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Some properties of the solution to fractional heat equation with a fractional Brownian noise

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Abstract

In this paper, we consider the stochastic heat equation of the form

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + \frac{\partial^2 B}{\partial t\,\partial x},$$

where $\frac{\partial^2 B}{\partial t \partial x}$ is a fractional Brownian sheet with Hurst indices $H_1, H_2 \in (\frac{1}{2}, 1)$ and $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is a fractional Laplacian operator with $1 < \alpha \le 2$. In particular, when $H_2 = \frac{1}{2}$ we show that the temporal process $\{u(t,\cdot), 0 \le t \le T\}$ admits a nontrivial p-variation with $p = \frac{2\alpha}{2\alpha H_1 - 1}$ and study its local nondeterminism and existence of the local time.

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Keywords: fractional Brownian sheet; p-variation; local nondeterminism; local time

1 Introduction

The stochastic calculus of Gaussian processes is not only an important research direction in stochastic analysis, but also an important instrument. Many important Gaussian processes such as fractional Brownian motion, sub-fractional Brownian motion, bi-fractional Brownian motion and weighted-fractional Brownian motion have be studied. Some surveys and a complete list of literature for fBm could be found in Alós *et al.* [1], Nualart [2] and the references therein. On the other hand, stochastic heat equations driven by Gaussian noises are a recent research direction in probability theory and stochastic analysis, and many interesting studies have been given. We mention the works of Bo *et al.* [3], Chen *et al.* [4], Duncan *et al.* [5], Hajipour and Malek [6], M Hu *et al.* [7], Y Hu [8–10], Jiang *et al.* [11], Liu and Yan [12], Nualart and Ouknine [13], Tindel *et al.* [14], Walsh [15], Yang and Baleanu [16] and the references therein. Moreover, the solutions of linear stochastic heat equations with additive Gaussian noises are some Gaussian fields. Such a stochastic heat equation on $\mathbb{R}$ can be written as

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \mathcal{L}u + \frac{\partial^2}{\partial t\,\partial x} B(t,x), & t \in [0,T], x \in \mathbb{R}, \\ u(0,x) = 0, & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a quasi-differential operator and B is a two-parameter Gaussian field. Therefore, it seems interesting to study the properties and calculus for the solutions of equation (1.1) as some special Gaussian process.

When $\mathcal{L} = \Delta$ and B is a white noise, the solution of (1.1) satisfies

$$Eu(t,x)u(s,x) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{s+t} - \sqrt{|t-s|}\right)$$

for all $s,t \in [0,T]$ and $x \in \mathbb{R}$. In this case, the temporal process $\{u(t,\cdot), t \in [0,T]\}$ is a bi-fractional Brownian motion, and it admits a nontrivial quartic variation. More works can be found in Mueller and Tribe [17], Pospisil and Tribe [18], Sun and Yan [19], Swanson [20] and the references therein. When $\mathcal{L} = \Delta$ and B is a fractional noise with Hurst index $\frac{1}{2} < H < 1$, the solution of (1.1) satisfies

$$Eu(t,x)u(s,x) = \frac{H(2H-1)}{2\sqrt{2\pi}} \int_0^t \int_0^s |u-v|^{2H-2}(s+t-u-v)^{-\frac{1}{2}}\,du\,dv$$

for all $s,t \in [0,T]$ and $x \in \mathbb{R}$, which shows that the temporal process $\{u(t,\cdot), t \in [0,T]\}$ is a self-similar Gaussian process with the index $H - \frac{1}{4}$. Moreover, Ouahhabi and Tudor [21] studied the local nondeterminism and joint continuity of its local times of the solution to (1.1). When $\mathcal{L} = \Delta$ and B is a fractional noise in time with correlated spatial structure, Tudor and Xiao [22] studied various path properties of the solution process u with respect to the time and space variable. When $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ and B is a white noise, Cui *et al.* [23] and Wu [24] studied some properties and stochastic calculus of the solution of (1.1).

Motivated by the above results, in this paper we consider also equation (1.1) when $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ and W is a fractional Brownian sheet with Hurst indices $H_1, H_2 \in (\frac{1}{2}, 1)$. Our main objectives are to introduce the local nondeterminism, existence of the local time and p-variation of the solution. In Section 2, we give some basic notations on the fractional Laplacian operator $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ and the fractional Brownian sheet. In Section 3 we consider the time regularity of solution $u(t,x)$ to (1.1) with $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ and a fractional Brownian sheet B. In particular, when $H_2 = \frac{1}{2}$ we show that the temporal process $\{u(t,\cdot), t \in [0,T]\}$ of the solution satisfies

$$C_1|t-s|^{2H_1-\frac{1}{\alpha}} \le E\left|u(s,x) - u(t,x)\right|^2 \le C_2|t-s|^{2H_1-\frac{1}{\alpha}}$$

for any $t,s \in [0,T], x \in \mathbb{R}$. As a corollary, we see that the temporal process $\{u(t,\cdot), t \in [0,T]\}$ is nontrivial p-variation with $p = \frac{2\alpha}{2\alpha H_1 - 1}$. The existence of the local nondeterminism and the local times of the solution will be discussed in Section 4, respectively.

2 Preliminaries

In this section, we briefly recall some basic results for the Green function of the operator $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ and fractional Brownian sheet. We refer to Chen and Kumagai [25], Russo and Tudor [26] and the references therein for more details. Throughout this paper, for simplicity we let C stand for a positive constant and its value may be different in different appearances; and sometimes we also stress that it depends on some constants. For $x,y,z \in \mathbb{R}$, we denote $x_+ = \max(x,0)$ and

$$\mathcal{J}^+(x,y,z) = (x-y)_+^{z-\frac{3}{2}}, \qquad \mathcal{J}(x,y,z) = (x-y)^{z-\frac{3}{2}}, \qquad \mathcal{K}(x,y,z) = |x-y|^{2z-2}.$$

2.1 Fractional Laplacian operator $\Delta_\alpha = -(-\Delta)^{\alpha/2}$

Consider a symmetric α-stable motion $X = \{X_t, t \geq 0\}$ with $\alpha \in (0,2)$ on $\mathbb{R}$, and let its transition density function be $G_\alpha(x,t)$. Then we have

$$\int_{\mathbb{R}} G_\alpha(x,t)e^{izx}\,dx = e^{-t|z|^\alpha}$$

for all $t \geq 0$ and $z \in \mathbb{R}$, and $G_\alpha(x,t)$ is the fundamental solution of equation

$$\frac{\partial u}{\partial t} = \Delta_\alpha u.$$

Certainly, the kernel G_α is also called the heat kernel of the operator Δ_α. Denote

$$\mathcal{G}_\alpha(s,x;t,y) = G_\alpha(y-x,t-s)$$

for all $x,y \in \mathbb{R}$ and $s,t \geq 0$. It follows that

$$C_1^{-1}\left((t-s)^{-\frac{d}{\alpha}} \wedge \frac{a^\alpha(t-s)}{|x-y|^{d+\alpha}}\right) \leq \mathcal{G}_\alpha(s,y;t,x) \leq C_1\left((t-s)^{-\frac{d}{\alpha}} \wedge \frac{a^\alpha(t-s)}{|x-y|^{d+\alpha}}\right), \tag{2.1}$$

$$\left|\frac{\partial \mathcal{G}_\alpha(s,y;t,x)}{\partial t}\right| \leq \frac{C}{t-s}\mathcal{G}_\alpha(s,y;t,x) \tag{2.2}$$

for all $x,y \in \mathbb{R}, t > s \geq 0$ and some constant $C, C_1 > 1$, where $x \wedge y = \min\{x,y\}$ for $x,y \in \mathbb{R}$.

2.2 Fractional Brownian sheet

Recall that a two-parameter fractional Brownian sheet $B = \{B(t,x), t \in [0,T], x \in \mathbb{R}\}$ is a mean zero Gaussian random field with the covariance function

$$\begin{aligned}\Re_{H_1}(s,t)\Re_{H_2}(x,y) &= E\big(B(t,x)B(s,y)\big)\\ &= \frac{1}{2}\big(s^{2H_1} + t^{2H_1} - |s-t|^{2H_1}\big) \times \frac{1}{2}\big(|x|^{2H_2} + |y|^{2H_2} - |x-y|^{2H_2}\big)\end{aligned}$$

with $H_1, H_2 \in (0,1)$. Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{(s,t]\times(x,y]}$ on $[0,T] \times \mathbb{R}$ with respect to the scalar product

$$\langle 1_{[0,t]\times[0,x]}\rangle_{\mathcal{H}} = \Re_{H_1}(s,t)\Re_{H_2}(x,y),$$

where $1_{[0,t]\times[0,x]} = 1_{[0,t]\times[x,0]}$ if $x \leq 0$. Define a linear mapping Φ on $\mathcal{E}$ by

$$\varphi = 1_{[0,t]\times[0,x]} \longmapsto B(t,x) = \int_0^T \int_{\mathbb{R}} \varphi(s,y)B(ds,dy).$$

Then the mapping is an isometry between $\mathcal{E}$ and the Gaussian space associated with B. Moreover, the mapping can be extended to $\mathcal{H}$, and it is called the Wiener integral with respect to B which is denoted by

$$B(\varphi) := \int_0^T \int_{\mathbb{R}} \varphi(s,y)B(ds,dy), \quad \varphi \in \mathcal{H}.$$

Proposition 2.1 *If $\rho \in \mathcal{H}$, then*

$$\begin{aligned}\int_{[0,1]}\int_{\mathbb{R}} \rho(s,y)B(ds,dy) \\ = \int_{\mathbb{R}^2} W(ds,dy)\int_{\mathbb{R}} \rho(t,y)\mathcal{J}^{+}(t,s,H)1_{(0,1)}(t)\,dt,\end{aligned} \tag{2.3}$$

where $s \in \mathbb{R}, y \in \mathbb{R}$ and $W(s,y)$ is a space-time white noise.

Representation (2.3) can be obtained by using the moving average expression of the fractional Brownian motion. Notice that a similar transfer formula can be written using the representation of the fractional Brownian motion as Wiener integral on a finite interval (see, e.g., Nualart [2]). Denote

$$\Lambda_H(t,s;x,y) = 4H_1H_2(2H_1-1)(2H_2-1)\mathcal{K}(t,s,H_1)\mathcal{K}(x,y,H_2)$$

for any $0 \le s < t \le T$ and $x, y \in \mathbb{R}$. Thus, from Bo *et al.* [3], Jiang *et al.* [27] and Wei [28] one can give the following statements:

- For $H > \frac{1}{2}$, we have

$$L^{\frac{1}{H}}\big([0,T]\times\mathbb{R}\big) \subset \mathcal{H}.$$

- For $\varphi, \psi \in \mathcal{H}$, we have $E[B(\varphi)] = 0$ and

$$E\big[B(\varphi)B(\psi)\big] = \int_{[0,T]^2} dv\,du \int_{\mathbb{R}^2} \varphi(u,x)\psi(v,y)\Lambda_H(u,v;x,y)\,dy\,dx.$$

- If $H > \frac{1}{2}$ and $f, g \in L^{\frac{1}{H}}([a,b])$, then

$$\int_a^b\int_a^b f(u)g(v)\mathcal{K}(u,v,H)\,du\,dv \le C_H\|f\|_{L^{\frac{1}{H}}([a,b])}\|g\|_{L^{\frac{1}{H}}([a,b])}.$$

3 Some basic estimates of the solution

Given a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, P)$, where $\mathscr{F}_t$ is the σ-algebra generated by B up to time t. In this section, we introduce some basic estimates of the solution of the equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta_\alpha u + \frac{\partial^2 B}{\partial t\,\partial x}(t,x), \quad t \in [0,T], x \in \mathbb{R} \tag{3.1}$$

with initial value $u(0,x) = 0$, where B is a two-parameter fractional Brownian sheet with Hurst index $H_1, H_2 \in (\frac{1}{2}, 1)$. Clearly, the unique solution to (3.1) can be written as (see Walsh [15])

$$u(t,x) = \int_0^t\int_{\mathbb{R}} \mathcal{G}_\alpha(s,y;t,x)B(ds,dy) \tag{3.2}$$

for all $t \in [0,T]$ and $x \in \mathbb{R}$.

Proposition 3.1 *The unique solution* (3.2) *satisfies*

$$\sup_{t\in[0,T],x\in\mathbb{R}} E\big|u(t,x)\big|^p < \infty$$

for all $T > 0$, $\alpha \in (1,2)$, $H_1, H_2 \in (\frac{1}{2},1)$ *and* $p \geq 2$.

Proof Clearly, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left((t-s)^{-\frac{1}{\alpha}} \wedge \frac{t-s}{|x-y|^{1+\alpha}}\right)^{\frac{1}{H_2}} dy \\
&\quad = 2\int_0^{(t-s)^{\frac{1}{\alpha}}} (t-s)^{-\frac{1}{\alpha H_2}}\, dy + 2\int_{(t-s)^{\frac{1}{\alpha}}}^{\infty} \left(\frac{t-s}{u^{1+\alpha}}\right)^{\frac{1}{H_2}} du \\
&\quad = C(t-s)^{\frac{H_2-1}{\alpha H_2}}
\end{aligned}$$

for all $t > s > 0$ and $x \in \mathbb{R}$. It follows that

$$\begin{aligned}
\big\|\mathcal{G}_\alpha(s,\cdot;t,x)\big\|_{L^{\frac{1}{H_2}}(\mathbb{R})} &= \left(\int_{\mathbb{R}} \mathcal{G}_\alpha(s,y;t,x)^{\frac{1}{H_2}}\, dy\right)^{H_2} \\
&\leq C\left(\int_{\mathbb{R}} \left((t-s)^{-\frac{1}{\alpha}} \wedge \frac{(t-s)}{|x-y|^{1+\alpha}}\right)^{\frac{1}{H_2}} dy\right)^{H_2} \\
&\leq C(t-s)^{\frac{H_2-1}{\alpha}}
\end{aligned}$$

for all $t > s > 0$ and $x \in \mathbb{R}$, which implies that

$$\begin{aligned}
&E\big|u(t,x)\big|^p \\
&\quad = E\left|\int_0^t \int_{\mathbb{R}} \mathcal{G}_\alpha(s,z;t,x) B(ds,dz)\right|^p \\
&\quad \leq C\left(\int_{[0,t]^2} ds_1\, ds_2 \int_{\mathbb{R}^2} \mathcal{G}_\alpha(s_1,z_1;t,x)\Lambda_H(s_1,s_2;z_1,z_2)\mathcal{G}_\alpha(s_2,z_2;t,x)\, dz_1\, dz_2\right)^{\frac{p}{2}} \\
&\quad = C\left(\int_{[0,t]^2} \mathcal{K}(s_1,s_2,H_1)\, ds_1\, ds_2 \int_{\mathbb{R}^2} \mathcal{K}(z_1,z_2,H_2)\mathcal{G}_\alpha(s_1,z_1;t,x)\mathcal{G}_\alpha(s_2,z_2;t,x)\, dz_1\, dz_2\right)^{\frac{p}{2}} \\
&\quad \leq C\left(\int_{[0,t]^2} \mathcal{K}(s_1,s_2,H_1)\big\|\mathcal{G}_\alpha(s_1,\cdot;t,x)\big\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\big\|\mathcal{G}_\alpha(s_2,\cdot;t,x)\big\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\, ds_1\, ds_2\right)^{\frac{p}{2}} \\
&\quad \leq C\left(\int_0^t \left(\big\|\mathcal{G}_\alpha(s,\cdot;t,x)\big\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\right)^{\frac{1}{H_1}} ds\right)^{pH_1} \leq Ct^{\frac{(\alpha H_1+H_2-1)p}{\alpha}}
\end{aligned}$$

for all $t > s > 0$ and $x \in \mathbb{R}$. Thus, we have showed that

$$\sup_{t\in[0,T],x\in\mathbb{R}} E\big|u(t,x)\big|^p < \infty,$$

and the proposition follows. □

Now, we give the time regularity of solution (3.2) and sharp upper and lower bounds for the L^2-norm of increments.

Theorem 3.2 *Let $u(t,x)$ be the solution of* (3.1). *We then have that*

$$E\left|u(s,x)-u(t,x)\right|^2 \le C_2|t-s|^{2\vartheta} \tag{3.3}$$

for any $t,s\in[0,T], x\in\mathbb{R}$ and $\vartheta\in(0,\frac{\alpha H_1+H_2-1}{\alpha})$. In particular, when $H_2=\frac{1}{2}$, we have

$$C_1|t-s|^{2H_1-\frac{1}{\alpha}} \le E\left|u(s,x)-u(t,x)\right|^2 \le C_2|t-s|^{2H_1-\frac{1}{\alpha}}$$

for any $t,s\in[0,T], x\in\mathbb{R}$.

In order to prove the theorem, we need the following lemma.

Lemma 3.1 *There exists a constant $C>0$ such that*

$$\int_{\mathbb{R}}\left(\left|\frac{\partial\mathcal{G}_\alpha}{\partial t}(r,z;t,x)\right|^{\vartheta}\left|\mathcal{G}_\alpha(r,z;t,x)\right|^{1-\vartheta}\right)^{\frac{1}{H_2}}dz \le C(t-r)^{\frac{H_2-\alpha\vartheta-1}{\alpha H_2}} \tag{3.4}$$

for all $0<r<t\le T$, $x\in\mathbb{R}$ and $\vartheta\in(0,1)$. Moreover, when $\vartheta<\frac{\alpha H_1+H_2-1}{\alpha}$, we have

$$\int_0^t\left(\int_{\mathbb{R}}\left(\left|\frac{\partial\mathcal{G}_\alpha}{\partial t}(r,z;t,x)\right|^{\vartheta}\left|\mathcal{G}_\alpha(r,z;t,x)\right|^{1-\vartheta}\right)^{\frac{1}{H_2}}dz\right)^{\frac{H_2}{H_1}}dr \le C \tag{3.5}$$

for all $t\in[0,T]$ and $x\in\mathbb{R}$.

Proof Denote $D_z=\{|x-z|<(t-r)^{\frac{1}{\alpha}}\}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\left|\frac{\partial\mathcal{G}_\alpha}{\partial t}(r,z;t,x)\right|^{\vartheta}\left|\mathcal{G}_\alpha(r,z;t,x)\right|^{1-\vartheta}\right)^{\frac{1}{H_2}}dz\\
&\quad\le\int_{D_z}\left(\left|\frac{(t-r)^{-\frac{1}{\alpha}}}{t-r}\right|^{\frac{\vartheta}{H_2}}\cdot\left|(t-r)^{-\frac{1}{\alpha}}\right|^{\frac{1-\vartheta}{H_2}}\right)dz\\
&\qquad+\int_{\overline{D}_z}\left(\left|\frac{\frac{t-r}{|x-z|^{1+\alpha}}}{t-r}\right|^{\frac{\vartheta}{H_2}}\cdot\left|\frac{t-r}{|x-z|^{1+\alpha}}\right|^{\frac{1-\vartheta}{H_2}}\right)dz\\
&\quad\le C(t-r)^{\frac{H_2-\alpha\vartheta-1}{\alpha H_2}}
\end{aligned}$$

for all $t>r>0$ and $x\in\mathbb{R}$, and (3.4) and (3.5) follow. □

Lemma 3.2 *When $\frac{1}{2}<H_1<1$, we have*

$$\int_{[0,1]^2}dv\,dr(1-r)^{-\frac{1}{\alpha}}\mathcal{K}(r,v,H_1)\int_0^{\frac{r\wedge v}{v\vee r}}z^{H_1-\frac{3}{2}}(1-z)^{1-2H_1}\,dz<\infty.$$

Proof By some elementary calculations and the properties of beta functions, the consequence is obvious. □

Proof of Theorem 3.2 We shall divide the proof into two steps.

Step 1. We first consider the upper bound. Denote

$$\begin{aligned} |A_1(t,s,x)| &= \left|\int_0^s \int_{\mathbb{R}} \big(\mathcal{G}_\alpha(r,z;t,x) - \mathcal{G}_\alpha(r,z;s,x)\big)B(dz,dr)\right|, \\ |A_2(t,s,x)| &= \left|\int_s^t \int_{\mathbb{R}} \mathcal{G}_\alpha(r,z;t,x)B(dz,dr)\right| \end{aligned}$$

for each $x \in \mathbb{R}$ and $0 \le s < t \le T$. Then we have

$$|u(t,x) - u(s,x)| \le |A_1(t,s,x)| + |A_2(t,s,x)|$$

for each $x \in \mathbb{R}$ and $0 \le s < t \le T$. Moreover, for every $\vartheta \in (0,1)$, we let

$$\begin{aligned} |A_{1,1}(t,s,x)| &= \big\| |\mathcal{G}_\alpha(\cdot,\cdot;t,x) - \mathcal{G}_\alpha(\cdot,\cdot;s,x)|^\vartheta \cdot |\mathcal{G}_\alpha(\cdot,\cdot;t,x)|^{1-\vartheta} \big\|_{\mathcal{H}}^2, \\ |A_{1,2}(t,s,x)| &= \big\| |\mathcal{G}_\alpha(\cdot,\cdot;t,x) - \mathcal{G}_\alpha(\cdot,\cdot;s,x)|^\vartheta \cdot |\mathcal{G}_\alpha(\cdot,\cdot;s,x)|^{1-\vartheta} \big\|_{\mathcal{H}}^2 \end{aligned}$$

for each $x \in \mathbb{R}$ and $0 \le s < t \le T$. Then we have

$$\begin{aligned} E|A_1(t,s,x)|^2 &\le C\|\mathcal{G}_\alpha(\cdot,\cdot;t,x) - \mathcal{G}_\alpha(\cdot,\cdot;s,x)\|_{\mathcal{H}}^2 \\ &\le C\big\| |\mathcal{G}_\alpha(\cdot,\cdot;t,x) - \mathcal{G}_\alpha(\cdot,\cdot;s,x)|^\vartheta \cdot |\mathcal{G}_\alpha(\cdot,\cdot;t,x) - \mathcal{G}_\alpha(\cdot,\cdot;s,x)|^{1-\vartheta} \big\|_{\mathcal{H}}^2 \\ &\le C\big(|A_{1,1}(t,s,x)| + |A_{1,2}(t,s,x)|\big) \end{aligned}$$

for all $x \in \mathbb{R}$ and $0 \le s < t \le T$. Using (2.1), Proposition 2.1, Lemma 3.1 and the mean-value theorem, for $\eta \in (s,t)$, one can get

$$\begin{aligned} &|A_{1,1}(t,s,x)| \\ &\quad = \left\| \left|\frac{\partial \mathcal{G}_\alpha}{\partial t}(\cdot,\cdot;\eta,x)\right|^\vartheta |t-s|^\vartheta \cdot |\mathcal{G}_\alpha(\cdot,\cdot;t,x)|^{1-\vartheta} \right\|_{\mathcal{H}}^2 \\ &\quad = |t-s|^{2\vartheta} \int_{[0,t]^2} dr_1\,dr_2 \int_{\mathbb{R}^2} \left|\frac{\partial \mathcal{G}_\alpha}{\partial t}(r_1,z_1;\eta,x)\right|^\vartheta |\mathcal{G}_\alpha(r_1,z_1;t,x)|^{1-\vartheta} \\ &\qquad \times \Lambda_H(r_1,r_2;z_1,z_2)\left|\frac{\partial \mathcal{G}_\alpha}{\partial t}(r_2,z_2;\eta,x)\right|^\vartheta |\mathcal{G}_\alpha(r_2,z_2;t,x)|^{1-\vartheta}\,dz_1\,dz_2 \\ &\quad \le C|t-s|^{2\vartheta}\left(\int_0^T \left(\int_{\mathbf{R}} \left(\left|\frac{\partial \mathcal{G}_\alpha}{\partial t}(r,z;t,x)\right|^\vartheta |\mathcal{G}_\alpha(r,z;t,x)|^{1-\vartheta}\right)^{\frac{1}{H_2}} dz\right)^{\frac{H_2}{H_1}} dr\right)^{2H_1} \\ &\quad \le C|t-s|^{2\vartheta} \end{aligned}$$

for all $x \in \mathbb{R}$ and $0 \le s < t \le T$, which gives

$$|A_{1,1}(t,s,x)| \le C|t-s|^{2\vartheta}$$

for all $x \in \mathbb{R}$ and $0 \le s < t \le T$. Similarly, one can prove that

$$|A_{1,2}(t,s,x)| \le C|t-s|^{2\vartheta}$$

for all $x \in \mathbb{R}$ and $0 \leq s < t \leq T$. It follows that

$$E\left|A_1(t,s,x)\right|^p \leq C|t-s|^{2\vartheta} \tag{3.6}$$

when $\vartheta \in (0, \frac{\alpha H_1 + H_2 - 1}{\alpha})$.

On the other hand, we have that

$$\begin{aligned}
&E\left|A_2(t,s,x)\right|^2 \\
&\quad = E\left|\int_s^t \int_{\mathbb{R}} \mathcal{G}_\alpha(r,z;t,x)B(dr,dz)\right|^2 \\
&\quad \leq C\int_{[s,t]^2} dr_1\,dr_2 \int_{\mathbb{R}^2} \mathcal{G}_\alpha(r_1,z_1;t,x)\Lambda_H(r_1,r_2;z_1,z_2)\mathcal{G}_\alpha(r_2,z_2;t,x)\,dz_1\,dz_2 \\
&\quad = C\int_{[s,t]^2} \mathcal{K}(r_1,r_2,H_1)\,dr_1\,dr_2 \int_{\mathbb{R}^2} \mathcal{K}(z_1,z_2,H_2) \\
&\qquad \times \mathcal{G}_\alpha(r_1,z_1;t,x)\mathcal{G}_\alpha(r_2,z_2;t,x)\,dz_1\,dz_2 \\
&\quad \leq C\int_{[s,t]^2} \mathcal{K}(r_1,r_2,H_1)\left\|\mathcal{G}_\alpha(r_1,\cdot;t,x)\right\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\left\|\mathcal{G}_\alpha(r_2,\cdot;t,x)\right\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\,dr_1\,dr_2 \\
&\quad \leq C\left(\int_s^t \left(\left\|\mathcal{G}_\alpha(r,\cdot;t,x)\right\|_{L^{\frac{1}{H_2}}(\mathbb{R})}\right)^{\frac{1}{H_1}}dr\right)^{2H_1} \\
&\quad \leq C|t-s|^{2\vartheta}
\end{aligned} \tag{3.7}$$

for all $x \in \mathbb{R}$ and $0 \leq s < t \leq T$. Combining (3.6) and (3.7), we get

$$E\left|u(t,x) - u(s,x)\right|^2 \leq C|t-s|^{2\vartheta}$$

for all $x \in \mathbb{R}$ and $0 \leq s < t \leq T$.

Step 2. We consider the lower bound. We have that

$$\begin{aligned}
u(t,x) - u(s,y) = {} & \int_0^1 \int_{\mathbb{R}} \mathcal{G}_\alpha(\omega,y;t,x)1_{(0,1)}(\omega)B(d\omega,dy) \\
& - \int_0^1 \int_{\mathbb{R}} \mathcal{G}_\alpha(\omega,y;s,x)1_{(0,s)}(\omega)B(d\omega,dy)
\end{aligned}$$

for $s,t \in [0,T]$ and $x,y \in \mathbb{R}$. Let B be fractional in time and white in space, that is, $H_1 \in (\frac{1}{2},1), H_2 = \frac{1}{2}$. By the transfer rule (2.3) we have

$$\begin{aligned}
&u(t,x) - u(s,y) \\
&\quad = \int_{\mathbb{R}^2} W(d\omega,dy)\int_{\mathbb{R}} dv\mathcal{J}^+(v,\omega,H_1)\mathcal{G}_\alpha(v,y;t,x)1_{(0,t)}(v) \\
&\qquad - \int_{\mathbb{R}^2} W(d\omega,dy)\int_{\mathbb{R}} dv\mathcal{J}^+(v,\omega,H_1)\mathcal{G}_\alpha(v,y;s,x)1_{(0,s)}(v)
\end{aligned}$$

for $s,t \in [0,T]$ and $x,y \in \mathbb{R}$. Denote

$$\mathcal{O}_{r,1}(v) = \int_{\mathbb{R}} dy\mathcal{G}_\alpha(v,y;t,x)\mathcal{G}_\alpha(r,y;t,x),$$

$$\mathcal{O}_{r,2}(v) = \int_s^{v\wedge r} \mathcal{J}(v,\omega,H_1)\mathcal{J}(r,\omega,H_1)\,d\omega$$

for $v,s,t \in [0,T]$ and $w,x,y \in \mathbb{R}$. By the isometry of the Brownian motion W and $\mathcal{G}_\alpha(v,y;s,x) = 0$, when $v > s$, it follows that

$$\begin{aligned}
&E\big|u(t,x) - u(s,x)\big|^2 \\
&\quad= \int_{\mathbb{R}^2} d\omega\,dy\biggl(\int_{\mathbb{R}} \mathcal{J}^+(v,\omega,H_1)\bigl(\mathcal{G}_\alpha(v,y;t,x)1_{(0,t)}(v) - \mathcal{G}_\alpha(v,y;s,x)1_{(0,s)}(v)\bigr)\,dv\biggr)^2 \\
&\quad\ge \int_s^t \int_{\mathbb{R}} d\omega\,dy\biggl(\int_{\mathbb{R}} \mathcal{J}^+(v,\omega,H_1)\bigl(\mathcal{G}_\alpha(v,y;t,x)1_{(0,t)}(v) - \mathcal{G}_\alpha(v,y;s,x)1_{(0,s)}(v)\bigr)\,dv\biggr)^2 \\
&\quad= \int_s^t \int_{\mathbb{R}} d\omega\,dy\biggl(\int_\omega^t \mathcal{J}^+(v,\omega,H_1)\mathcal{G}_\alpha(v,y;t,x)\,dv\biggr)^2 \\
&\quad\ge \int_s^t \int_{\mathbb{R}} d\omega\,dy \int_{[\omega,t]^2} dr\,dv\mathcal{G}_\alpha(v,y;t,x)\mathcal{J}(v,\omega,H_1)\mathcal{G}_\alpha(r,y;t,x)\mathcal{J}(r,a,H_1) \\
&\quad= \int_{[s,t]^2}\biggl(\int_s^{v\wedge r} \mathcal{J}(v,\omega,H_1)\mathcal{J}(r,\omega,H_1)\,d\omega\biggr)\,dv\,dr \int_{\mathbb{R}} dy\mathcal{G}_\alpha(v,y;t,x)\mathcal{G}_\alpha(r,y;t,x) \\
&\quad\ge \int_{[s,t]^2} \mathcal{O}_{r,1}(v)\mathcal{O}_{r,2}(v)\,dv\,dr
\end{aligned}$$

for $s,t \in [0,T]$ and $x,y \in \mathbb{R}$. Denote

$$D_1 = \bigl\{|y-x| < (t-v)^{\frac{1}{\alpha}}\bigr\}, \qquad D_2 = \bigl\{|y-x| \ge (t-r)^{\frac{1}{\alpha}}\bigr\}$$

for every $x,y \in \mathbb{R}$, $t > r > 0$ and $t > v > 0$. Some elementary calculations can show that

$$\begin{aligned}
\mathcal{O}_{r,1}(v) &\ge C\int_{\mathbb{R}}\biggl(\frac{t-v}{|y-x|^{1+\alpha}} \wedge (t-v)^{-\frac{1}{\alpha}}\biggr)\biggl(\frac{t-r}{|y-x|^{1+\alpha}} \wedge (t-r)^{-\frac{1}{\alpha}}\biggr)\,dy \\
&= C\int_{D_1} (t-v)^{-\frac{1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\,dy + C\int_{\bar{D_1}\bar{D_2}} \frac{t-v}{|y-x|^{1+\alpha}}(t-r)^{-\frac{1}{\alpha}}\,dy \\
&\quad + C\int_{D_2} \frac{t-v}{|y-x|^{1+\alpha}}\,\frac{t-r}{|y-x|^{1+\alpha}}\,dy \\
&\ge C(t-r)^{-\frac{1}{\alpha}}
\end{aligned} \tag{3.8}$$

for $0 < r < v$. Similarly, when $r > v$, we have

$$\begin{aligned}
\mathcal{O}_{r,1}(v) &\ge C\int_{\mathbb{R}}\biggl(\frac{t-v}{|y-x|^{1+\alpha}} \wedge (t-v)^{-\frac{1}{\alpha}}\biggr)\biggl(\frac{t-r}{|y-x|^{1+\alpha}} \wedge (t-r)^{-\frac{1}{\alpha}}\biggr)\,dy \\
&= C\int_{D_1} (t-v)^{-\frac{1}{\alpha}}(t-r)^{-\frac{1}{\alpha}}\,dy + C\int_{\bar{D_1}\bar{D_2}} (t-v)^{-\frac{1}{\alpha}}\frac{t-r}{|y-x|^{1+\alpha}}\,dy \\
&\quad + C\int_{D_2} \frac{t-v}{|y-x|^{1+\alpha}}\,\frac{t-r}{|y-x|^{1+\alpha}}\,dy \\
&\ge C(t-v)^{-\frac{1}{\alpha}}.
\end{aligned} \tag{3.9}$$

Moreover, when $0 < r < \nu$, setting $z = (r - \omega)/(\nu - \omega)$, we have

$$\mathcal{O}_{r,2}(\nu) = (\nu - r)^{2H_1-2} \int_0^{\frac{r-s}{\nu-s}} z^{H_1-\frac{3}{2}}(1-z)^{1-2H_1}\,dz, \tag{3.10}$$

and when $r > \nu$, let $z = (\nu - \omega)/(r - \omega)$, we have

$$\mathcal{O}_{r,2}(\nu) = (r - \nu)^{2H_1-2} \int_0^{\frac{\nu-s}{r-s}} z^{H_1-\frac{3}{2}}(1-z)^{1-2H_1}\,dz. \tag{3.11}$$

Let

$$\Upsilon_1(s,r,\nu) = \frac{r-s}{\nu-s}\mathbb{1}_{\{r<\nu\}} + \frac{\nu-s}{r-s}\mathbb{1}_{\{r>\nu\}},$$
$$\Upsilon_2(r,\nu) = \frac{r}{\nu}\mathbb{1}_{\{r<\nu\}} + \frac{\nu}{r}\mathbb{1}_{\{r>\nu\}}$$

for all $t > r > 0$ and $t > \nu > 0$. It follows from the substitutions $(r,\nu) \to (r+s, \nu+s)$ and $(r,\nu) \to ((t-s)r, (t-s)\nu)$ that

$$\begin{aligned}
E\big|u(t,x) - u(s,x)\big|^2 &\geq \int_{[s,t]^2} d\nu\,dr(t-r)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r<\nu\}}\Lambda\big(\Upsilon_1(s,r,\nu)\big) \\
&\quad + \int_{[s,t]^2} d\nu\,dr(t-\nu)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r>\nu\}}\Lambda\big(\Upsilon_1(s,r,\nu)\big) \\
&= \int_{[0,t-s]^2} d\nu\,dr(t-s-r)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r<\nu\}}\Lambda\big(\Upsilon_2(r,\nu)\big) \\
&\quad + \int_{[0,t-s]^2} d\nu\,dr(t-s-\nu)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r>\nu\}}\Lambda\big(\Upsilon_2(r,\nu)\big) \\
&= (t-s)^{2H_1-\frac{1}{\alpha}} \int_{[0,1]^2} d\nu\,dr(1-r)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r<\nu\}}\Lambda\big(\Upsilon_2(r,\nu)\big) \\
&\quad + (t-s)^{2H_1-\frac{1}{\alpha}} \int_{[0,1]^2} d\nu\,dr(1-\nu)^{-\frac{1}{\alpha}}\mathcal{K}(r,\nu,H_1)\mathbb{1}_{\{r>\nu\}}\Lambda\big(\Upsilon_2(r,\nu)\big) \\
&\geq C(t-s)^{2H_1-\frac{1}{\alpha}}
\end{aligned}$$

for all $t > s > 0$ and $x \in \mathbb{R}$, where

$$\Lambda(x) = \int_0^x z^{H_1-\frac{3}{2}}(1-z)^{1-2H_1}\,dz, \quad x \in [0,1].$$

This completes the proof. □

At the end of this section, we give the p-variations of solution (3.2). For a continuous process $U = \{U_t; 0 \leq t < T\}$, we define

$$\mathcal{V}_{p,n}(U;T) := \sum_{k=1}^{n} |U_{t_k} - U_{t_{k-1}}|^p,$$

where $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ is an arbitrary partition of $[0,T]$ such that $\max_k |t_k - t_{k-1}|$ tends to zero as $n \to \infty$. The process U is said to be of bounded p-variation with

$p \geq 1$ on the interval $[0, T]$ if

$$\mathcal{V}_p(U; T) := \lim_{n\to\infty} \mathcal{V}_{p,n}(U; T)$$

exists in L^1 as $n \to \infty$.

Theorem 3.3 *Let $u(t,x)$ be the solution of* (3.1) *with $H_1 \in (\frac{1}{2}, 1)$ and $H_2 = \frac{1}{2}$. Denote $W_x = u(t,x), t \in [0,T]$ for $x \in \mathbb{R}$. Then there exists a constant $\beta > 0$ depending only on H_1, T and α such that*

$$\mathcal{V}_p(W_x; T) = \beta$$

if $p = \frac{2\alpha}{2\alpha H_1 - 1}$.

When $\alpha = 2$ and $H_1 = H_2 = \frac{1}{2}$, we know that the temporal process $W_x = u(t,x), t \in [0,T]$ for $x \in \mathbb{R}$ admits a nontrivial quartic variation (see, for example, Swanson [20]). Thus, the above theorem is a natural extension.

Proof of Theorem 3.3 Let $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be an arbitrary partition of $[0,T]$ such that $\max_k\{t_k - t_{k-1}\}$ tends to zero as $n \to \infty$. By Theorem 3.2 we have that

$$\begin{aligned} E\big(\mathcal{V}_p(W_x; T)\big) &= E\left(\sum_{k=1}^{n} |W_{t_k,x} - W_{t_{k-1},x}|^p\right) \\ &= \sum_{k=1}^{n} E|W_{t_k,x} - W_{t_{k-1},x}|^p \\ &= C_p \sum_{k=1}^{n} \big(E|W_{t_k,x} - W_{t_{k-1},x}|^2\big)^{\frac{p}{2}} \\ &\asymp C_p \sum_{k=1}^{n} |t_k - t_{k-1}|^{\frac{p(2\alpha H_1 - 1)}{2\alpha}} \\ &\asymp CT, \end{aligned}$$

which shows that the p-variation of the temporal process W_x is nontrivial if $p = \frac{2\alpha}{2\alpha H_1 - 1}$ for all $x \in \mathbb{R}$, where the notation $f \asymp h$ denotes

$$cf(x) \leq h(x) \leq Cf(x)$$

in the common domain of definition for f and h. This completes the proof. □

4 Existence and regularity of the local times of the solution

We devote this section to discussion on the existence and regularity of the local time of the temporal process $W_x = \{u(t,x), t \in [0,T]\}$ of solution (3.2). For convenience we take $x = 1$ and $T = 1$. Denote $u(t,1) = u(t), t \in [0,1]$.

Let $X = \{X(t), t \in I\}$ be a real-valued separable stochastic process. For every pair of linear Borel sets $\mathcal{B} \subset \mathbb{R}_+$ and $K \subset [0,1]$, the occupation measure of X on $\mathcal{B}$ is defined as follows:

$$\nu_K(\mathcal{B}) = \mathcal{L}\big\{s \in K : X(s) \in \mathcal{B}\big\}, \quad \mathcal{B} \in \mathscr{B}(\mathbb{R}),$$

where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure. If, for fixed K, ν_K is absolutely continuous as a measure of $\mathcal{B}$, we say that $X(t)$ has local time on K. The local time is defined as the Radon-Nykodim derivative of ν_K

$$\ell(K,x) = \frac{d}{d\mathcal{L}}\nu_K(x), \quad x \in \mathbb{R}.$$

We will use the notation

$$\ell(t,x) := \ell\big([0,t],x\big), \quad t \in \mathbb{R}_+, x \in \mathbb{R}.$$

Moreover, $\ell(t,x)$ satisfies the following occupation density formula:

$$\int_K f\big(X(t)\big)\,dt = \int_{\mathbf{R}} f(x)\ell(K,x)\,dx$$

for every Borel set K in I and for every measurable function $f : \mathbb{R} \to \mathbb{R}$, see Geman and Horowitz [29].

We prove the existence of the local time of u. The result is a consequence of the left-hand side of inequality (3.3) and a result in Berman [30]. We first need to show that the temporal process $\{u(t,x), t \in [0,1]\}$ is local nondeterminism. The concept of local nondeterminism was first introduced by Berman [31] to unify and extend his methods for studying the existence and joint continuity of local times of real-valued Gaussian processes. We refer to Cuzick and DuPreez [32], Xiao [33] and the references therein for more details and some extensions.

Definition 4.1 Let I be a closed interval on $\mathbb{R}_+$ and $Y = \{Y(t), t \in I\}$ be a stochastic process. For fixed $\kappa \in (0,1)$ and all $s,t \in \mathbb{R}_+$, we define the metric

$$\nu_\kappa(s,t) = |t-s|^\kappa. \tag{4.1}$$

Then Y is said to be local ν_κ-nondeterministic on I if there exists a constant $C > 0$ such that for any integer $n \geq 1$ and for all points $t_1, \ldots, t_n \in I$,

$$\operatorname{Var}\big(Y(t_n)|Y(t_1),\ldots,Y(t_{n-1})\big) \geq C|t_n - t_{n-1}|^{2\kappa}. \tag{4.2}$$

The concept of local nondeterminism was extended by Cuzick [34] who defined local ν_τ-nondeterminism. As an immediate consequence of Definition 4.1, $Y(t)$ has strong local ν_τ-nondeterminism on I if and only if there exist $C, r_0 > 0$ such that

$$\operatorname{Var}\big(Y(t)|Y(s), s \in T, r \leq |t-s| \leq r_0\big) \geq C\nu_\kappa(r) \tag{4.3}$$

for all $t \in I$ and $0 < r \leq \min(t, r_0)$.

Proposition 4.1 *Let $\{u(t,x), t \in [0,1], x \in \mathbb{R}\}$ be the solution of* (3.1), *and let ν_κ be given by* (4.1) *with $\kappa = H_1 - \frac{1}{2\alpha}$. Then the temporal process $W_x = \{u(t,x), t \in [0,T]\}$ is strong local ν_κ-nondeterministic for every fixed $x \in \mathbb{R}$.*

Proof Let $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < 1$ be arbitrary points in $(0,1)$ and $\kappa_1, \ldots, \kappa_{n-1} \in \mathbb{R}$. The local nondeterministic property will follow if we prove that

$$E\left(u(t_n,x) - \sum_{i=1}^{n-1} \kappa_i u(t_i,x)\right)^2 \geq C|t_n - t_{n-1}|^{2H_1-\frac{1}{\alpha}}.$$

Using the transfer formula (2.3), we have

$$\begin{aligned}
&u(t_n,x) - \sum_{i=1}^{n-1} \kappa_i u(t_i,y) \\
&= \int_0^1 \int_{\mathbb{R}} \mathcal{G}_\alpha(s,y;t_n,x) 1_{(0,t_n)}(s) B(ds,dy) \\
&\quad - \int_0^1 \int_{\mathbb{R}} \sum_{i=1}^{n-1} \kappa_i \mathcal{G}_\alpha(s,y;t_i,x) 1_{(0,t_i)}(s) B(ds,dy) \\
&= \int_0^1 \int_{\mathbb{R}} W(ds,dy) \int_{\mathbb{R}} \mathcal{J}^+(v,s,H_1) \mathcal{G}_\alpha(v,y;t_n,x) 1_{(0,t_n)}(v)\,dv \\
&\quad - \int_0^1 \int_{\mathbb{R}} W(ds,dy) \int_{\mathbb{R}} \mathcal{J}^+(v,s,H_1) \sum_{i=1}^{n-1} \kappa_i \mathcal{G}_\alpha(v,y;t_i,x) 1_{(0,t_i)}(v)\,dv,
\end{aligned}$$

where B is a two-dimensional Brownian sheet. By the isometry of the stochastic integral with respect to B, bounding below the integral over the interval (t_{n-1}, t_n) and (3.3), it follows that

$$\begin{aligned}
&E\left(u(t_n,x) - \sum_{i=1}^{n-1} \kappa_i u(t_i,y)\right)^2 \\
&\geq \int_{t_{n-1}}^{t_n} ds \int_{\mathbb{R}} dy \left(\int_s^{t_n} dv \mathcal{G}_\alpha(v,y;t_n,x) \mathcal{J}^+(v,s,H_1)\right)^2 \\
&\geq C(t_n - t_{n-1})^{2H_1-\frac{1}{\alpha}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2 *The process $\{u(t), t \in [0,1]\}$ has a local time $\ell([a,b],x), x \in \mathbb{R}$. Moreover, on each time interval $K = [a,b] \subset [0,\infty)$,*

$$E \int_{\mathbb{R}} \ell(K,x)^2\,dx < \infty, \quad a.s.$$

Moreover, the local time admits the following L^2-integral representation:

$$\ell(K,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izx} \int_K e^{izu(s)}\,ds\,dz. \tag{4.4}$$

Proof By Berman [30] (see also Lemma 8.1 in Xiao [33]), for any continuous and zero-mean Gaussian process $X = (X(t), t \in [0,T])$ with bounded variance function, the condition

$$\int_{[0,T]^2} \frac{ds\,dt}{\sqrt{E[X(t)-X(s)]^2}} < \infty$$

is sufficient for the local time of X to exist and to be square integrable. According to Theorem 3.2, for all $K = [a,b]$ interval of $[0,1]$, we have

$$\int_{K^2} \frac{ds\,dt}{\sqrt{E(u(t)-u(s))^2}} < C \int_{K^2} \frac{ds\,dt}{\sqrt{(t-s)^{2H_1-\frac{1}{\alpha}}}} < \infty. \tag{4.5}$$

Formula (4.4) is a consequence of Lemma 8.1 in Xiao [33]. □

At the end, let us prove now the joint continuity of the local time of u.

Theorem 4.3 *For any integer $k \geq 2$, there exists a finite constant $C_k > 0$ such that, for all $t \in [0,1], \delta \in (0,1), x, x' \in \mathbb{R}$, and*

$$0 < \zeta < \frac{1 - H_1 + \frac{1}{2\alpha}}{2H_1 - \frac{1}{\alpha}}, \tag{4.6}$$

it holds

$$E\big[\ell(t+\delta,x) - \ell(t,x) - \ell\big(t+\delta,x'\big) + \ell\big(t,x'\big)\big]^k \leq C_k \big|x - x'\big|^{\zeta k} \delta^{k(1-(H_1-\frac{1}{2\alpha})(1-\zeta))}. \tag{4.7}$$

Proof From (4.4), for any $x, x' \in \mathbb{R}, \mathcal{A}(t) := t < t_1 < \cdots < t_k < t+\delta \in [0,1]$, let $v_j = z_j - z_{j+1}, j = 1, \ldots, k-1$ and $v_k = z_k$, let $\varepsilon_j = 0, 1,$ or 2, and $\sum_{j=1}^k \varepsilon_j = k$, we have

$$\begin{aligned}
&E\big[\ell(t+\delta,x) - \ell(t,x) - \ell\big(t+\delta,x'\big) + \ell\big(t,x'\big)\big]^k \\
&\quad = \frac{1}{(2\pi)^k} \int_{[t,t+\delta]^k} \prod_{j=1}^k ds_j \int_{\mathbb{R}^k} \prod_{j=1}^k \big(e^{-iv_j x} - e^{-iv_j x'}\big) E\big(e^{i\sum_{j=1}^k v_j u(s_j)}\big) \prod_{j=1}^k dv_j \\
&\quad \leq C_k \big|x - x'\big|^{k\zeta} \int_{[t,t+\delta]^k} \prod_{j=1}^k ds_j \int_{\mathbb{R}^k} \prod_{j=1}^k |v_j|^\zeta E\big(e^{i\sum_{j=1}^k v_j u(s_j)}\big) \prod_{j=1}^k dv_j \\
&\quad \leq C_k \big|x - x'\big|^{k\zeta} \int_{\mathcal{A}(t)} \prod_{j=1}^k dt_j \int_{\mathbb{R}^k} \prod_{j=1}^k |z_j|^{\zeta\varepsilon_j} \\
&\qquad \times \exp\left(-\frac{C_k}{2} \sum_{j=1}^k z_j^2 E\big(u(t_j) - u(t_{j-1})\big)^2\right) \prod_{j=1}^k dz_j \\
&\quad \leq C_k \big|x - x'\big|^{k\zeta} \int_{\mathcal{A}(t)} \prod_{j=1}^k \mathbb{E}\big(u(t_j) - u(t_{j-1})\big)^{-1-\zeta\varepsilon_j} \prod_{j=1}^k dt_j,
\end{aligned}$$

where we use the elementary inequalities $|1 - e^{i\eta}| \leq 2^{1-\zeta}|\eta|^\zeta$ and $|a-b|^\zeta \leq |a|^\zeta + |b|^\zeta$ for all $0 < \zeta < 1$ and any $\eta, a, b \in \mathbb{R}$.

According to Theorem 3.3, we get

$$E\big(u(t_j)-u(t_{j-1})\big)^2 \geq C_1|t_j-t_{j-1}|^{2H_1-\frac{1}{\alpha}},$$

it follows that

$$\begin{aligned}
&\int_{\mathcal{A}(t)}\prod_{j=1}^{k}E\big(u(t_j)-u(t_{j-1})\big)^{-1-\zeta\varepsilon_j}\prod_{j=1}^{k}dt_j \\
&\quad\leq \int_{\mathcal{A}(t)}\prod_{j=1}^{k}(t_j-t_{j-1})^{-\frac{1}{2}(2H_1-\frac{1}{\alpha})(1+\zeta\varepsilon_j)}\prod_{j=1}^{k}dt_j \\
&\quad\leq c_k\delta^{k-\prod_{j=1}^{k}\frac{1}{2}(2H_1-\frac{1}{\alpha})(1+\zeta\varepsilon_j)} \\
&\quad\leq c_k h^{k-k(H_1-\frac{1}{2\alpha})(1+\zeta)} \\
&\quad= c_k h^{k(1-(H_1-\frac{1}{2\alpha}))(1+\zeta)}
\end{aligned}$$

for $k\geq 1, \delta>0$ and $0<\zeta<\frac{1-H_1+\frac{1}{2\alpha}}{2H_1-\frac{1}{\alpha}}$, then

$$E\big[\ell(t+\delta,x)-\ell(t,x)-\ell\big(t+\delta,x'\big)+\ell\big(t,x'\big)\big]^k \leq C_k\big|x-x'\big|^{\zeta k}\delta^{k(1-(H_1-\frac{1}{2\alpha})(1+\zeta))}.$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DFX and LTY carried out the mathematical studies, participated in the sequence alignment, drafted the manuscript, participated in the design of the study and performed the proof of results. All authors read and approved the final manuscript.

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