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Non-permanence for three-species Lotka-Volterra cooperative difference systems

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Abstract
The permanence for three-species cooperative difference systems of Lotka-Volterra is considered. By constructing some suitable regions, we prove that a cooperative system cannot be permanent. Furthermore, we find the orbits starting from our regions approach the coordinate axes as *n* tends to infinity. This is somewhat similar to the May-Leonard behavior in the case of competition models.

Keywords: Lotka-Volterra system; cooperative; difference system; permanence

1 Introduction

In theoretical ecology, the models governed by difference equations are used to characterize the interactions of species with non-overlapping generations. For example, Ricker [1] introduced a discrete-time population model to forecast fish stock recruitment given by

$$x(k+1) = x(k)\exp\left(r - \frac{x(k)}{K}\right), \tag{1.1}$$

where $x(k)$ represents the number of individuals of generation k, r is the intrinsic growth rate and K is interpreted as the carrying capacity of the environment. It is well known that the dynamics of model (1.1) may be extremely complex [2]. However, one of the most important questions from a biological point of view is whether or not all species in a multi-species community can be permanent.

The discrete-time model of Ricker type of n species is modeled by the following equation:

$$x_i(k+1) = x_i(k)\exp\left(r_i - \sum_{j=1}^{n} a_{ij}x_j(k)\right), \tag{1.2}$$

where $x_i(k)$ represents the densities of the i species at kth generation, a_{ij} measures the intensity of intra-specific competition or the interspecific competition. These equations are also called the discrete-time Lotka-Volterra systems. Permanence is an important concept of mathematical biology concerning the survival of a population which has been studied in many papers [3–7]. Here, we give its definition as follows.

Definition 1.1 System (1.2) is said to be permanent if there is a compact set E in the interior of R_+^n such that, for each positive initial position, the orbit of system (1.2) through this initial position eventually enters and remains in E. Equivalently, there exist constants $M, m > 0$, such that, for each initial value $(x_1(0), x_2(0), \dots, x_n(0)) \in \operatorname{int} R^n_+$, the orbits of system (1.2) starting from the initial value satisfy

$$0 < m \le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M, \quad 1 \le i \le n. \tag{1.3}$$

For system (1.2) in the n-dimensional competitive and prey-predator cases, in [3], Hofbauer *et al.* obtained conditions for permanence of the system (1.2). For system (1.2) in the two-species prey-predator case, Hutson and Moran [5] gave necessary and sufficient conditions for its permanence. For the two-species competition system of equation (1.2), the permanence was considered by Lu and Wang [6]. For higher-dimensional case of equation (1.2), a three-species competitive system of the May-Leonard type was studied by Roeger in [8]. Global stability result of system (1.2) was given by Wang and Lu in [9]. For system (1.2) in the two-dimensional cooperative case, an example was given by Hofbauer *et al.* [3] to show that system (1.2) may not be permanent. By applying the approach of Hofbauer *et al.* in [3], Lu and Wang [6] proved that any two-species cooperative system of equation (1.2) cannot be permanent. For the high-dimensional cooperative case, the technique developed in [4, 6] cannot easily be extended, Wei *et al.* [7] gave an incomplete result of non-permanence for system (1.2) in the n-species cooperative case.

The above results proposed by previous studies [3, 5, 6] were extended by Kon [10] to a general two-species Kolmogorov system. By virtue of the average Lyapunov functions approach developed by Hofbauer *et al.* [3], he gave some permanence conditions and applied them to prey-predator and competitive cases. However, in the cooperative case, he suggested that the conditions of permanence may fail based on the results obtained in [6]. By introducing the constants θ_{ij}, system (1.2) in the two-species case can be extended as follows:

$$\begin{aligned} x_{n+1} &= x_n \exp\bigl(r_1 - a_{11}x_n^{\theta_{11}} + a_{12}y_n^{\theta_{12}}\bigr), \\ y_{n+1} &= y_n \exp\bigl(r_2 + a_{21}x_n^{\theta_{21}} - a_{22}y_n^{\theta_{22}}\bigr). \end{aligned} \tag{1.4}$$

Equation (1.4) is called the Gilpin and Ayala type model; it was proposed by Gilpin and Ayala in [11]. For two- and high-dimensional competitive and prey-predator cases of (1.4), the dynamical behavior was studied by Kon [10] and Chen *et al.* in [12, 13]. For system (1.4) in the cooperative case, the permanence result was left unsolved in [10].

In this paper, we consider the following three-dimensional discrete-time Lotka-Volterra systems:

$$\begin{aligned} x_{n+1} &= x_n \exp(r_1 - a_{11}x_n + a_{12}y_n + a_{13}z_n), \\ y_{n+1} &= y_n \exp(r_2 + a_{21}x_n - a_{22}y_n + a_{23}z_n), \\ z_{n+1} &= z_n \exp(r_3 + a_{31}x_n + a_{32}y_n - a_{33}z_n). \end{aligned} \tag{1.5}$$

The main object in the present paper is to complete the proof of Wei *et al.* [7] in the three-dimensional case. Furthermore, we also consider the permanence of model (1.4) in some special case. Our main results are stated as follows.

Theorem 1.1 *If* $a_{ij} > 0$ $(i,j = 1,2)$ *and* $\theta_{11} = \theta_{21} > 0$, $\theta_{12} = \theta_{22} > 0$ *are satisfied, then system* (1.4) *is not permanent.*

Theorem 1.2 *If system* (1.5) *is a cooperative one, i.e.* $a_{ij} > 0$ $(i,j = 1,2,3)$, *then it is not permanent.*

2 Preliminaries and some lemmas

To show the main results, firstly, we will give some lemmas which will be used in the sequel.

Let

$$A = \begin{pmatrix} -a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{33} \end{pmatrix},$$

where $a_{ij} > 0$.

By virtue of the Perron-Frobenius theorem, Theorem 15.1.1 and 15.1.2 in [4] and Lemma 2 in [7], it can be easily seen that we have the following.

Lemma 2.1 *If matrix A with* $a_{ij} > 0$ *satisfies* $\det(A) > 0$, *then A has a real eigenvalue* $\lambda > 0$ *and a corresponding eigenvector* $p = (p_1, p_2, p_3) > 0$ *with* $\sum_{i=1}^{3} p_i = 1$, *that is,*

$$p \cdot A = \lambda p,$$

furthermore,

$$\begin{aligned} &-a_{11}p_1 + a_{21}p_2 + a_{31}p_3 > 0, \\ &a_{12}p_1 - a_{22}p_2 + a_{32}p_3 > 0, \\ &a_{13}p_1 + a_{23}p_2 - a_{33}p_3 > 0. \end{aligned} \tag{2.1}$$

Remark 2.1 Arrow [14] and Szidarovszky [15] extended the Perron-Frobenius theorem to Metzler matrices. Here a matrix A is called a Metzler matrix if its off-diagonal entries are non-negative. Thus, Lemma 2.1 can also be obtained from Theorem 4′ in [14] or Theorem 7.12 in [15].

Lemma 2.2 ([4]) *If* $u = (u_1, u_2 \cdots u_n) \geq 0$, $v = (v_1, v_2 \cdots v_n) \geq 0$ *and* $\sum_{i=1}^{n} v_i = 1$, *then*

$$\prod_{i=1}^{n} u_i^{v_i} \leq \sum_{i=1}^{n} u_i v_i.$$

Lemma 2.3 *By considering the relationships between* a_{31} *and* a_{21}, a_{23} *and* a_{13}, a_{32} *and* a_{12}, *the interaction matrix A can be classified into the following five classes*:

(1) $a_{31} \geq a_{21}$, $a_{23} \leq a_{13}$;
(2) $a_{32} \geq a_{12}$, $a_{23} \geq a_{13}$, $a_{31} < a_{21}$; $a_{32} \geq a_{12}$, $a_{23} > a_{13}$, $a_{31} \geq a_{21}$;
(3) $a_{32} < a_{12}$, $a_{31} < a_{21}$; $a_{32} = a_{12}$, $a_{23} < a_{13}$, $a_{31} < a_{21}$; $a_{32} < a_{12}$, $a_{23} > a_{13}$, $a_{31} = a_{21}$;
(4) $a_{12} < a_{32}$, $a_{31} < a_{21}$, $a_{23} < a_{13}$;
(5) $a_{12} > a_{32}$, $a_{31} > a_{21}$, $a_{23} > a_{13}$.

Remark 2.2 Clearly, the above five cases are a complete classification of interaction matrix A in terms of the relationship between a_{31} and a_{21}, a_{23} and a_{13}, a_{32} and a_{12}.

For convenience, we rewrite system (1.5) into the following form:

$$\begin{aligned} x' &= x\exp(r_1 - a_{11}x + a_{12}y + a_{13}z), \\ y' &= y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z), \\ z' &= z\exp(r_3 + a_{31}x + a_{32}y - a_{33}z). \end{aligned} \tag{2.2}$$

Lemma 2.4 *If* $\det(A) < 0$ *and* $a_{31} \geq a_{21}$, $a_{23} \leq a_{13}$, *then the map defined by equation* (2.2) *satisfies*

$$B_N \cap I_{(M,M)} \to B_N \cap \mathrm{IV}_{(M_{\tilde{\varepsilon}},M_{\tilde{\varepsilon}})}, \tag{2.3}$$

$$B_N \cap \mathrm{IV}_{(M,M)} \to B_N \cap I_{(M_{\tilde{\varepsilon}},M_{\tilde{\varepsilon}})}, \tag{2.4}$$

where $\tilde{\varepsilon} > 1$ *and*

$$\begin{aligned} B_N &= \left\{(x,y,z) \in R_+^3 | x^\alpha y^\beta z^\gamma \leq N\right\}, \\ I_{(M,M)} &= \left\{(x,y,z) \in R_+^3 | x \geq M, y \geq M, z \geq y\exp(-ay), y \leq \frac{a-\delta}{a_{22}}x\right\}, \\ \mathrm{IV}_{(M,M)} &= \left\{(x,y,z) \in R_+^3 | y \geq M, z \geq M, x \geq y\exp(-ay), y \leq \frac{a-\delta}{a_{22}}z\right\}. \end{aligned}$$

Here $N > 0$ *is arbitrarily fixed*, $a = \frac{1}{2}\min_{i\neq j}\{a_{ij}\}$, $M > 0$ *is sufficiently large and* $0 < \delta < a$ *is a constant.*

Proof Let $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ be defined as

$$\begin{aligned} \hat{\alpha} &= (A_{11} + A_{12} + A_{13})/\left|\det(A)\right|, \\ \hat{\beta} &= (A_{21} + A_{22} + A_{23})/\left|\det(A)\right|, \\ \hat{\gamma} &= (A_{31} + A_{32} + A_{33})/\left|\det(A)\right|, \end{aligned} \tag{2.5}$$

where A_{ii} and A_{ij} $(i \neq j)$ are the algebraic cofactors of $-a_{ii}$ and a_{ij} $(i \neq j)$, respectively.

Along the orbits of (2.2), we have

$$x'^{\hat{\alpha}} y'^{\hat{\beta}} z'^{\hat{\gamma}} = x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \exp(\hat{\triangle} - x - y - z),$$

where $\hat{\triangle} = r_1\hat{\alpha} + r_2\hat{\beta} + r_3\hat{\gamma}$.

Let $(x,y,z) \in B_N \cap I_{(M,M)}$. Then, for large enough $M > 0$, we obtain

$$x'^{\hat{\alpha}} y'^{\hat{\beta}} z'^{\hat{\gamma}} \leq x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \exp(\hat{\triangle} - 2M) \leq x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \leq N. \tag{2.6}$$

Moreover, since $x \to x\exp(-ax)$ is monotonically decreasing for $x \geq M > \frac{1}{a}$, we infer that

$$y \geq M \geq M\exp(-aM) \geq x\exp(-ax),$$

together with the definition of region I, it follows that

$$\begin{aligned}
y' &= y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z) \\
&\geq x\exp(-ax)\exp(r_2 + a_{21}x - a_{22}y + a_{23}z) \\
&\geq x\exp(r_2 + ax - a_{22}y + a_{23}z) \\
&\geq x\exp\bigl(r_2 + ax - (a-\delta)x + a_{23}z\bigr) \geq x\exp(r_2 + \delta M) \geq x\exp\left(\frac{\delta}{2}M\right) \quad (M \text{ large}) \\
&\triangleq \tilde{\varepsilon}x \geq \tilde{\varepsilon}M \geq M > \frac{1}{a}.
\end{aligned} \tag{2.7}$$

From (2.6), we see that if $(x,y,z) \in B_N \cap I_{(M,M)}$, then $z \leq N^{\frac{1}{\hat{\gamma}}} M^{-(\frac{\hat{\alpha}+\hat{\beta}}{\hat{\gamma}})}$. Thus, we conclude that

$$\begin{aligned}
z' &= z\exp(r_3 + a_{31}x + a_{32}y - a_{33}z) \\
&\geq y\exp(-ay)\exp(r_3 + a_{31}x + a_{32}y - a_{33}z) \\
&\geq y\exp(r_3 + a_{31}x + ay - a_{33}z) \\
&\geq y\exp\left(r_3 + a_{31}M + aM - a_{33}\frac{N^{\frac{1}{\hat{\gamma}}}}{M^{\frac{\hat{\alpha}+\hat{\beta}}{\hat{\gamma}}}}\right) \\
&\geq y\exp\left(a_{31}M + \frac{a}{2}M\right) \geq y\exp\left(\frac{3a}{2}M\right) \geq y\exp\left(\frac{\delta}{2}M\right) \quad (M \text{ large}) \\
&\triangleq \tilde{\varepsilon}y \geq \tilde{\varepsilon}M \geq M.
\end{aligned} \tag{2.8}$$

By the fact that $y \to y\exp(-ay)$ is monotonically decreasing for $y > \frac{1}{a}$, it follows that

$$\begin{aligned}
x'/y'\exp\bigl(-ay'\bigr) &\geq x\exp(r_1 - a_{11}x + a_{12}y + a_{13}z)/\bigl(\tilde{\varepsilon}x\exp(-a\tilde{\varepsilon}x)\bigr) \\
&= \frac{1}{\tilde{\varepsilon}}\exp\bigl[r_1 + (a\tilde{\varepsilon} - a_{11})x + a_{12}y + a_{13}z\bigr] \\
&\geq \frac{1}{\tilde{\varepsilon}}\exp\bigl[r_1 + (a\tilde{\varepsilon} - a_{11})M + a_{12}M\bigr] \\
&> \exp\left[\left(a\tilde{\varepsilon} - a_{11} + a - \frac{\delta}{2}\right)M\right] \quad (M \text{ large}) \\
&> \exp\left[\frac{aM}{2}\exp\left(\frac{\delta}{2}M\right)\right] > 1 \quad (M \text{ large}), \\
z'/y' &= z\exp(r_3 + a_{31}x + a_{32}y - a_{33}z)/\bigl[y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)\bigr] \\
&\geq y\exp(-ay)\exp(r_3 + a_{31}x + a_{32}y - a_{33}z)/\bigl[y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)\bigr] \\
&= \exp\bigl[(r_3 - r_2) + (a_{31} - a_{21})x + (a + a_{22})y - (a_{33} + a_{23})z\bigr] \\
&\geq \exp\left[(r_3 - r_2) + (a_{31} - a_{21})M + (a + a_{22})M - (a_{33} + a_{23})\frac{N^{\frac{1}{\hat{\gamma}}}}{M^{\frac{\hat{\alpha}+\hat{\beta}}{\hat{\gamma}}}}\right] \\
&\geq \exp\left[(a_{31} - a_{21})M + \left(\frac{a}{2} + a_{22}\right)M\right] > \frac{a_{22}}{a-\delta} \quad (M \text{ large}).
\end{aligned} \tag{2.9}$$

Clearly, (2.6)-(2.9) imply $(x',y',z') \in B_N \cap \mathrm{IV}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$.

Let $(x,y,z) \in B_N \cap \mathrm{IV}_{(M,M)}$, then, for large enough $M > 0$,

$$x'^{\hat{\alpha}} y'^{\hat{\beta}} z'^{\hat{\gamma}} \le x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \exp(\triangle - 2M) \le x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \le N. \tag{2.10}$$

Moreover,

$$\begin{aligned}
x' &= x\exp(r_1 - a_{11}x + a_{12}y + a_{13}z) \\
&= y\exp(-ay)\exp(r_1 - a_{11}x + a_{12}y + a_{13}z) \\
&\ge y\exp(r_1 - a_{11}x + ay + a_{13}z) \\
&\ge y\exp\left(r_1 - a_{11}\frac{N^{\frac{1}{\hat{\alpha}}}}{M^{\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}}}} + aM + a_{13}M\right) \\
&\ge y\exp\left(\frac{a}{2}M + a_{13}M\right) \ge y\exp\left(\frac{3a}{2}M\right) \ge y\exp\left(\frac{\delta}{2}M\right) \quad (M \text{ large}) \\
&= \tilde{\varepsilon}y \ge \tilde{\varepsilon}M > M, \\
y' &= y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z) \\
&\ge z\exp(-az)\exp(r_2 + a_{21}x - a_{22}y + a_{23}z) = z\exp(r_2 + a_{21}x - a_{22}y + az) \\
&\ge z\exp(r_2 + \delta z) \ge z\exp\left(\frac{\delta}{2}M\right) \quad (M \text{ large}) \\
&= \tilde{\varepsilon}z \ge \tilde{\varepsilon}M \ge M > \frac{1}{a}.
\end{aligned} \tag{2.11}$$

Since $y \to y\exp(-ay)$ is monotonically decreasing for $y > \frac{1}{a}$, we have

$$\begin{aligned}
z'/y'\exp(-ay') &\ge z\exp(r_3 + a_{31}x + a_{32}y - a_{33}z)/(\tilde{\varepsilon}z\exp(-a\tilde{\varepsilon}z)) \\
&= \frac{1}{\tilde{\varepsilon}}\exp[r_3 + a_{31}x + a_{32}y + (a\tilde{\varepsilon} - a_{33})z] \\
&\ge \frac{1}{\tilde{\varepsilon}}\exp[(a\tilde{\varepsilon} - a_{33} + a)M] \quad (M \text{ large}) \\
&= \exp\left[\left(a\exp\left(\frac{\delta M}{2}\right) - a_{33} + a - \frac{\delta}{2}\right)M\right] \\
&\ge \exp\left[\left(\frac{a}{2}\exp\left(\frac{\delta M}{2}\right)\right)M\right] > 1,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
x'/y' &= x\exp(r_1 - a_{11}x + a_{12}y + a_{13}z)/[y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)] \\
&= y\exp(-ay)\exp(r_1 - a_{11}x + a_{12}y + a_{13}z)/[y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)] \\
&\ge y\exp(r_1 - a_{11}x + ay + a_{13}z)/[y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)] \\
&= \exp[(r_1 - r_2) - (a_{11} + a_{21})x + (a + a_{22})y + (a_{13} - a_{23})z] \\
&\ge \exp\left[(r_1 - r_2) - (a_{31} - a_{21})\frac{N^{\frac{1}{\hat{\alpha}}}}{M^{\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}}}} + (a + a_{22})M + (a_{13} - a_{23})M\right] \\
&\ge \exp\left[\left(\frac{a}{2} + a_{22} + a_{13} - a_{23}\right)M\right] > \frac{a_{22}}{a - \delta} \quad (M \text{ large}).
\end{aligned} \tag{2.13}$$

According to (2.14)-(2.13), we have $(x',y',z') \in B_N \cap \mathrm{I}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$. □

Lemma 2.5 *If* $\det(A) < 0$ *and* $a_{12} < a_{32}$, $a_{31} < a_{21}$, $a_{23} < a_{13}$, *then the map defined by equation* (2.2) *satisfies*

$$B_N \cap \mathrm{III}'_{(M,M)} \to B_N \cap \mathrm{V}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \tag{2.14}$$

$$B_N \cap \mathrm{V}'_{(M,M)} \to B_N \cap \mathrm{I}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \tag{2.15}$$

$$B_N \cap \mathrm{I}'_{(M,M)} \to B_N \cap \mathrm{III}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \tag{2.16}$$

where $\tilde{\varepsilon} > 1$ *and*

$$\begin{aligned}
\mathrm{I}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | x \geq M, y \geq M, z \geq y\exp(-ay), y \leq \frac{1}{b_1}x\right\}, \\
\mathrm{III}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | y \geq M, z \geq M, x \geq z\exp(-az), z \leq \frac{1}{b_3}y\right\}, \\
\mathrm{V}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | x \geq M, z \geq M, y \geq x\exp(-ax), x \leq \frac{1}{b_5}z\right\},
\end{aligned} \tag{2.17}$$

where $a = \frac{1}{2}\min_{i\neq j}\{a_{ij}\}$, $M > 0$ *is sufficiently large and*

$$\begin{aligned}
b_1 &= \max\{a_{22}/(a-\delta), (a_{22}+a_{32})/(a_{21}-a_{31}-\delta_1)\}, \\
b_3 &= \max\{a_{33}/(a-\delta), (a_{33}+a_{13})/(a_{32}-a_{12}-\delta_3)\}, \\
b_5 &= \max\{a_{11}/(a-\delta), (a_{11}+a_{21})/(a_{13}-a_{23}-\delta_5)\},
\end{aligned}$$

with constants $0 < \delta < a, 0 < \delta_1 < a_{21} - a_{31}, 0 < \delta_3 < a_{32} - a_{12}$ *and* $0 < \delta_5 < a_{13} - a_{23}$.

Proof Since the proofs of (2.15) and (2.16) are essentially the same as the proof of (2.14), it suffices to prove that (2.14) is satisfied, that is, we need only verify that there exists a constant $\tilde{\varepsilon} > 1$ such that the following holds true for the map defined by (2.2):

$$B_N \cap \mathrm{III}'_{(M,M)} \to B_N \cap \mathrm{V}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}.$$

Let $(x,y,z) \in B_N \cap \mathrm{III}'_{(M,M)}$, for large enough $M > 0$, we have

$$x'^{\hat{\alpha}} y'^{\hat{\beta}} z'^{\hat{\gamma}} \leq x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \exp(\triangle - 2M) \leq x^{\hat{\alpha}} y^{\hat{\beta}} z^{\hat{\gamma}} \leq N, \tag{2.18}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ are defined as in (2.5).

Notice that $y\exp(-ay)$ is monotonically decreasing when $y \geq \frac{1}{a}$, and we obtain for $(x,y,z) \in B_N \cap \mathrm{III}'_{(M,M)}$,

$$y \geq M \geq \frac{1}{a} \quad \text{and} \quad y\exp(-ay) \leq M\exp(-aM) \leq M \leq z.$$

Furthermore, from (2.18), we deduce that if $(x,y,z) \in B_N \cap \mathrm{III}'_{(M,M)}$, then

$$x \leq N^{\frac{1}{\hat{\alpha}}} M^{-(\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}})}.$$

Therefore,

$$
\begin{aligned}
x' &\ge z\exp(-az)\exp(r_1 - a_{11}x + a_{12}y + a_{13}z) \\
&\ge z\exp(r_1 - a_{11}x + a_{12}y + az) \\
&\ge y\exp(-ay)\exp(r_1 - a_{11}x + a_{12}y + az) \\
&\ge y\exp(r_1 - a_{11}x + ay + az) \\
&\ge y\exp\left(r_1 - a_{11}\frac{N^{\frac{1}{\hat{\alpha}}}}{M^{\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}}}} + aM + aM\right) \quad (M \text{ large}) \\
&\ge y\exp\left(a_{12}M + \frac{a}{2}M\right) \\
&\ge y\exp\left(\frac{3a}{2}M\right) \ge y\exp\left(\frac{\delta}{2}M\right) \quad (M \text{ large}) \\
&\triangleq \tilde{\varepsilon}y \ge \tilde{\varepsilon}M > M, \\
z' &\ge y\exp(-ay)\exp(r_3 + a_{31}x + a_{32}y - a_{33}z) \\
&\ge y\exp(r_3 + a_{31}x + ay - a_{33}z) \\
&\ge y\exp(r_3 + \delta y) \ge y\exp\left(\frac{\delta M}{2}\right) \quad (M \text{ large}) \\
&\triangleq \tilde{\varepsilon}y \ge \tilde{\varepsilon}M \ge M.
\end{aligned}
\tag{2.19}
$$

From the above it follows that

$$
\begin{aligned}
y'/x'\exp\bigl(-ax'\bigr) &\ge y\exp(r_2 + a_{21}x - a_{22}y + a_{23}z)/\bigl(\tilde{\varepsilon}y\exp(-a\tilde{\varepsilon}y)\bigr) \\
&= \frac{1}{\tilde{\varepsilon}}\exp\bigl[r_2 + (a\tilde{\varepsilon}_1 - a_{22})y + a_{23}z\bigr] \\
&\ge \frac{1}{\tilde{\varepsilon}}\exp\bigl[(a\tilde{\varepsilon}_1 - a_{22} + a)M\bigr] \quad (M \text{ large}) \\
&\ge \exp\left(-\frac{\delta}{2}M\right)\exp\left[\left(a\exp\left(\frac{\delta}{2}M\right) - a_{22} + a\right)M\right] \quad (M \text{ large}) \\
&= \exp\left[\left(a\exp\left(\frac{\delta}{2}M\right) - a_{22} + a - \frac{\delta}{2}\right)M\right] \quad (M \text{ large}) \\
&\ge \exp\left[\frac{aM}{2}\exp\left(\frac{\delta}{2}M\right)\right] > 1 \quad (M \text{ large}),
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
z'/x' &= z\exp(r_3 + a_{31}x + a_{32}y - a_{33}z)/\bigl[x\exp(r_1 - a_{11}x + a_{12}y + a_{13}z)\bigr] \\
&= \bigl(z\exp\bigl[(r_3 - r_1) + (a_{31} + a_{11})x + (a_{32} - a_{12})y - (a_{33} + a_{13})z\bigr]\bigr)/x \\
&\ge \left(z\exp\left[(r_3 - r_1) + (a_{31} + a_{11})x + (a_{32} - a_{12})y - \left(\frac{a_{33} + a_{13}}{b_5}\right)y\right]\right)/x \\
&= \bigl(z\exp\bigl[(r_3 - r_1) + \delta_3 y\bigr]\bigr)/x \ge \bigl(M\exp\bigl[(r_3 - r_1) + \delta_3 y\bigr]\bigr)/x \\
&\ge \left(M^{\frac{\hat{\alpha}+\hat{\beta}+\hat{\gamma}}{\hat{\alpha}}}\exp\left(\frac{3\delta_3}{4}M\right)\right)\Big/N^{\frac{1}{\hat{\alpha}}} \ge b_5 \quad (M \text{ large}).
\end{aligned}
$$

Thus, (2.18)-(2.20) show that

$$B_N \cap \mathrm{III}'_{(M,M)} \to B_N \cap \mathrm{V}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$$

for the map given by (2.2).

By using the same procedure as in the proof of (2.14), we see that the map defined by (2.2) satisfies (2.15) and (2.16). □

3 Proof of main results

3.1 Proof of Theorem 1.1

Proof By introducing the transformation

$$\begin{aligned} u_n &= x_n^{\theta_{11}}, \\ v_n &= y_n^{\theta_{22}}, \end{aligned} \tag{3.1}$$

system (1.4) becomes

$$\begin{aligned} u_{n+1} &= u_n \exp(\theta_{11} r_1 - \theta_{11} a_{11} u_n + \theta_{11} a_{12} v_n), \\ v_{n+1} &= v_n \exp(\theta_{22} r_2 + \theta_{22} a_{21} u_n - \theta_{22} a_{22} v_n). \end{aligned} \tag{3.2}$$

The proof proceeds in a similar manner to that employed in Theorem 1 in [6]. □

3.2 Proof of Theorem 1.2

Proof By considering the sign of $\det(A)$, we will divide our proof in three cases.

Case I. $\det(A) = 0$. Define a continuous function $V(x,y,z)$ by

$$V(x,y,z) = x^{A_{11}} y^{A_{21}} z^{A_{31}}, \tag{3.3}$$

where A_{11}, A_{21} and A_{31} are the algebraic cofactors of $-a_{11}$, a_{21} and a_{31}, respectively.

It is obvious that $A_{ij} > 0$ since $a_{ij} > 0$. Along the orbits of (2.2), we have

$$V(x',y',z') = V(x,y,z)\exp(\triangle), \tag{3.4}$$

where $\triangle = A_{11} r_1 + A_{21} r_2 + A_{31} r_3$.

Case I.1. If $\triangle = 0$, then $V(x,y,z)$ is invariant along the orbits of (2.2). In this case, for any given compact set E in the interior of R^3_+, we can take sufficiently large C such that

$$\{(x,y,z) | V(x,y,z) = C\} \cap E = \emptyset,$$

which implies the non-permanence of system (1.5).

Case I.2. If $\triangle < 0$ (> 0), then $\exp(\triangle) < 1$ (> 1). It follows from (3.4) that, for any given compact set E in the interior of R^3_+, there is a positive orbit of (2.2) leaving ultimately E, which will not return to E afterwards. Therefore, system (1.5) remains not permanent in this case.

Case II. Suppose $\det(A) > 0$. From Lemma 2.1, we see that there exist constant $p_1, p_2, p_3 > 0$, such that (2.1) holds.

Construct the following continuous function $V(x,y,z)$:

$$V(x,y,z) = x^{\alpha} y^{\beta} z^{\gamma}, \tag{3.5}$$

where $\alpha = p_1$, $\beta = p_2$, $\gamma = p_3$.

Calculating along the orbits of (2.2), we obtain

$$V(x',y',z') = V(x,y,z)\exp(\tilde{\Delta} + \mu_1 x + \mu_2 y + \mu_3 z), \tag{3.6}$$

where $\tilde{\Delta} = r_1\alpha + r_2\beta + r_3\gamma$ and

$$\begin{aligned} &\mu_1 = -a_{11}p_1 + a_{21}p_2 + a_{31}p_3 > 0,\\ &\mu_2 = a_{12}p_1 - a_{22}p_2 + a_{32}p_3 > 0,\\ &\mu_3 = a_{13}p_1 + a_{23}p_2 - a_{33}p_3 > 0. \end{aligned} \tag{3.7}$$

Since $x > 0$, $y > 0$, $z > 0$, let us denote by μ the number defined by $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. Then together with Lemma 2.2, it is shown that

$$\mu_1 x + \mu_2 y + \mu_3 z \geq \mu(x+y+z) \geq \mu\tilde{k}\big(x^{\alpha} y^{\beta} z^{\gamma}\big)^{\tilde{\mu}},$$

where $\tilde{\mu} = (\alpha+\beta+\gamma)^{-1}$ and $0 < \tilde{k} < \min\{(\alpha\tilde{\mu})^{-1}, (\beta\tilde{\mu})^{-1}, (\gamma\tilde{\mu})^{-1}\}$.

Therefore, we have

$$V(x',y',z') \geq V(x,y,z)\exp\big[\bar{\Delta} + \mu\tilde{k}\big(V(x,y,z)\big)^{\tilde{\mu}}\big].$$

Since $\tilde{k}\mu > 0$, $\tilde{\mu} > 0$, one can take $M > 0$ large enough to satisfy $\varepsilon = \bar{\Delta} + \mu\tilde{k}M^{\tilde{\mu}} > 1$ and considering the regions $U_M = \{(x,y,z) \in R_+^3 | V(x,y,z) \geq M\}$, (2.2) maps U_M into $U_{M\varepsilon}$. Therefore, all the orbits starting in U_M are unbounded and system (1.5) is not permanent.

Case III. $\det(A) < 0$. According to Lemma 2.3, our proof in this case can be divided into five cases.

Case III.1. $a_{31} \geq a_{21}$, $a_{23} \leq a_{13}$.

From Lemma 2.4, we claim that (2.2) maps

$$B_N \cap \mathrm{I}_{(M,M)} \to B_N \cap \mathrm{IV}_{(M_{\tilde{\varepsilon}}, M_{\tilde{\varepsilon}})}.$$

Similarly,

$$B_N \cap \mathrm{IV}_{(M,M)} \to B_N \cap \mathrm{I}_{(M_{\tilde{\varepsilon}}, M_{\tilde{\varepsilon}})},$$

where $\tilde{\varepsilon} > 1$. By iteration, this shows that all orbits starting in $B_N \cap \mathrm{I}_{(M,M)}$ are unbounded and system (1.5) is not permanent.

Case III.2. $a_{32} \geq a_{12}$, $a_{23} \geq a_{13}$, $a_{31} < a_{21}$; $a_{32} \geq a_{12}$, $a_{23} > a_{13}$, $a_{31} \geq a_{21}$.

In this case we define the regions $\mathrm{II}_{(M,M)}$ and $\mathrm{V}_{(M,M)}$ as follows:

$$\mathrm{II}_{(M,M)} = \left\{(x,y,z) \in R^3_+ \middle| x \geq M, y \geq M, z \geq x\exp(-ax), x \leq \frac{a-\delta}{a_{11}}y\right\},$$

$$\mathrm{V}_{(M,M)} = \left\{(x,y,z) \in R^3_+ \middle| x \geq M, z \geq M, y \geq x\exp(-ax), x \leq \frac{a-\delta}{a_{11}}z\right\}.$$

It is easy to check that Case III.2 satisfies $a_{32} \geq a_{12}$, $a_{23} \geq a_{13}$. Observe that Case III.1 and $a_{32} \geq a_{12}$, $a_{23} \geq a_{13}$ are in a symmetric form. The proof of this case follows in a similar manner to Case III.1, so we see that the map defined by equation (2.2) satisfies

$$B_N \cap \mathrm{II}_{(M,M)} \to B_N \cap \mathrm{V}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$$

and

$$B_N \cap \mathrm{V}_{(M,M)} \to B_N \cap \mathrm{II}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$$

with $\tilde{\varepsilon} > 1$. By iteration, this shows that all orbits starting in $B_N \cap \mathrm{II}_{(M,M)}$ are unbounded, which implies that system (1.5) is not permanent.

Case III.3. $a_{32} < a_{12}$, $a_{31} < a_{21}$; $a_{32} = a_{12}$, $a_{23} < a_{13}$, $a_{31} < a_{21}$; $a_{32} < a_{12}$, $a_{23} > a_{13}$, $a_{31} = a_{21}$.

In this case let us define regions $\mathrm{III}_{(M,M)}$ and $\mathrm{IV}_{(M,M)}$ as follows:

$$\mathrm{III}_{(M,M)} = \left\{(x,y,z) \in R^3_+ \middle| y \geq M, z \geq M, x \geq z\exp(-az), z \leq \frac{a-\delta}{a_{33}}y\right\},$$

$$\mathrm{IV}_{(M,M)} = \left\{(x,y,z) \in R^3_+ \middle| y \geq M, z \geq M, x \geq y\exp(-ay), y \leq \frac{a-\delta}{a_{22}}z\right\}.$$

It is obvious that Case III.3 satisfies $a_{12} \geq a_{32}$, $a_{21} \geq a_{31}$. By virtue of the symmetry, the theorem can be proved by the same method as employed in the proof of Case III.1 and hence we know that (2.2) satisfies

$$B_N \cap \mathrm{III}_{(M,M)} \to B_N \cap \mathrm{VI}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$$

and

$$B_N \cap \mathrm{VI}_{(M,M)} \to B_N \cap \mathrm{III}_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}$$

with $\tilde{\varepsilon} > 1$. By iteration, this shows that all orbits starting in $B_N \cap \mathrm{II}_{(M,M)}$ are unbounded. This leads to non-permanence of system (1.5).

Case III.4. $a_{12} < a_{32}$, $a_{31} < a_{21}$, $a_{23} < a_{13}$.

From Lemma 2.5, we infer that

$$B_N \cap \mathrm{III}'_{(M,M)} \to B_N \cap \mathrm{V}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})},$$

$$B_N \cap \mathrm{V}'_{(M,M)} \to B_N \cap \mathrm{I}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})},$$

$$B_N \cap \mathrm{I}'_{(M,M)} \to B_N \cap \mathrm{III}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})},$$

where $\tilde{\varepsilon} > 1$. By iteration, all orbits starting in $B_N \cap \mathrm{III}'_{(M,M)}$ are unbounded. Hence, system (1.5) is not permanent.

Case III.5. $a_{12} > a_{32}$, $a_{31} > a_{21}$, $a_{23} > a_{13}$.

Define the regions as follows:

$$\begin{aligned} \mathrm{II}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | x \geq M, y \geq M, z \geq x\exp(-ax), x \leq \frac{1}{b_2}y\right\}, \\ \mathrm{IV}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | y \geq M, z \geq M, x \geq y\exp(-ay), y \leq \frac{1}{b_4}z\right\}, \\ \mathrm{VI}'_{(M,M)} &= \left\{(x,y,z) \in R^3_+ | x \geq M, z \geq M, y \geq z\exp(-az), z \leq \frac{1}{b_6}x\right\}, \end{aligned} \tag{3.8}$$

where $a = \frac{1}{2}\min\{a_{ij}\}_{(i\neq j)}$, $M > 0$ is sufficiently large and

$$\begin{aligned} b_2 &= \max\bigl\{a_{11}/(a-\delta), (a_{11}+a_{21})/(a_{12}-a_{32}-\delta_2)\bigr\}, \\ b_4 &= \max\bigl\{a_{22}/(a-\delta), (a_{22}+a_{12})/(a_{23}-a_{13}-\delta_4)\bigr\}, \\ b_6 &= \max\bigl\{a_{33}/(a-\delta), (a_{33}+a_{23})/(a_{31}-a_{21}-\delta_6)\bigr\}, \end{aligned} \tag{3.9}$$

with $0 < \delta < a$, $0 < \delta_2 < a_{12} - a_{32}$, $0 < \delta_4 < a_{23} - a_{13}$, $0 < \delta_6 < a_{31} - a_{21}$. Similarly to the proof of Case III.4, we obtain

$$\begin{aligned} &B_N \cap \mathrm{IV}'_{(M,M)} \to B_N \cap \mathrm{II}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \\ &B_N \cap \mathrm{II}'_{(M,M)} \to B_N \cap \mathrm{VI}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \\ &B_N \cap \mathrm{VI}'_{(M,M)} \to B_N \cap \mathrm{IV}'_{(M\tilde{\varepsilon},M\tilde{\varepsilon})}, \end{aligned} \tag{3.10}$$

where $\tilde{\varepsilon} > 1$. By iteration, all orbits starting in $B_N \cap \mathrm{IV}'_{(M,M)}$ are unbounded and Theorem 1.2 holds true in this case.

This completes the proof of the theorem. □

Remark 3.1 The method we used here is to find some regions so that the orbits of (1.5) starting from them approach the axes as n tends to infinity. For example, in Case III.1, we constructed the regions B_N, $\Im'$, III', V' satisfying the requirement that the orbits starting in region $B_N \cap \mathrm{III}'$ jump to the region $B_N \cap \mathrm{V}'$ and then jump to the region $B_N \cap \mathrm{I}'$ and finally jump to the region $B_N \cap \mathrm{III}'$ again. This is somewhat similar to a phenomenon proposed by May and Leonard [16] in the three competition models.

4 Concluding remarks

We have seen that systems (1.4) and (1.5) in the cooperative case cannot be permanent by modifying the techniques used in Lu and Wang [6] and Hofbauer *et al.* [3]. Our approach is to find a suitable set of initial positions satisfying the requirement that the positive orbits of (1.5) starting from this set approach the x-axis, y-axis and z-axis as n tends to infinity. This is somewhat similar to that in [8, 16] where the orbits of a three-dimensional discrete competition models approach the boundary cycle. The orbits of (1.5) starting from the given regions satisfy

$$\begin{aligned}
&\liminf_{n\to+\infty} x_n = 0 \quad \text{and} \quad \limsup_{n\to+\infty} x_n = +\infty, \\
&\liminf_{n\to+\infty} y_n = 0 \quad \text{and} \quad \limsup_{n\to+\infty} y_n = +\infty, \\
&\liminf_{n\to+\infty} z_n = 0 \quad \text{and} \quad \limsup_{n\to+\infty} z_n = +\infty.
\end{aligned} \tag{4.1}$$

Hence, sufficiently small statistical fluctuations can lead to the extinction of any species. This means that the three species reveal the great risk of extinction in practice although they are cooperative and each one can be permanent in the absence of the other two.

For the matrix A, we have shown that system (1.5) is not permanent in any case with all the elements a_{ij} $(i,j = 1,2,3)$ positive. In fact, by using the technique similar to the proof of Theorem 1.2, we can also obtain the non-permanence of system (1.5). This is the following theorem without limitation for a_{ii} $(i = 1,2,3)$.

Theorem 4.1 *If for the matrix A, for all i, j such that $a_{ij} > 0$ $(i \neq j)$ $(i,j = 1,2,3)$, then system* (1.5) *is not permanent.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equally made contributions. All authors read and approved the final manuscript.

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