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On nonlocal fractional sum-difference boundary value problems for Caputo fractional functional difference equations with delay

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Abstract

In this article, we study the existence and uniqueness results for a nonlocal fractional sum-difference boundary value problem for a Caputo fractional functional difference equation with delay, by using the Schauder fixed point theorem and the Banach contraction principle. Finally, we present some examples to display the importance of these results.

MSC: 39A05; 39A12

Keywords: Caputo fractional functional difference equations; boundary value problem; delay; existence

1 Introduction

In this paper, we consider a nonlocal fractional sum boundary value problem for a Caputo fractional functional difference equation with delay of the form

$$\begin{aligned} &\Delta_C^{\alpha}u(t) = \mathcal{F}\big[t+\alpha-1, u_{t+\alpha-1}, \Delta_C^{\beta}u(t+\alpha-\beta)\big], \quad t \in \mathbb{N}_{0,T} := \{0,1,\dots,T\}, \\ &\Delta_C^{\gamma}u(\alpha-\gamma-1) = 0, \qquad u(T+\alpha) = \rho\Delta^{-\omega}u(\eta+\omega), \end{aligned} \tag{1.1}$$

and $u_{\alpha-3} = \psi$, where $\rho \neq \frac{\Gamma(\omega)((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(T+\alpha)[(T-\alpha+5)(2-\gamma)-2])}{\sum_{s=\alpha-3}^{\eta}(\eta+\omega-\sigma(s))^{\underline{\omega-1}}((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(\eta+\omega)[(\eta+\omega-2\alpha+5)(2-\gamma)-2])}$, $\alpha \in (2,3)$, $\beta,\gamma,\omega \in (0,1)$, $\eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}$ are given constants, $\mathcal{F} \in C(\mathbb{N}_{\alpha-3,T+\alpha} \times C_r \times \mathbb{R},\mathbb{R})$ and ψ is an element of the space

$$C_r^{+}(\alpha-3) := \big\{\psi \in C_r : \psi(\alpha-3) = 0, \Delta_C^{\beta}\psi(s-\beta+1) = 0, s \in \mathbb{N}_{\alpha-r-3,\alpha-3}\big\}.$$

For $r \in \mathbb{N}_{0,T+3}$ we denote C_r is the Banach space of all continuous functions $\psi : \mathbb{N}_{\alpha-r-3,\alpha-3} \to \mathbb{R}$ endowed with the norm

$$\|\psi\|_{C_r} = \max_{s\in\mathbb{N}_{\alpha-r-3,\alpha-3}} \big|\psi(s)\big|.$$

If $u : \mathbb{N}_{\alpha-r-3,\alpha-3} \to \mathbb{R}$, then for any $t \in \mathbb{N}_{\alpha-3,T+\alpha}$ we denote by u_t the element of C_r defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in \mathbb{N}_{-r,0}.$$

Fractional difference calculus or discrete fractional calculus is a very new field for mathematicians. Basic definitions and properties of fractional difference calculus can be found in the book [1]. Some real-world phenomena are being studied with the assistance of fractional difference operators, one may refer to [2, 3] and the references therein. Good papers related to discrete fractional boundary value problems can be found in [4–21] and the references cited therein.

At present, the development of boundary value problems for fractional difference equations which show an operation of the investigative function. The study may also have other functions related to the ones we are interested in. These creations are incorporating with nonlocal conditions which are both extensive and more complex. For example, Goodrich [6] considered the discrete fractional boundary value problem

$$\begin{cases} -\Delta^{\nu} y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), \quad t \in \mathbb{N}_{1,b+1}, \\ y(\nu-2) = \sum_{i=1}^{N} F_i^1(y(t_i^1)), \\ y(\nu+b+1) = \sum_{i=1}^{M} F_i^2(y(t_i^2)), \end{cases} \tag{1.2}$$

where $b > 0$ is an integer, and $\lambda > 0$ is a parameter, $\nu \in (1,2]$ is a real number, $f : \mathbb{N}_{\nu,\nu+b} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $F_i^1, F_i^2 : \mathbb{R} \to [0,\infty]$ are continuous functions for each i and satisfy some growth conditions to be specified later, and $\{t_i^1\}_{i=1}^N, \{t_i^2\}_{i=1}^M \subseteq \mathbb{N}_{\nu-1,\nu+b}$. The existence of positive solutions are obtained by Krasnosel'skii fixed point theorem.

Reunsumrit *et al.* [18] obtained sufficient conditions for the existence of positive solutions for the three-point fractional sum boundary value problem for Caputo fractional difference equations via an argument with a shift

$$\begin{aligned} &\Delta_C^{\alpha} u(t) + a(t+\alpha-1) f\big(u\big(\theta(t+\alpha-1)\big)\big) = 0, \quad t \in \mathbb{N}_{0,T}, \\ &u(\alpha-3) = \Delta^2 u(\alpha-3) = 0, \\ &u(T+\alpha) = \lambda \Delta^{-\beta} u(\eta+\beta), \end{aligned} \tag{1.3}$$

where $2 < \alpha \leq 3$, $0 < \beta \leq 1$, $\eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}$, Δ_C^{α} is the Caputo fractional difference operator of order α, and $f : [0,\infty) \to [0,\infty)$ is a continuous function. The existence of at least one positive solution is proved by using Krasnoselskii's fixed point theorem.

Recently, Sitthiwirattham [16] investigated three-point fractional sum boundary value problems for sequential fractional difference equations of the forms

$$\begin{cases} \Delta_C^{\alpha}[\phi_p(\Delta_C^{\beta} x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \\ \Delta_C^{\beta} x(\alpha-1) = 0, \qquad x(\alpha+\beta+T) = \rho \Delta^{-\gamma} x(\eta+\gamma), \end{cases} \tag{1.4}$$

where $t \in \mathbb{N}_{0,T}$, $0 < \alpha, \beta \leq 1$, $1 < \alpha+\beta \leq 2$, $0 < \gamma \leq 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1}$, ρ is a constant, $f : \mathbb{N}_{\alpha+\beta-2,\alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ϕ_p is the p-Laplacian operator. Existence and uniqueness of solutions are obtained by the Banach fixed point theorem and Schaefer's fixed point theorem.

The results mentioned above are the motivation for this research. The plan of this paper is as follows. In Section 2 we recall some definitions and basic lemmas. Also, we derive a representation for the solution of (1.1) by converting the problem to an equivalent summation equation. In Section 3, we prove existence and uniqueness results of the problem (1.1) by using the Schauder fixed point theorem and the Banach contraction principle. Some illustrative examples are presented in Section 4.

2 Preliminaries

In the following, there are notations, definitions, and lemma which are used in the main results. We briefly recall the necessary concepts from the discrete fractional calculus; see [1] for further information.

Definition 2.1 We define the generalized falling function by $t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, for any t and α for which the right-hand side is defined. If $t+1-\alpha$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\alpha}} = 0$.

Lemma 2.1 ([11]) *Assume the following factorial functions are well defined. If $t \leq r$, then $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.2 For $\alpha > 0$ and f defined on $\mathbb{N}_a := \{a, a+1, \ldots\}$, the αth-order fractional sum of f is defined by

$$\Delta^{-\alpha} f(t) = \Delta_a^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} \bigl(t-\sigma(s)\bigr)^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.3 For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the αth-order Caputo fractional difference of f is defined by

$$\Delta_C^{\alpha} f(t) := \Delta_a^{-(N-\alpha)} \Delta^N f(t) = \frac{1}{\Gamma(N-\alpha)} \sum_{s=a}^{t-(N-\alpha)} \bigl(t-\sigma(s)\bigr)^{\underline{N-\alpha-1}} \Delta^N f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < \alpha < N$. If $\alpha = N$, then $\Delta_C^{\alpha} f(t) = \Delta^N f(t)$.

Lemma 2.2 ([9]) *Assume that $\alpha > 0$ and f defined on $\mathbb{N}_a$. Then*

$$\Delta_{a+N-\alpha}^{-\alpha} \Delta_C^{\alpha} y(t) = y(t) + C_0 + C_1 (t-a)^{\underline{1}} + C_2 (t-a)^{\underline{2}} + \cdots + C_{N-1} (t-a)^{\underline{N-1}},$$

for some $C_i \in \mathbb{R}$, $0 \leq i \leq N-1$ and $0 \leq N-1 < \alpha \leq N$.

To define the solution of the boundary value problem (1.1) we need the following lemma that deals with a linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.3 *Let* $\rho \neq \frac{\Gamma(\omega)((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(T+\alpha)[(T-\alpha+5)(2-\gamma)-2])}{\sum_{s=\alpha-3}^{\eta}(\eta+\omega-\sigma(s))^{\underline{\omega-1}}((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(\eta+\omega)[(\eta+\omega-2\alpha+5)(2-\gamma)-2])}$, $\alpha \in (2,3)$, $\gamma,\omega \in (0,1)$ *and* $f \in C(\mathbb{N}_{\alpha-1,T+\alpha-1},\mathbb{R})$ *be given. Then the problem*

$$\Delta_C^{\alpha} u(t) = f(t+\alpha-1), \quad t \in \mathbb{N}_{0,T}, \tag{2.1}$$

$$\begin{cases} u(\alpha-3) = 0, \\ \Delta_C^{\gamma} u(\alpha-\gamma-1) = 0, \\ u(T+\alpha) = \rho\Delta^{-\omega} u(\eta+\omega), \quad \eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}, \end{cases} \tag{2.2}$$

has the unique solution

$$\begin{aligned} u(t) &= \frac{1}{\Lambda}\big[(\alpha-3)\big((\alpha-2)(2-\gamma)+2\big) + 2t^{\underline{1}}\big((3-\alpha)(2-\gamma)-1\big) + t^{\underline{2}}\big] \\ &\quad \times \Bigg[\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}} f(s+\alpha-1) - \frac{\rho}{\Gamma(\omega)\Gamma(\alpha)} \\ &\quad \times \sum_{s=0}^{\eta-\alpha}\sum_{\xi=s}^{\eta-\alpha}\big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}}\big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}} f(s+\alpha-1)\Bigg] \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\big(t-\sigma(s)\big)^{\underline{\alpha-1}} f(s+\alpha-1), \end{aligned} \tag{2.3}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$, *where*

$$\begin{aligned} \Lambda &= (3-\alpha)\big[(\alpha-2)(2-\gamma)+2\big] + (T+\alpha)\big[2-(T-\alpha+5)(2-\gamma)\big] \\ &\quad + \frac{\rho}{\Gamma(\omega)}\sum_{s=\alpha-3}^{\eta}\big(\eta+\omega-\sigma(s)\big)^{\underline{\omega-1}}\big((\alpha-3)\big[(\alpha-2)(2-\gamma)+2\big] + (\eta+\omega) \\ &\quad \times \big[(\eta+\omega-2\alpha+5)(2-\gamma)-2\big]\big). \end{aligned} \tag{2.4}$$

Proof Using the fractional sum of order $\alpha \in (2,3)$ for (2.1) and from Lemma 2.2, we obtain

$$u(t) = C_1 + C_2 t^{\underline{1}} + C_3 t^{\underline{2}} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\big(t-\sigma(s)\big)^{\underline{\alpha-1}} f(s+\alpha-1), \tag{2.5}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

By substituting $t = \alpha-3$ into (2.5) and employing the first condition of (2.2), we obtain

$$C_1 + C_2(\alpha-3)^{\underline{1}} + C_3(\alpha-3)^{\underline{2}} = 0. \tag{2.6}$$

Using the Caputo fractional difference of order $0 < \gamma < 1$ for (2.5), we obtain

$$\begin{aligned} \Delta_C^{\gamma} u(t) &= \frac{1}{\Gamma(1-\gamma)}\sum_{s=\alpha-3}^{t+\gamma-1}\big(t-\sigma(s)\big)^{\underline{-\gamma}}\big[C_2 + 2C_3 s^{\underline{1}}\big] \\ &\quad + \frac{1}{\Gamma(1-\gamma)\Gamma(\alpha-1)}\sum_{s=\alpha-1}^{t+\gamma-1}\sum_{\xi=0}^{s-\alpha+1}\big(t-\sigma(s)\big)^{\underline{-\gamma}}\big(s-\sigma(\xi)\big)^{\underline{\alpha-2}} f(\xi+\alpha-1), \end{aligned} \tag{2.7}$$

for $t \in \mathbb{N}_{\alpha-\gamma-2,T+\alpha-\gamma+1}$.

By substituting $t = \alpha - \gamma - 1$ into (2.7) and employing the second condition of (2.2) implies

$$(2-\gamma)C_2 + 2\big[1 + (\alpha - 3)(2-\gamma)\big]C_3 = 0. \tag{2.8}$$

Finally, taking the fractional sum of order $0 < \omega < 1$ for (2.5), we obtain

$$\begin{aligned}
\Delta^{-\omega}u(t) &= \frac{1}{\Gamma(\omega)}\sum_{s=\alpha-3}^{t-\omega}\big(t-\sigma(s)\big)^{\underline{\omega-1}}\big[C_1 + C_2 s^{\underline{1}} + C_3 s^{\underline{2}}\big] \\
&\quad + \frac{1}{\Gamma(\omega)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\omega}\sum_{\xi=0}^{s-\alpha}\big(t-\sigma(s)\big)^{\underline{\omega-1}}\big(s-\sigma(\xi)\big)^{\underline{\alpha-1}}f(\xi+\alpha-1) \\
&= \frac{1}{\Gamma(\omega)}\sum_{s=\alpha-3}^{t-\omega}\big(t-\sigma(s)\big)^{\underline{\omega-1}}\big[C_1 + C_2 s^{\underline{1}} + C_3 s^{\underline{2}}\big] \\
&\quad + \frac{1}{\Gamma(\omega)\Gamma(\alpha)}\sum_{s=0}^{t-\alpha-\omega}\sum_{\xi=s}^{t-\alpha-\omega}\big(t-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}} \\
&\quad \times \big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}f(s+\alpha-1),
\end{aligned} \tag{2.9}$$

for $t \in \mathbb{N}_{\alpha-\gamma+\omega-2,T+\alpha-\gamma+\omega+1}$.

By substituting $t = T + \alpha, \eta + \omega$ into (2.5) ,(2.9), respectively, and employing the last condition of (2.2) implies

$$\begin{aligned}
&\left[1 - \frac{\rho}{\Gamma(\omega)}\sum_{s=\alpha-3}^{\eta}\big(\eta+\omega-\sigma(s)\big)^{\underline{\omega-1}}\right]C_1 \\
&\quad + \left[(T+\alpha)^{\underline{1}} - \frac{\rho}{\Gamma(\omega)}\sum_{s=\alpha-3}^{\eta}\big(\eta+\omega-\sigma(s)\big)^{\underline{\omega-1}}(\eta+\omega)^{\underline{1}}\right]C_2 \\
&\quad + \left[(T+\alpha)^{\underline{2}} - \frac{\rho}{\Gamma(\omega)}\sum_{s=\alpha-3}^{\eta}\big(\eta+\omega-\sigma(s)\big)^{\underline{\omega-1}}(\eta+\omega)^{\underline{2}}\right]C_3 \\
&= -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}\big(t+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}f(s+\alpha-1) \\
&\quad + \frac{\rho}{\Gamma(\omega)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta-\alpha}\sum_{\xi=s}^{s-\alpha}\big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}} \\
&\quad \times \big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}f(s+\alpha-1).
\end{aligned} \tag{2.10}$$

The constants C_1, C_2 and C_3 can be obtained by solving the system of equations (2.6), (2.8) and (2.10),

$$\begin{aligned}
C_1 &= \frac{1}{\Lambda}(\alpha-3)\big[(\alpha-2)(2-\gamma)+2\big]\mathcal{Q}[f], \\
C_2 &= \frac{2}{\Lambda}\big[(3-\alpha)(2-\gamma)-1\big]\mathcal{Q}[f], \\
C_3 &= \frac{1}{\Lambda}\mathcal{Q}[f],
\end{aligned}$$

where Λ is defined by (2.4) and the functional $\mathcal{Q}[f]$ is defined as

$$\begin{aligned}\mathcal{Q}[f] &= \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}f(s+\alpha-1)-\frac{\rho}{\Gamma(\omega)\Gamma(\alpha)}\\&\quad\times\sum_{s=0}^{\eta-\alpha}\sum_{\xi=s}^{\eta-\alpha}\big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}}\big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}f(s+\alpha-1).\end{aligned}$$

Substituting the constants C_1, C_2 and C_3 into (2.5), we obtain (2.3). □

Corollary 2.1 *Problem* (2.1)-(2.2) *has a unique solution of the form*

$$u(t)=\sum_{s=0}^{T}G(t,s)h(s+\alpha-1), \tag{2.11}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}g_1(t,s), & s\in\mathbb{N}_{0,t-\alpha}\cap\mathbb{N}_{0,\eta-\alpha},\\ g_2(t,s), & s\in\mathbb{N}_{t-\alpha+1,\eta-\alpha},\\ g_3(t,s), & s\in\mathbb{N}_{\eta-\alpha+1,t-\alpha},\\ g_4(t,s), & s\in\mathbb{N}_{t-\alpha+1,T}\cap\mathbb{N}_{\eta-\alpha+1,T},\end{cases} \tag{2.12}$$

with $g_i(t,s)$, $1\le i\le 4$, *as*

$$\begin{aligned}g_1(t,s) &=: \frac{1}{\Lambda}\big[(\alpha-3)\big((\alpha-2)(2-\gamma)+2\big)+2\big((3-\alpha)(2-\gamma)-1\big)t^{\underline{1}}+t^{\underline{2}}\big]\\&\quad\times\Bigg[\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}-\frac{\rho}{\Gamma(\omega)}\sum_{\xi=s}^{\eta-\alpha}\big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}}\big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}\Bigg]\\&\quad+\big(t-\sigma(s)\big)^{\underline{\alpha-1}},\\ g_2(t,s) &=: \frac{1}{\Lambda}\big[(\alpha-3)\big((\alpha-2)(2-\gamma)+2\big)+2\big((3-\alpha)(2-\gamma)-1\big)t^{\underline{1}}+t^{\underline{2}}\big]\\&\quad\times\Bigg[\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}-\frac{\rho}{\Gamma(\omega)}\sum_{\xi=s}^{\eta-\alpha}\big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}}\big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}\Bigg],\\ g_3(t,s) &=: \frac{1}{\Lambda}\big[(\alpha-3)\big((\alpha-2)(2-\gamma)+2\big)+2\big((3-\alpha)(2-\gamma)-1\big)t^{\underline{1}}+t^{\underline{2}}\big]\\&\quad\times\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}+\big(t-\sigma(s)\big)^{\underline{\alpha-1}},\\ g_4(t,s) &=: \frac{1}{\Lambda}\big[(\alpha-3)\big((\alpha-2)(2-\gamma)+2\big)+2\big((3-\alpha)(2-\gamma)-1\big)t^{\underline{1}}+t^{\underline{2}}\big]\\&\quad\times\big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}}.\end{aligned}$$

Lemma 2.4 ([22]) *A bounded set in* $\mathbb{R}^n$ *is relatively compact*; *a closed bounded set in* $\mathbb{R}^n$ *is compact.*

Lemma 2.5 ([22]) *If a set is closed and relatively compact, then it is compact.*

Lemma 2.6 (Schauder fixed point theorem [22]) *Assume that K is a convex compact set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.*

3 Main results

In this section, we wish to establish the existence results for the problem (1.1). To accomplish this, we define the Banach space

$$\mathcal{X} = \{u : u \in C(\mathbb{N}_{\alpha-r-3,T+\alpha}, \mathbb{R}), \Delta_C^{\beta} u \in C(\mathbb{N}_{\alpha-\beta-r-2,T+\alpha-\beta+1}, \mathbb{R}), 0 < \beta < 1\}$$

with the norm defined by

$$\|u\|_{\mathcal{X}} = \|u\| + \|\Delta_C^{\beta} u\|, \tag{3.1}$$

where $\|u\| = \max_{t\in\mathbb{N}_{\alpha-r-3,T+\alpha}} |u(t)|$ and $\|\Delta_C^{\beta} u\| = \max_{t\in\mathbb{N}_{\alpha-r-3,T+\alpha}} |\Delta_C^{\nu} u(t-\beta+1)|$.

For $u_{\alpha-3} = \psi$, in view of the definitions of u_t and ψ, we obtain

$$u_{\alpha-3} = u_{\alpha-3}(\theta) = u(\theta+\alpha-3) = \psi(\theta+\alpha-3) \quad \text{for } \theta \in \mathbb{N}_{-r,0}. \tag{3.2}$$

Thus, we have

$$u(t) = \psi(t) \quad \text{for } t \in \mathbb{N}_{\alpha-r-3,\alpha-3}. \tag{3.3}$$

Since $\mathcal{F} \in C(\mathbb{N}_{\alpha-3,T+\alpha} \times C_r \times \mathbb{R}, \mathbb{R})$, set $\mathcal{F}[t, u_t, \Delta_C^{\beta} u(t-\beta+1)] := f(t)$ in Lemma 2.3. We see by Lemma 2.3 that a function u is a solution of boundary value problem (1.1) if and only if it satisfies

$$u(t) = \begin{cases} \sum_{s=0}^{T} G(t,s)\mathcal{F}[s+\alpha-1, u_{s+\alpha-1}, \Delta_C^{\beta} u(s+\alpha-\beta)], & t \in \mathbb{N}_{\alpha-3,T+\alpha}, \\ \psi(t), & t \in \mathbb{N}_{\alpha-r-3,\alpha-3}. \end{cases} \tag{3.4}$$

Define an operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ as follows:

$$(\mathcal{T}u)(t) = \begin{cases} \sum_{s=0}^{T} G(t,s)\mathcal{F}[s+\alpha-1, u_{s+\alpha-1}, \Delta_C^{\beta} u(s+\alpha-\beta)], & t \in \mathbb{N}_{\alpha-3,T+\alpha}, \\ \psi(t), & t \in \mathbb{N}_{\alpha-r-3,\alpha-3}, \end{cases} \tag{3.5}$$

and

$$\Theta = \max_{t\in\mathbb{N}_{\alpha-r-3,\alpha-3}} \left\{ \sum_{s=0}^{T} \left| G(t,s)\phi(s+\alpha-1) \right| \right\}, \tag{3.6}$$

$$\widetilde{\Theta} = \max_{t\in\mathbb{N}_{\alpha-r-3,\alpha-3}} \left\{ \sum_{s=0}^{T} \left| {}_t\Delta G(t-\beta+1,s)\phi(s+\alpha-1) \right| \right\}, \tag{3.7}$$

$$\begin{aligned} \Upsilon = {} & \frac{1}{|\Lambda|\Gamma(2-\beta)} \left| \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)} \right| \big\{ (T+\alpha)\Gamma(2-\beta) \\ & \times \big[(3-\gamma)T + \alpha + 2(1-\gamma)\big] + 2(T+\alpha) + 2\big[(3-\alpha)(2-\gamma)-1\big]\big\} \\ & + \frac{(T+\alpha)^{\underline{\alpha+1}}}{\Gamma(\alpha)} \left\{ 1 + \frac{\alpha(T+\alpha-\beta+1)^{\underline{1-\beta}}}{(T+1)\Gamma(2-\beta)} \right\}. \end{aligned} \tag{3.8}$$

Theorem 3.1 *Assume the following properties*:

(A1) *There exists a nonnegative function* $\phi \in C(\mathbb{N}_{\alpha-3,T+\alpha})$ *such that*

$$\left|\mathcal{F}[t,x,y]\right| \leq \phi(t) + \lambda_1|x|^{\tau_1} + \lambda_2|y|^{\tau_2},$$

for each $x \in C_r$, $y \in \mathbb{R}$ *where* λ_1, λ_2 *are negative constants and* $0 < \tau_1, \tau_2 < 1$; *or*

(A2) *there exists a nonnegative function* $\phi \in C(\mathbb{N}_{\alpha-3,T+\alpha})$ *such that*

$$\left|\mathcal{F}[t,x,y]\right| \leq \phi(t) + \lambda_1|x|^{\tau_1} + \lambda_2|y|^{\tau_2},$$

for each $x \in C_r$, $y \in \mathbb{R}$ *where* λ_1, λ_2 *are negative constants and* $\tau_1, \tau_2 > 1$.

Then boundary value problem (1.1) *has at least one solution*.

Proof We shall use the Schauder fixed point theorem to prove that the operator $\mathcal{T}$ defined by (3.5) has a fixed point. We divide the proof into three steps.

Step I. Verify $\mathcal{T}$ maps bounded sets into bounded sets.

Suppose (A1) holds, choose

$$L \geq \max\left\{3\left(\Theta + \widetilde{\Theta}\frac{(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\right), (3\lambda_1\Upsilon)^{\frac{1}{1-\tau_1}}, (3\lambda_2\Upsilon)^{\frac{1}{1-\tau_2}}\right\}, \tag{3.9}$$

and define the $\mathcal{P} = \{u \in \mathcal{X} : \|u\| \leq L, L > 0\}$.

For any $u \in \mathcal{P}$, we obtain

$$\begin{aligned}
&\left|(\mathcal{T}u)(t)\right| \\
&\quad= \left|\sum_{s=0}^{T} G(t,s)\mathcal{F}\left[s+\alpha-1, u_{s+\alpha-1}, \Delta_C^{\beta}u(s+\alpha-\beta)\right]\right| \\
&\quad\leq \sum_{s=0}^{T}\left|G(t,s)\phi(s+\alpha-1)\right| + \left(\lambda_1|u_{s+\alpha-1}|^{\tau_1} + \lambda_2\left|\Delta_C^{\beta}u(s+\alpha-\beta)\right|^{\tau_2}\right) \\
&\qquad\times\left\{\frac{1}{|\Lambda|}\left|(\alpha-3)\left((\alpha-2)(2-\gamma)+2\right) + 2\left((3-\alpha)(2-\gamma)-1\right)t^{\underline{1}} + t^{\underline{2}}\right|\right. \\
&\qquad\times\left|\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}\left(T+\alpha-\sigma(s)\right)^{\underline{\alpha-1}} - \frac{\rho}{\Gamma(\omega)\Gamma(\alpha)}\right. \\
&\qquad\times\left.\left.\sum_{s=0}^{\eta-\alpha}\sum_{\xi=s}^{\eta-\alpha}\left(\eta+\omega-\alpha-\sigma(\xi)\right)^{\underline{\omega-1}}\left(\xi+\alpha-\sigma(s)\right)^{\underline{\alpha-1}}\right| + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}\left(t-\sigma(s)\right)^{\underline{\alpha-1}}\right\} \\
&\quad\leq \Theta + \left(\lambda_1|u_{s+\alpha-1}|^{\tau_1} + \lambda_2\left|\Delta_C^{\beta}u(s+\alpha-\beta)\right|^{\tau_2}\right)\left\{\frac{1}{|\Lambda|}\left|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\right|\right. \\
&\qquad\times\left.\left[(T+\alpha)\left[(3-\gamma)T+\alpha+2(1-\gamma)\right]\right] + \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right\}.
\end{aligned}$$

Next, we consider

$$\begin{aligned}
&\left|({}_t\Delta\mathcal{T}u)(t)\right| \\
&\quad\leq \sum_{s=0}^{T}\left|{}_t\Delta G(t,s)\right|\left|\mathcal{F}\left[s+\alpha-1, u_{s+\alpha-1}, \Delta_C^{\beta}u(s+\alpha-\beta)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{s=0}^{T}\bigl|{}_t\Delta G(t,s)\phi(s+\alpha-1)\bigr| + \bigl(\lambda_1|u_{s+\alpha-1}|^{\tau_1} + \lambda_2\bigl|\Delta_C^{\beta}u(s+\alpha-\beta)\bigr|^{\tau_2}\bigr)\\
&\quad\times\Biggl\{\frac{1}{|\Lambda|}\bigl|2\bigl((3-\alpha)(2-\gamma)-1\bigr)+2t^{\underline{1}}\bigr|\Biggl|\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}\bigl(T+\alpha-\sigma(s)\bigr)^{\underline{\alpha-1}}\\
&\quad-\frac{\rho}{\Gamma(\omega)\Gamma(\alpha)}\sum_{s=0}^{\eta-\alpha}\sum_{\xi=s}^{\eta-\alpha}\bigl(\eta+\omega-\alpha-\sigma(\xi)\bigr)^{\underline{\omega-1}}\bigl(\xi+\alpha-\sigma(s)\bigr)^{\underline{\alpha-1}}\Biggr|+\frac{t^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Biggr\}\\
&\leq \widetilde{\Theta} + \bigl(\lambda_1|u_{s+\alpha-1}|^{\tau_1} + \lambda_2\bigl|\Delta_C^{\beta}u(s+\alpha-\beta)\bigr|^{\tau_2}\bigr)\\
&\quad\times\Biggl\{\frac{2}{|\Lambda|}\Biggl|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}-\frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\Biggr|\\
&\quad\times\bigl[T+\alpha+(3-\alpha)(2-\gamma)-1\bigr]+\frac{(T+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\Biggr\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|\bigl(\Delta_C^{\beta}\mathcal{T}u\bigr)(t-\beta+1)\bigr|\\
&\quad\leq\frac{1}{\Gamma(1-\beta)}\sum_{s=0}^{t}\bigl(t-\beta+1-\sigma(s)\bigr)^{\underline{-\beta}}\bigl|(\Delta\mathcal{T}u)(s)\bigr|\\
&\quad\leq\frac{(t-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\Biggl\{\widetilde{\Theta}+\frac{(\lambda_1|u_{s+\alpha-1}|^{\tau_1}+\lambda_2|\Delta_C^{\beta}u(s+\alpha-\beta)|^{\tau_2})}{\Gamma(2-\beta)}\Biggl[\frac{(T+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\\
&\qquad+\frac{1}{|\Lambda|}\Biggl|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}-\frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\Biggr|\bigl|2(T+\alpha)+2\bigl[(3-\alpha)(2-\gamma)-1\bigr]\bigr|\Biggr]\Biggr\}\\
&\quad\leq\frac{(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\Biggl\{\widetilde{\Theta}+\frac{(\lambda_1|u_{s+\alpha-1}|^{\tau_1}+\lambda_2|\Delta_C^{\beta}u(s+\alpha-\beta)|^{\tau_2})}{\Gamma(2-\beta)}\Biggl[\frac{(T+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\\
&\qquad+\frac{2}{|\Lambda|}\Biggl|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}-\frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\Biggr|\bigl[T+\alpha+(3-\alpha)(2-\gamma)-1\bigr]\Biggr]\Biggr\}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\bigl\|\mathcal{T}u(t)\bigr\|_{\mathcal{X}}&\leq\Theta+\frac{\widetilde{\Theta}(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\\
&\quad+\bigl(\lambda_1|u_{s+\alpha-1}|^{\tau_1}+\lambda_2\bigl|\Delta_C^{\beta}u(s+\alpha-\beta)\bigr|^{\tau_2}\bigr)\Upsilon\\
&\leq\frac{L}{3}+\bigl(\lambda_1|u_{s+\alpha-1}|^{\tau_1}+\lambda_2\bigl|\Delta_C^{\beta}u(s+\alpha-\beta)\bigr|^{\tau_2}\bigr)\Upsilon\\
&\leq\frac{L}{3}+\frac{L}{3}+\frac{L}{3}=L,
\end{aligned}\tag{3.10}
$$

which implies that $\mathcal{T}:\mathcal{P}\to\mathcal{P}$.

For the second cases, if (A2) holds, choose

$$
L\geq\max\Biggl\{3\biggl(\Theta+\widetilde{\Theta}\frac{(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\biggr),\biggl(\frac{1}{3\lambda_1\Upsilon}\biggr)^{\frac{1}{1-\tau_1}},\biggl(\frac{1}{3\lambda_2\Upsilon}\biggr)^{\frac{1}{1-\tau_2}}\Biggr\},\tag{3.11}
$$

and by the same argument as above, we obtain

$$\begin{aligned}\|\mathcal{T}u(t)\|_{\mathcal{X}} &\leq \Theta + \frac{\widetilde{\Theta}(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)} + \bigl(\lambda_1|u_{s+\alpha-1}|^{\tau_1} + \lambda_2\bigl|\Delta_C^{\beta}u(s+\alpha-\beta)\bigr|^{\tau_2}\bigr)\Upsilon \\ &\leq \frac{L}{3}+\frac{L}{3}+\frac{L}{3} = L, \end{aligned} \tag{3.12}$$

which implies that $\mathcal{T}:\mathcal{P}\to\mathcal{P}$.

Step II. The continuity of the operator $\mathcal{T}$ follows from the continuity of $\mathcal{F}$ and G.

Step III. By Lemma 2.4 and Lemma 2.5, $\mathcal{P}$ is compact.

Hence, by the Schauder fixed point theorem, we can conclude that problem (1.1) has at least one solution. The proof is completed. □

The second result is the existence and uniqueness of a solution to problem (1.1), by using the Banach contraction principle.

Theorem 3.2 *Assume the following properties*:

(A3) *There exists a constant* $\kappa > 0$ *such that*

$$\bigl|\mathcal{F}[t,u_1,u_2] - \mathcal{F}[t,v_1,v_2]\bigr| \leq \kappa\bigl(|u_1-v_1| + |u_2-v_2|\bigr),$$

for each $u_1, v_1 \in C_r$, $u_2, v_2 \in \mathbb{R}$.

(A4) $\kappa(\Omega_1+\Omega_2) < 1$, *where*

$$\begin{aligned}\Omega_1 &= \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|}\left|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\right| \\ &\quad\times (T+\alpha)\bigl[(3-\gamma)T+\alpha+2(1-\gamma)\bigr], \end{aligned} \tag{3.13}$$

$$\begin{aligned}\Omega_2 &= \frac{(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)}\left|\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)}\right| \\ &\quad\times\left\{\frac{2}{|\Lambda|}\bigl[T+\alpha+(3-\alpha)(2-\gamma)-1\bigr] + \frac{(T+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)}\right\}. \end{aligned} \tag{3.14}$$

Then the problem (1.1) *has a unique solution*.

Proof Consider the operator $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ defined by (3.5). Clearly, the fixed point of the operator $\mathcal{T}$ is the solution of boundary value problem (1.1). We will use the Banach contraction principle to prove that $\mathcal{T}$ has a fixed point. We first show that $\mathcal{T}$ is a contraction. For each $t\in\mathbb{N}_{\alpha-3,T+\alpha}$, we have

$$\begin{aligned}&\bigl|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\bigr| \\ &\quad= \sum_{s=0}^{T}\bigl|G(t,s)\bigr|\bigl|\mathcal{F}\bigl[s+\alpha-1,u_{s+\alpha-1},\Delta_C^{\beta}u(s+\alpha-\beta)\bigr] \\ &\qquad - \mathcal{F}\bigl[s+\alpha-1,v_{s+\alpha-1},\Delta_C^{\beta}v(s+\alpha-\beta)\bigr]\bigr| \\ &\quad\leq \kappa\|u-v\|_{\mathcal{X}}\Bigl\{\bigl|(\alpha-3)\bigl((\alpha-2)(2-\gamma)+2\bigr) + 2\bigl((3-\alpha)(2-\gamma)-1\bigr)t^{\underline{1}} + t^{\underline{2}}\bigr|\end{aligned}$$

$$
\begin{aligned}
&\times \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{T} \big(T+\alpha-\sigma(s)\big)^{\underline{\alpha-1}} - \frac{\rho}{\Gamma(\omega)\Gamma(\alpha)} \right. \\
&\left. \times \sum_{s=0}^{\eta-\alpha} \sum_{\xi=s}^{\eta-\alpha} \big(\eta+\omega-\alpha-\sigma(\xi)\big)^{\underline{\omega-1}} \big(\xi+\alpha-\sigma(s)\big)^{\underline{\alpha-1}} \right| + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} \big(t-\sigma(s)\big)^{\underline{\alpha-1}} \Bigg\} \\
&\leq \kappa \|u-v\|_{\mathcal{X}} \left\{ \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{1}{|\Lambda|} \left| \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)} \right| \right. \\
&\left. \quad \times (T+\alpha)\big[(3-\gamma)T+\alpha+2(1-\gamma)\big] \right\} \\
&= \kappa \|u-v\|_{\mathcal{X}} \Omega_1.
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
&\big|\big(\Delta_C^{\beta}\mathcal{T}u\big)(t-\beta+1) - \big(\Delta_C^{\beta}\mathcal{T}v\big)(t-\beta+1)\big| \\
&\leq \left| \frac{1}{\Gamma(1-\beta)} \sum_{s=0}^{t} \big(t-\beta+1-\sigma(s)\big)^{\underline{-\beta}} \big[(\Delta\mathcal{T}u)(s) - (\Delta\mathcal{T}v)(s)\big] \right| \\
&\leq \frac{1}{\Gamma(1-\beta)} \sum_{s=0}^{t} \big(t-\beta+1-\sigma(s)\big)^{\underline{-\beta}} \Bigg[\sum_{\xi=0}^{T} \big|{}_s\Delta G(s,\xi)\big| \\
&\quad \times \big|\mathcal{F}\big[\xi+\alpha-1, u_{\xi+\alpha-1}, \Delta_C^{\beta}u(\xi+\alpha-1-\beta+1)\big] - \mathcal{F}\big[\xi+\alpha-1, v_{\xi+\alpha-1}, \\
&\quad \Delta_C^{\beta}v(\xi+\alpha-1-\beta+1)\big]\big| \Bigg] \\
&\leq \kappa \|u-v\|_{\mathcal{X}} \frac{1}{\Gamma(1-\beta)} \sum_{s=0}^{t} \big(t-\beta+1-\sigma(s)\big)^{\underline{-\beta}} \left[\sum_{\xi=0}^{T} \big|{}_s\Delta G(s,\xi)\big| \right] \\
&\leq \kappa \|u-v\|_{\mathcal{X}} \frac{(T+\alpha-\beta+1)^{\underline{1-\beta}}}{\Gamma(2-\beta)} \left| \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)} - \frac{\rho(T+\alpha+\omega)}{\Gamma(T)\Gamma(\alpha+\omega+1)} \right| \\
&\quad \times \left\{ \frac{2}{|\Lambda|}\big[T+\alpha+(3-\alpha)(2-\gamma)-1\big] + \frac{(T+\alpha)^{\underline{\alpha-1}}}{\Gamma(\alpha)} \right\} \\
&= \kappa \|u-v\|_{\mathcal{X}} \Omega_2.
\end{aligned}
$$

Obviously, for each $t \in \mathbb{N}_{\alpha-r-3,\alpha-3}$, we have $|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| = 0$.

Therefore, we obtain

$$
\big\|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)\big\|_{\mathcal{X}} \leq \kappa \|x-y\|_{\mathcal{X}} (\Omega_1 + \Omega_2).
$$

By (A4) implies $\mathcal{T}$ is a contraction. Hence, by the Banach contraction principle, we see that $\mathcal{T}$ has a fixed point which is a unique solution of the problem (1.1). □

4 Examples

In this section, to illustrate our results, we consider some examples.

Example 4.1 Consider the following fractional difference boundary value problem:

$$\begin{aligned} &\Delta_C^{\frac{5}{2}}u(t) = \left(t+\frac{3}{2}\right)^{\underline{2}} + \frac{e^{-(t+\frac{3}{2})}}{(t+\frac{5}{2})^3}|u_{t+\frac{3}{2}}|^{\tau_1} + e^{-2(t+\frac{5}{2})}\left|\Delta_C^{\frac{2}{3}}u\left(t+\frac{11}{6}\right)\right|^{\tau_2}, \quad t\in\mathbb{N}_{0,5}, \\ &u\left(-\frac{1}{2}\right) = \Delta_C^{\frac{3}{4}}u\left(\frac{3}{4}\right) = 0, \qquad u\left(\frac{15}{2}\right) = \frac{1}{2}\Delta^{-\frac{4}{3}}u\left(\frac{43}{10}\right). \end{aligned} \tag{4.1}$$

Set $\alpha = \frac{5}{2}$, $\beta = \frac{2}{3}$, $\gamma = \frac{3}{4}$, $\omega = \frac{4}{5}$, $\eta = \frac{7}{2}$, $T = 5$,

$$\begin{aligned} \rho &= \frac{1}{2} \neq 1.068 \\ &= \frac{\Gamma(\omega)((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(T+\alpha)[(T-\alpha+5)(2-\gamma)-2])}{\sum_{s=\alpha-3}^{\eta}(\eta+\omega-\sigma(s))^{\underline{\omega-1}}((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(\eta+\omega)[(\eta+\omega-2\alpha+5)(2-\gamma)-2])}, \end{aligned}$$

and $\mathcal{F}[t,u_t,\Delta_C^{\beta}u_{t-\beta+1}] = t^{\underline{2}} + \frac{e^{-t}}{(t+1)^3}|u_t|^{\tau_1} + e^{-2(t+1)}|\Delta_C^{\frac{2}{3}}u_{t+\frac{1}{3}}|^{\tau_2}$.

For $t\in\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we have

$$\left|\mathcal{F}\left[t,u_t,\Delta_C^{\beta}u_{t-\beta+1}\right]\right| \le \left(\frac{15}{2}\right)^{\underline{2}} + \frac{8}{\sqrt{e}}|u_t|^{\tau_1} + \frac{1}{e}\left|\Delta_C^{\frac{2}{3}}u\left(t+\frac{1}{3}\right)\right|^{\tau_2},$$

so $|\phi(t)| \le \frac{195}{4}$, $\lambda_1 = \frac{8}{\sqrt{e}}$, $\lambda_2 = \frac{1}{e}$. For $0<\tau_1,\tau_2<1$, (A1) is satisfied and for $\tau_1,\tau_2>1$, (A2) is satisfied. Therefore, by Theorem 3.1, boundary value problem (4.1) has at least one solution.

Example 4.2 Consider the following fractional difference boundary value problem:

$$\begin{aligned} &\Delta_C^{\frac{5}{2}}u(t) = \frac{|u_{t+\frac{3}{2}}| + |\Delta_C^{\frac{2}{3}}u(t+\frac{11}{6})|}{(t+\frac{403}{2})^2[1+|u_{t+\frac{3}{2}}|+|\Delta_C^{\frac{2}{3}}u(t+\frac{11}{6})|]}, \quad t\in\mathbb{N}_{0,5}, \\ &u\left(-\frac{1}{2}\right) = \Delta_C^{\frac{3}{4}}u\left(\frac{3}{4}\right) = 0, \qquad u\left(\frac{15}{2}\right) = \frac{1}{2}\Delta^{-\frac{4}{3}}u\left(\frac{43}{10}\right). \end{aligned} \tag{4.2}$$

Set $\alpha = \frac{5}{2}$, $\beta = \frac{2}{3}$, $\gamma = \frac{3}{4}$, $\omega = \frac{4}{5}$, $\eta = \frac{7}{2}$, $T = 5$,

$$\begin{aligned} \rho &= \frac{1}{2} \neq 1.068 \\ &= \frac{\Gamma(\omega)((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(T+\alpha)[(T-\alpha+5)(2-\gamma)-2])}{\sum_{s=\alpha-3}^{\eta}(\eta+\omega-\sigma(s))^{\underline{\omega-1}}((\alpha-3)[(\alpha-2)(2-\gamma)+2]+(\eta+\omega)[(\eta+\omega-2\alpha+5)(2-\gamma)-2])}, \end{aligned}$$

and $\mathcal{F}[t,u_t,\Delta_C^{\beta}u_{t-\beta+1}] = \frac{|u_t|+|\Delta_C^{\frac{2}{3}}u(t+\frac{1}{3})|}{(t+200)^2[1+|u_t|+|\Delta_C^{\frac{2}{3}}u(t+\frac{1}{3})|]}$.

For $t\in\mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we have

$$\left|\mathcal{F}\left[t,u_t,\Delta_C^{\beta}u\right] - \mathcal{F}\left[t,v_t,\Delta_C^{\beta}v\right]\right| \le \frac{4}{159{,}201}\left[|u_t-v_t| + \left|\Delta_C^{\frac{2}{3}}u\left(t+\frac{1}{3}\right) - \Delta_C^{\frac{2}{3}}v\left(t+\frac{1}{3}\right)\right|\right],$$

and for $t \in \mathbb{N}_{-\frac{1}{2},\frac{15}{2}}$, we have

$$\left|\mathcal{F}\left[t,u_t,\Delta_C^\beta u\right]-\mathcal{F}\left[t,v_t,\Delta_C^\beta v\right]\right| \le \frac{4}{148{,}225}\left[|u_t-v_t|+\left|\Delta_C^{\frac{2}{3}}u\left(t+\frac{1}{3}\right)-\Delta_C^{\frac{2}{3}}v\left(t+\frac{1}{3}\right)\right|\right],$$

so (A3) holds with $\kappa = \frac{4}{159{,}201}$. Also, we can show that

$$|\Lambda| \approx 28.719, \qquad \Omega_1 \approx 116.027, \qquad \Omega_2 \approx 1{,}202.136,$$

and

$$\kappa(\Omega_1+\Omega_2) \approx 0.0331 < 1.$$

Therefore (A4) holds, by Theorem 3.2, boundary value problem (4.2) has a unique solution.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that they carried out all the work in this manuscript and read and approved the final manuscript.

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