\in\mathbb{R}^+$, $k,n\in\mathbb{N}$, $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$*

$${}_k^sI_{a^+}^{\beta}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x)=\left({}_k^CD_{s;a^+}^{\alpha-\beta}f\right)(x). \tag{25}$$

Corollary 2.10 *For continuous f on $[0,\infty)$ and $\alpha,\beta,s\in\mathbb{R}^+$, $k,m,n\in\mathbb{N}$. Then $\forall 0<a<x$ and $k(n-1)<\alpha<nk$, $k(m-1)<\beta<mk$, $\alpha+\beta<nk$,*

$${}_k^CD_{s;a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\beta}f\right)(x)=\frac{1}{k^n}\left({}_k^CD_{s;a^+}^{\alpha+\beta}f\right)(x). \tag{26}$$

Corollary 2.11 *For continuous f on $[0,\infty)$ and $\alpha,\beta,s\in\mathbb{R}^+$, $k,n,m\in\mathbb{N}$ and $n=[\alpha]+1$, $m=[\beta]+1$, $\alpha+\beta<nk$. Then $\forall 0<a<x$*

$${}_k^CD_{s;a^+}^{\alpha}\left({}_k^CD_{s;a^+}^{\beta}f\right)(x)={}_k^CD_{s;a^+}^{\beta}\left({}_k^CD_{s;a^+}^{\alpha}f\right)(x). \tag{27}$$

Corollary 2.12 *Let f be continuous on $[0,\infty)$, and let $\alpha,s\in\mathbb{R}^+$, $k,n\in\mathbb{N}$, $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$*

$${}_k^CD_{s;a^+}^{\alpha}\left[c_1g(x)+c_2h(x)\right]=c_1\left({}_k^CD_{s;a^+}^{\alpha}g\right)(x)+c_2\left({}_k^CD_{s;a^+}^{\alpha}h\right)(x). \tag{28}$$

Example 2.13 Let $a=0$, $s=1$, $\alpha=3$, $k=2$, $n=2$ and $f(x)=x^2$. Then $\forall x>0$, using the result (10), we have

$${}_k^CD^{\alpha}\left[f(x)\right]=D^{\frac{\alpha}{k}}\left[f(x)\right]=D^{\frac{3}{2}}\left[f(x)\right]=\frac{1}{2\Gamma_2(4-3)}\int_0^x (x-y)^{-\frac{1}{2}}2\,dy.$$

Using the result (2), we have

$$D^{\frac{3}{2}}\left[f(x)\right]=2\sqrt{\frac{2x}{\pi}}.$$

Example 2.14 Let $\alpha,s,\beta\in\mathbb{R}^+$ and $k,n\in\mathbb{N}$. Then $\forall x>0$

$${}_k^CD_{s;0^+}^{\alpha}\left[(x)^{\frac{\beta}{k}}\right]=\frac{(s)^{(\frac{\alpha-nk}{k})}\Gamma_k(\frac{\beta}{s}-nk+k)\Gamma(\frac{\beta}{k}+1)x^{\frac{\beta-\alpha}{k}}}{\Gamma_k(\frac{\beta}{s}-\alpha+k)\Gamma(\frac{\beta}{k}-n+1)}. \tag{29}$$

Solution Using the result (10) in the LHS of equation (29), we have

$$ {}_k^C D_{s;0^+}^{\alpha}\big[(x)^{\frac{\beta}{k}}\big] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)} \int_0^x \big(x^s - t^s\big)^{\frac{nk-\alpha}{k}-1} t^{s-1}\big(t^{1-s}\big)^n \left(\frac{d}{dx}\right)^n (t)^{\frac{\beta}{k}}\, dt. $$

By taking the nth derivative and then by substituting $y = \frac{t^s}{x^s}$

$$ {}_k^C D_{s;0^+}^{\alpha}\big[(x)^{\frac{\beta}{k}}\big] = \frac{(s)^{(\frac{\alpha-nk}{k})}\Gamma(\frac{\beta}{k}+1)x^{\frac{\beta-\alpha}{k}}}{k\Gamma_k(nk-\alpha)\Gamma(\frac{\beta}{k}-n+1)} \int_0^1 (1-y)^{\frac{nk-\alpha}{k}-1}(y)^{(\frac{\beta-nsk+sk}{sk})-1}\, dt. $$

Using the results (3) and (4), we get the required result.

Theorem 2.15 *Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$. Then $\forall 0 < a < t$*

$$ M\big[{}_k^C D_{s;a^+}^{\alpha} f(t)\big] = \frac{(s)^{(\frac{\alpha-nk}{k})}\Gamma_k(\alpha - nk + k - \frac{ks_1}{s})}{\Gamma_k(k-\frac{ks_1}{s})}\left[M\big(f^{(n)}(t)\big)\left(s_1 - \frac{s\alpha}{k} + n\right)\right], \\ \operatorname{Re}(s_1) > 0. \tag{30} $$

Proof Using the results (10) and (15) in the LHS of equation (30), we have

$$ M\big[{}_k^C D_{s;a^+}^{\alpha} f(t)\big] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)} \int_0^\infty t^{s_1-1}\left[\int_a^t \big(t^s - y^s\big)^{\frac{nk-\alpha}{k}-1} y^{s-1}\big(y^{1-s}\big)^n f^{(n)}(y)\, dy\right] dt. $$

By using Fubini's theorem,

$$ M\big[{}_k^C D_{s;a^+}^{\alpha} f(t)\big] = \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)} \int_0^\infty y^{s-1}\big(y^{1-s}\big)^n f^{(n)}(y)\left[\int_y^\infty \big(t^s - y^s\big)^{\frac{nk-\alpha}{k}-1} t^{s_1-1}\, dt\right] dy. $$

By substituting $t^s = \frac{y^s}{z}$,

$$ M\big[{}_k^C D_{s;a^+}^{\alpha} f(t)\big] = \frac{(s)^{(\frac{\alpha-nk}{k})}}{k\Gamma_k(nk-\alpha)} \int_0^\infty y^{s_1-\frac{s\alpha}{k}+n-1} f^{(n)}(y)\left[\int_0^1 (1-z)^{n-\frac{\alpha}{k}-1} z^{\frac{\alpha}{k}-n-\frac{s_1}{s}}\, dz\right] dy. $$

Using the results (3) and (4), we get

$$ M\big[{}_k^C D_{s;a^+}^{\alpha} f(t)\big] = \frac{(s)^{(\frac{\alpha-nk}{k})}\Gamma_k(\alpha - nk + k - \frac{ks_1}{s})}{\Gamma_k(k-\frac{ks_1}{s})} \int_0^\infty y^{s_1-\frac{s\alpha}{k}+n-1} f^{(n)}(y)\, dy. $$

Using the result (15), we get the required result. □

Example 2.16 Let $\alpha, s \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$. Then $\forall x > 0$, using the result (30), we can get

$$ M\big[{}_k^C D_{s;a^+}^{\alpha}\big(e^{-x}\big)\big] = \frac{(-1)^n(s)^{(\frac{\alpha-nk}{k})}\Gamma_k(\alpha - nk + k - \frac{ks_1}{s})\Gamma(s_1 - \frac{s\alpha}{k} + n)}{\Gamma_k(k-\frac{ks_1}{s})}, \quad \operatorname{Re}(s_1) > 0. \tag{31} $$

Example 2.17 Consider the differential equation

$$D^{\frac{3}{2}}\big[f(x)\big] + bf(x) = 0, \quad 0 < a < x,$$

with the initial condition

$$\big[D^{-\frac{1}{2}}f(x)\big]_{x=a} = A.$$

We can solve this by taking $k = 2$, $n = 2$, $\alpha = 3$, $s = 1$ and using the result (30) and then finding the inverse Mellin transform.

Theorem 2.18 *Let f be continuous on $[0,\infty)$, and let $\alpha, s \in \mathbb{R}^+$, $k, n \in \mathbb{N}$, $k(n-1) < \alpha < nk$, $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}_0$, also $n = \alpha$ for $\alpha \in \mathbb{N}_0$, where $\mathbb{N}_0 = \{0,1,2,\ldots\}$. Let $f(x) \in C^n[a,b]$, then the generalized Caputo type k-fractional derivative $({}^c_kD^{\alpha}_{s;a^+}f)(x)$ of order α is bounded on $[a,b]$ $(-\infty < a < b < \infty)$.*

Proof (a) If $a \notin \mathbb{N}_0$, then define the subspaces $C_a[a,b]$ of the space $C^n[a,b]$ as

$$C_a[a,b] = \big\{f(x) \in C_a[a,b]; f(a) = 0, \|f\|C^n_{\alpha} = \|f\|C^n\big\}.$$

From equations (10) and (11), we have

$$\begin{aligned}\big({}^C_kD^{\alpha}_{s;a^+}f\big)(x) &= \frac{(s)^{(\frac{\alpha-nk+k}{k})}}{k\Gamma_k(nk-\alpha)}\int_a^x \big(x^s - t^s\big)^{\frac{nk-\alpha}{k}-1}t^{s-1}\left(t^{1-s}\frac{d}{dt}\right)^n f(t)\,dt \\ &= {}^s_kI^{nk-\alpha}_{a^+}\big(D^nf\big)(x).\end{aligned}$$

Let $g(x) = D^nf(x)$ be continuous on $x \in [a,b]$. We also know that ${}^s_kI^{nk-\alpha}_{a^+}[g(x)]$ exists for any $x \in [a,b]$ (see [23]). Hence, we get that $({}^c_kD^{\alpha}_{s;a^+}f)(x)$ is bounded from the space $C^n[a,b]$ to the subspace $C_a[a,b]$. Moreover, using equation (10), we have

$$\big\|{}^c_kD^{\alpha}_{s;a^+}f\big\|C_a \le \frac{(s)^{\frac{\alpha}{k}-n}\|f^{(n)}\|C}{|\Gamma_k(nk-\alpha)|[nk-\alpha+k]}(b-a)^{n-\frac{\alpha}{k}} < \infty,$$

which gives

$$\big\|{}^c_kD^{\alpha}_{s;a^+}f\big\|C_a \le M_{\alpha}\|f\|C^n,$$

where

$$M_{\alpha} = \frac{(\frac{b-a}{s})^{n-\frac{\alpha}{k}}}{|\Gamma_k(nk-\alpha)|[nk-\alpha+k]}.$$

(b) If $a \in \mathbb{N}_0$, then by following the steps of part (a), we get that $({}^c_kD^{\alpha}_{s;a^+}f)(x)$ is bounded from the space $C^n[a,b]$ to the subspace $C_a[a,b]$. Moreover,

$$\big\|{}^c_kD^{\alpha}_{s;a^+}f\big\|C \le \|f\|C^n.$$

□

Applications 2.19 Here, we give some applications of the generalized type k-fractional derivatives.

(i) They are used to solve Abel's integral equation. As we know, the solutions of many applied problems lead to integral equations, and these equations can be reduced to Abel's integral equation.

(ii) Viscoelasticity has the most extensive applications of fractional derivatives. The use of fractional derivatives for the mathematical modelling of viscoelastic material is quite natural.

(iii) Fractional order derivatives are applied to the modelling of diffusion in a specific type of porous medium. Partial differential equations of fractional order can also be solved.

(iv) The use of fractional order derivatives in control theory leads to better results and provides strong motivation for further development.

(v) Caputo type fractional derivatives are very effectively used in boundary value problems of applied mathematics in recent times.

(vi) The time derivatives of fractional order might describe the behavior of sound waves in rigid porous materials because the asymptotic expressions of stiffness and damping in this kind of materials are proportional to fractional powers of frequency.

(vii) The solutions of time-dependent viscous-diffusion fluid mechanics problems are determined by the Laplace, Fourier and Mellin transforms methods of fractional operators.

(viii) Fractional Order Controllers (FOC) are used in autonomous electric vehicles to tackle the path-tracking problems.

Example 2.20 Consider the differential equation

$$x^{\alpha+1}D^{\alpha+1}\big[y(x)\big] + x^{\alpha}D^{\alpha}\big[y(x)\big] + y(x) = f(x)$$

with conditions

$$y(0) = y^{(1)}(0) = 0, \qquad y(\infty) = y^{(1)}(\infty) = 0.$$

By taking $k = 1, s = 1$ and using the Mellin transform, we get

$$\left[\frac{\Gamma(1-s_1)(1-s_1-\alpha)}{\Gamma(1-s_1-\alpha)} + 1\right]Y(s_1) = F(s_1).$$

Simplifying this and then applying the inverse Mellin transform, we can get solution.

3 Conclusion

The results of generalized k-fractional derivative and generalized Caputo type k-fractional derivative are reduced to the k-calculus results for $s = 1$ and are also reduced to the classical calculus results for $s = 1$, $k = 1$. These generalized type k-fractional derivatives can be used to develop the fluid dynamics models. The governing equations of fluids like Maxwell fluid, second grade fluid etc., which are fractional order differential equations, can also be solved by using these derivatives.

Lastly, we conclude this paper by hoping that we can also extend these generalized type k-fractional derivatives and their results for $\alpha \in \mathbb{C}$ by analytical continuation.

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