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Existence of positive solutions for fractional differential equation involving integral boundary conditions with p-Laplacian operator

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Abstract

The existence of positive solutions is considered for a fractional differential equation with p-Laplacian operator in this article. By employing the Avery-Henderson fixed point theorem, a new result is obtained for the boundary value problems.
An example is also presented to illustrate the effectiveness of the main result.

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1 Introduction

Fractional calculus is the extension of integer order calculus to arbitrary order calculus. With the development of fractional calculus, fractional differential equations have wide applications in the modeling of different physical and natural science fields, such as fluid mechanics, chemistry, control system, heat conduction, etc. There are many papers concerning fractional differential equations with the p-Laplacian operator [1–6] and fractional differential equations with integral boundary conditions [7–11].

By means of the Guo-Krasnosel'skii fixed point theorem on cones, Han *et al.* [5] investigate positive solutions for the following problems for the generalized p-Laplacian operator:

$$
\begin{aligned}
&D^{\beta}_{0^+}\left(\phi\left(D^{\alpha}_{0^+}u(t)\right)\right) = \lambda f\left(u(t)\right), \quad 0 < t < 1,\\
&u(0) = u'(0) = u'(1) = 0,\\
&\phi\left(D^{\alpha}_{0^+}u(0)\right) = \left(\phi\left(D^{\alpha}_{0^+}u(1)\right)\right)' = 0,
\end{aligned}
$$

where $1 < \beta \leq 2$, $2 < \alpha \leq 3$, they obtain some new results of positive solutions for the aforementioned boundary value problem.

By means of the Avery-Henderson fixed point theorem and the Leggett-Williams fixed point theorem, Günendi and Yaslan [11] investigate positive solutions for the following problem with integral boundary conditions:

$$\begin{cases} -D_{0^+}^{\eta-2}(u''(t)) + f(u(t)) = 0, \quad t \in [0,1], \\ u''(0) = u'''(0) = \cdots = u^{(n-2)}(1) = 0, \qquad u'''(1) = 0, \\ \alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)\,\mathrm{d}s, \\ \gamma u(1) + \delta u'(1) = \sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s)\,\mathrm{d}s, \end{cases}$$

where $n-1 < \eta \le n$, $n \ge 3$, $\alpha, \beta, \gamma, \delta > 0$, $a_p, b_p \ge 0$ are given constants. They show the existence of multiple positive solutions for the aforementioned boundary value problems.

Motivated by the aforementioned work, this work discusses the existence of positive solutions for this fractional differential equation:

$$\begin{cases} D_{0^+}^{\beta}[\phi_p({}^{c}D_{0^+}^{\alpha} u(t))] + f(t, u(t)) = 0, \quad t \in (0,1), \\ \phi_p({}^{c}D_{0^+}^{\alpha} u(0)) = [\phi_p({}^{c}D_{0^+}^{\alpha} u(0))]' = \phi_p({}^{c}D_{0^+}^{\alpha} u(1)) = 0, \\ u''(0) = u'(1) = 0, \\ au(0) + bu'(0) = \int_0^1 g(t)u(t)\,\mathrm{d}t, \end{cases} \tag{1.1}$$

where $2 < \alpha \le 3$, $2 < \beta \le 3$ and $5 < \alpha + \beta \le 6$. $\phi_p(u) = |u|^{p-2}u$, $p > 1$. ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $D_{0^+}^{\beta}$ is the Riemann-Liouville fractional derivative.

We will always suppose the following conditions are satisfied:

(H_1) $g(t) : [0,1] \to [0, +\infty)$ with $g(t) \in L^1[0,1]$, $\int_0^1 g(t)\,\mathrm{d}t > 0$ and $\int_0^1 tg(t)\,\mathrm{d}t > 0$;
(H_2) $a, b \in (0, +\infty)$, $a > \int_0^1 g(t)\,\mathrm{d}t$ and $b > a$;
(H_3) $f(t, u) : [0,1] \times (0, \infty) \to (0, \infty)$ is continuous.

2 Background and definitions

To show the main result of this work, we give in the following some basic definitions and a theorem, which can be found in [12, 13].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1}\,\mathrm{d}x.$$

Definition 2.2 For a continuous function $y : (0, +\infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.3 For a continuous function $y:(0,+\infty)\to\mathbb{R}$, the Caputo derivative of fractional order $\alpha>0$ is defined as

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}y^{(n)}(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Theorem 2.1 (Avery-Henderson fixed point theorem [14]) *Let $(E,\|\cdot\|)$ be a Banach space, and $P\subset E$ be a cone. Let ψ and φ be increasing non-negative, continuous functionals on P, and ω be a non-negative continuous functional on P with $\omega(0)=0$, such that, for some $r_3>0$ and $M>0$, $\varphi(u)\le\omega(u)\le\psi(u)$, and $\|u\|\le M\varphi(u)$, for all $u\in\overline{P(\varphi,r_3)}$, where $P(\varphi,r_3)=\{u\in P:\varphi(u)<r_3\}$. Suppose that there exist positive numbers $r_1<r_2<r_3$, such that*

$$\omega(lu)\le l\omega(u)\quad \text{for } 0\le l\le 1, \text{and } u\in\partial P(\omega,r_2).$$

If $T:\overline{P(\varphi,r_3)}\to P$ is a completely continuous operator satisfying:

(C1) *$\varphi(Tu)>r_3$ for all $u\in\partial P(\varphi,r_3)$;*

(C2) *$\omega(Tu)<r_2$ for all $u\in\partial P(\omega,r_2)$;*

(C3) *$P(\psi,r_1)\neq\emptyset$, and $\psi(Tu)>r_1$ for all $u\in\partial P(\psi,r_1)$,*

then T has at least two fixed points u_1 and u_2 such that $r_1<\psi(u_1)$ with $\omega(u_1)<r_2$ and $r_2<\omega(u_2)$ with $\varphi(u_2)<r_3$.

3 Preliminary lemmas

Lemma 3.1 *The boundary value problem* (1.1) *is equivalent to the following equation*:

$$u(t)=d_0+d_1t+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)\,\mathrm{d}s, \tag{3.1}$$

where

$$\begin{aligned} d_0={}&\frac{1}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t(t-s)^{\alpha-1}v(s)\,\mathrm{d}s\,\mathrm{d}t\\ &+\frac{b-\int_0^1 tg(t)\,\mathrm{d}t}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)\,\mathrm{d}s, \end{aligned}\tag{3.2}$$

$$d_1=-\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)\,\mathrm{d}s, \tag{3.3}$$

$$v(s)=\phi_q\biggl(\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}\tau\biggr), \tag{3.4}$$

$$H(s,\tau)=\frac{1}{\Gamma(\beta)}\begin{cases}(s-s\tau)^{\beta-1}-(s-\tau)^{\beta-1}, & 0\le\tau\le s\le 1,\\ (s-s\tau)^{\beta-1}, & 0\le s\le\tau\le 1.\end{cases} \tag{3.5}$$

$\phi_q(s)$ is the inverse function of $\phi_p(s)$, a.e., $\phi_q(s)=|s|^{q-2}s$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof From $D_{0^+}^{\beta}[\phi_p({}^{\mathrm{c}}D_{0^+}^{\alpha}u(t))]+f(t,u(t))=0$, we get

$$\phi_p\bigl({}^{\mathrm{c}}D_{0^+}^{\alpha}u(t)\bigr)=-\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}\tau+c_1t^{\beta-1}+c_2t^{\beta-2}+c_3t^{\beta-3}.$$

In view of $\phi_p({}^{c}D_{0^+}^{\alpha}u(0)) = [\phi_p({}^{c}D_{0^+}^{\alpha}u(0))]' = 0$, we get $c_2 = c_3 = 0$, *i.e.*,

$$\phi_p\big({}^{c}D_{0^+}^{\alpha}u(t)\big) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f\big(\tau,u(\tau)\big)\,d\tau + c_1t^{\beta-1}. \tag{3.6}$$

Conditions $\phi_p({}^{c}D_{0^+}^{\alpha}u(1)) = 0$ imply that

$$c_1 = \frac{1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}f\big(\tau,u(\tau)\big)\,d\tau. \tag{3.7}$$

By use of (3.6) and (3.7), we get

$$\phi_p\big({}^{c}D_{0^+}^{\alpha}u(t)\big) = \int_0^1 H(t,\tau)f\big(\tau,u(\tau)\big)\,d\tau. \tag{3.8}$$

In view of (3.8), we obtain

$${}^{c}D_{0^+}^{\alpha}u(t) = \phi_q\bigg(\int_0^1 H(t,\tau)f\big(\tau,u(\tau)\big)\,d\tau\bigg). \tag{3.9}$$

Let

$$v(t) = \phi_q\bigg(\int_0^1 H(t,\tau)f\big(\tau,u(\tau)\big)\,d\tau\bigg),$$

by use of (3.9), we get

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\,ds + d_0 + d_1t + d_2t^2.$$

Conditions $u''(0) = 0$ imply that $d_2 = 0$, *i.e.*,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\,ds + d_0 + d_1t,$$

then we have

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}v(s)\,ds + d_1.$$

Conditions $u'(1) = 0$ imply that

$$d_1 = -\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,ds.$$

From $au(0) + bu'(0) = \int_0^1 g(t)u(t)\,dt$, we get

$$\begin{aligned} d_0 ={}& \frac{1}{[a-\int_0^1 g(t)\,dt]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t (t-s)^{\alpha-1}v(s)\,ds\,dt \\ &+ \frac{b-\int_0^1 tg(t)\,dt}{[a-\int_0^1 g(t)\,dt]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,ds. \end{aligned}$$

Therefore, we can obtain

$$\begin{aligned} u(t) &= d_0 + d_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s \\ &= \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s \\ &\quad - \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s. \end{aligned}$$

The proof is complete. □

Lemma 3.2 ([15]) *The function $H(s,\tau)$ defined by* (3.5) *is continuous on* $[0,1] \times [0,1]$ *and satisfy*

$$\frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)} \le H(s,\tau) \le \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)} \quad \text{for } s,\tau \in [0,1].$$

Let E be the real Banach space $C[0,1]$ with the maximum norm, define the operator $T : E \to E$ by

$$\begin{aligned} Tu(t) &= d_0 + d_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s \\ &= \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s \\ &\quad - \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s. \end{aligned}$$

Lemma 3.3 *For $u \in C[0,1]$ with $u(t) \ge 0$, $(Tu)(t)$ is non-increasing and non-negative.*

Proof Since

$$Tu(t) = d_0 + d_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s,$$

so we get

$$\begin{aligned} (Tu)'(t) &= d_1 + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} v(s)\,\mathrm{d}s \\ &= -\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} v(s)\,\mathrm{d}s \\ &\le 0. \end{aligned}$$

So $Tu(t)$ is non-increasing, then we have $\min_{t\in[0,1]} Tu(t) = Tu(1)$. We have

$$\begin{aligned}
Tu(1) &= d_0 + d_1 + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)\,\mathrm{d}s \\
&= \frac{1}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t (t-s)^{\alpha-1}v(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{b-\int_0^1 tg(t)\,\mathrm{d}t}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)\,\mathrm{d}s \\
&\geq \frac{1}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t (t-s)^{\alpha-1}v(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{[b-\int_0^1 tg(t)\,\mathrm{d}t]-[a-\int_0^1 g(t)\,\mathrm{d}t]}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}v(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)\,\mathrm{d}s \\
&\geq 0.
\end{aligned}$$

The proof is complete. □

4 Main results

Theorem 4.1 *Suppose that there exist numbers* $0 < r_1 < r_2 < r_3$ *such that* f *satisfies the following conditions*:

(H1) $f(t,u) > M_3$, *for* $t \in [0,1]$, $u \in [r_3, \frac{r_3}{k}]$;

(H2) $f(t,u) < M_2$, *for* $t \in [0,1]$, $u \in [0,r_2]$;

(H3) $f(t,u) > M_1$, *for* $t \in [0,1]$, $u \in [0,r_1]$,

where

$$\begin{aligned}
M_3 &= \frac{\Gamma(\beta)}{B(2,\beta)}\left(\frac{r_3}{L_3}\right)^{p-1}, \qquad M_2 = \frac{\Gamma(\beta-1)}{B(2,\beta)}\left(\frac{r_2}{L_2}\right)^{p-1}, \qquad M_1 = \frac{\Gamma(\beta)}{B(2,\beta)}\left(\frac{r_1}{L_1}\right)^{p-1}, \\
L_3 &= \frac{(b-a)B(\beta q-\beta-q+2,\alpha+q-2)}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} + \frac{B(\beta q-\beta-q+2,\alpha+q-1)}{\Gamma(\alpha)}, \\
L_2 &= \frac{\int_0^1 g(t)\,\mathrm{d}t}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha+1)} + \frac{b-\int_0^1 tg(t)\,\mathrm{d}t}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}, \\
L_1 &= \frac{(b-\int_0^1 tg(t)\,\mathrm{d}t)B(\beta q-\beta-q+2,\alpha+q-2)}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}.
\end{aligned}$$

Then the problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that* $r_1 < \psi(u_1)$ *with* $\omega(u_1) < r_2$ *and* $r_2 < \omega(u_2)$ *with* $\varphi(u_2) < r_3$.

Proof Define the cone $P \subset E$ by

$$P = \Bigl\{u \big| u \in E \text{ and } \min_{t\in[0,1]} u(t) \geq k\|u\|, t \in [0,1]\Bigr\},$$

where

$$k = \frac{[b - \int_0^1 tg(t)\,\mathrm{d}t] - [a - \int_0^1 g(t)\,\mathrm{d}t]}{b - \int_0^1 tg(t)\,\mathrm{d}t}, \quad 0 < k < 1.$$

For any $u \in P$, in view of Lemma 3.3, we get

$$\begin{aligned}
\min_{t\in[0,1]} \bigl|Tu(t)\bigr| &= \bigl|Tu(1)\bigr| = d_0 + d_1 + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)\,\mathrm{d}s \\
&= \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)\,\mathrm{d}s \\
&\geq \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{[b - \int_0^1 tg(t)\,\mathrm{d}t] - [a - \int_0^1 g(t)\,\mathrm{d}t]}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s \\
&\geq k \biggl\{ \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1} v(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)\,\mathrm{d}s \biggr\} \\
&= kTu(0) = k\|Tu\|.
\end{aligned}$$

Therefore, $T : P \to P$. In view of the Arzela-Ascoli theorem, we have $T : P \to P$ is completely continuous.

We define the functions on the cone P:

$$\varphi(u) = \min_{t\in[0,1]} \bigl|u(t)\bigr| = u(1), \qquad \omega(u) = \max_{t\in[0,1]} \bigl|u(t)\bigr| = u(0),$$

$$\psi(u) = \max_{t\in[0,1]} \bigl|u(t)\bigr| = u(0).$$

Obviously, we have $\omega(0) = 0$, $\varphi(u) \leq \omega(u) \leq \psi(u)$.

For any $u \in \overline{P(\varphi, r_3)}$, we get $\min_{t\in[0,1]} u(t) \geq k\|u\|$, that is, $\varphi(u) \geq k\|u\|$, therefore we obtain $\|u\| \leq \frac{1}{k}\varphi(u)$. For any $u \in \partial P(\omega, r_2)$, we get $\omega(lu) = l\omega(u)$ for $0 \leq l \leq 1$.

In the following, we prove that the conditions of Theorem 2.1 hold.

Firstly, let $u \in \partial P(\varphi, r_3)$, that is, $u \in [r_3, \frac{r_3}{k}]$ for $t \in [0,1]$. By means of (H1), we have

$$\begin{aligned}
v(s) &= \phi_q \biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau)\bigr)\,\mathrm{d}\tau \biggr) \\
&> \phi_q \biggl(M_3 \int_0^1 \frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\tau \biggr) \\
&= \biggl(\frac{M_3 s^{\beta-1}(1-s)B(2,\beta)}{\Gamma(\beta)} \biggr)^{q-1},
\end{aligned}$$

where $B(2,\beta) = \int_0^1 \tau(1-\tau)^{\beta-1}\,\mathrm{d}\tau$. So we get

$$\begin{aligned}
\varphi(Tu) &= \min_{t\in[0,1]}\bigl|Tu(t)\bigr| = Tu(1) = d_0 + d_1 + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\nu(s)\,\mathrm{d}s \\
&= \frac{1}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t (t-s)^{\alpha-1}\nu(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{b-\int_0^1 tg(t)\,\mathrm{d}t}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\nu(s)\,\mathrm{d}s \\
&\geq \frac{[b-\int_0^1 tg(t)\,\mathrm{d}t]-[a-\int_0^1 g(t)\,\mathrm{d}t]}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\nu(s)\,\mathrm{d}s \\
&> \frac{b-a}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\biggl(\frac{M_3 s^{\beta-1}(1-s)B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\biggl(\frac{M_3 s^{\beta-1}(1-s)B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1}\,\mathrm{d}s \\
&= \frac{b-a}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\biggl(\frac{M_3 B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1}\int_0^1 (1-s)^{\alpha-2}\bigl(s^{\beta-1}(1-s)\bigr)^{q-1}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\biggl(\frac{M_3 B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1}\int_0^1 (1-s)^{\alpha-1}\bigl(s^{\beta-1}(1-s)\bigr)^{q-1}\,\mathrm{d}s \\
&= \frac{b-a}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)}\biggl(\frac{M_3 B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1} B(\beta q-\beta-q+2,\alpha+q-2) \\
&\quad + \frac{1}{\Gamma(\alpha)}\biggl(\frac{M_3 B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1} B(\beta q-\beta-q+2,\alpha+q-1) \\
&= \biggl(\frac{M_3 B(2,\beta)}{\Gamma(\beta)}\biggr)^{q-1} L_3 = r_3.
\end{aligned}$$

Secondly, let $u \in \partial P(\omega, r_2)$, that is, $u \in [0, r_2]$ for $t \in [0,1]$. By means of (H2), we get

$$\begin{aligned}
\nu(s) &= \phi_q\biggl(\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau)\bigr)\,\mathrm{d}\tau\biggr) \\
&< \phi_q\biggl(M_2\int_0^1 \frac{\tau(1-\tau)^{\beta-1}}{\Gamma(\beta-1)}\,\mathrm{d}\tau\biggr) = \biggl(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\biggr)^{q-1}.
\end{aligned}$$

So we have

$$\begin{aligned}
\omega(Tu) &= \max_{t\in[0,1]}\bigl|Tu(t)\bigr| = Tu(0) = d_0 \\
&= \frac{1}{[a-\int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)}\int_0^1 g(t)\int_0^t (t-s)^{\alpha-1}\nu(s)\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s \\
&< \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^1 (1-s)^{\alpha-1} \left(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\right)^{q-1} \mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{[b - \int_0^1 tg(t)\,\mathrm{d}t]}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \left(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\right)^{q-1} \mathrm{d}s \\
&\le \frac{\int_0^1 g(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha+1)} \left(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\right)^{q-1} \\
&\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \left(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\right)^{q-1} \\
&= \left(\frac{M_2 B(2,\beta)}{\Gamma(\beta-1)}\right)^{q-1} L_2 = r_2.
\end{aligned}
$$

Finally, let $u \in \partial P(\psi, r_1)$, that is, $u \in [0, r_1]$ for $t \in [0,1]$. By means of (H3), we get

$$
\begin{aligned}
\nu(s) &= \phi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right) \\
&> \phi_q\left(M_1 \int_0^1 \frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)}\,\mathrm{d}\tau\right) \\
&= \left(\frac{M_1 s^{\beta-1}(1-s)B(2,\beta)}{\Gamma(\beta)}\right)^{q-1}.
\end{aligned}
$$

So we get

$$
\begin{aligned}
\psi(Tu) &= \max_{t\in[0,\eta]} \big|Tu(t)\big| = Tu(0) = d_0 \\
&= \frac{1}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha)} \int_0^1 g(t) \int_0^t (t-s)^{\alpha-1}\nu(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s \\
&\ge \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\nu(s)\,\mathrm{d}s \\
&> \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \left(\frac{M_1 s^{\beta-1}(1-s)B(2,\beta)}{\Gamma(\beta)}\right)^{q-1} \mathrm{d}s \\
&= \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \left(\frac{M_1 B(2,\beta)}{\Gamma(\beta)}\right)^{q-1} \int_0^1 (1-s)^{\alpha-2}\big(s^{\beta-1}(1-s)\big)^{q-1}\,\mathrm{d}s \\
&= \frac{b - \int_0^1 tg(t)\,\mathrm{d}t}{[a - \int_0^1 g(t)\,\mathrm{d}t]\Gamma(\alpha-1)} \left(\frac{M_1 B(2,\beta)}{\Gamma(\beta)}\right)^{q-1} B(\beta q - \beta - q + 2, \alpha + q - 2) \\
&= \left(\frac{M_1 B(2,\beta)}{\Gamma(\beta)}\right)^{q-1} L_1 = r_1.
\end{aligned}
$$

Therefore, in view of Theorem 2.1, we see that the problem (1.1) has at least two positive solutions u_1 and u_2 such that $r_1 < \psi(u_1)$ with $\omega(u_1) < r_2$ and $r_2 < \omega(u_2)$ with $\varphi(u_2) < r_3$. □

5 Example

In this section, we give a simple example to explain the main result.

Example 5.1 For the problem (1.1), Let $\alpha = 2.8$, $\beta = 2.3$, $a = 4$, $b = 10$, $p = 2$, $g(t) = t$, then we get $q = 2$, $\int_0^1 g(t)\,\mathrm{d}t = \frac{1}{2}$, $\int_0^1 tg(t)\,\mathrm{d}t = \frac{1}{3}$,

$$k = \frac{[b - \int_0^1 tg(t)\,\mathrm{d}t] - [a - \int_0^1 g(t)\,\mathrm{d}t]}{b - \int_0^1 tg(t)\,\mathrm{d}t} = \frac{37}{58} \approx 0.637931.$$

Let

$$f(t,u) = \begin{cases} 23, & t \in [0,1], u \in [0,9], \\ 23 + 600(u-9), & t \in [0,1], u \in [9,10], \\ 623, & t \in [0,1], u \in [10,+\infty). \end{cases}$$

From a direct calculation, we get

$$f(t,u) > M_3 \approx 583.266938 \quad \text{for } t \in [0,1], u \in \left[10, \frac{580}{37}\right];$$

$$f(t,u) < M_2 \approx 36.538326 \quad \text{for } t \in [0,1], u \in [0,9];$$

$$f(t,u) > M_1 \approx 21.322041 \quad \text{for } t \in [0,1], u \in [0,0.5].$$

In view of Theorem 4.1, we see that the aforementioned problem has at least two positive solutions u_1 and u_2 such that $0.5 < \psi(u_1)$ with $\omega(u_1) < 9$ and $9 < \omega(u_2)$ with $\varphi(u_2) < 10$.

Competing interests
The author declares that they have no competing interests.

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