Xie and Ma *Advances in Difference Equations* (2017) 2017:10
DOI 10.1186/s13662-016-1071-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The dynamics of a diffusive logistic model with nonlocal terms

Xianhua Xie* and Li Ma

*Correspondence:
xxianhua@sina.com.cn
Key Laboratory of Jiangxi Province for Numerical Simulation and Emulation Techniques, College of Mathematics and Computers, Gannan Normal University, Ganzhou, Jiangxi 341000, People's Republic of China

Abstract

It is well known that the set of positive solutions may contain crucial clues for the stationary patterns. In this paper, we consider a class of diffusive logistic equations with nonlocal terms subject to the Dirichlet boundary condition in a bounded domain. We study the existence of positive solutions under certain conditions on the parameters by using bifurcation theory. Finally, we illustrate the general results by applications to models with one-dimensional spatial domain.

MSC: Primary 35J15; secondary 92B20

Keywords: logistic model; positive steady-state solutions; *a priori* bounds; bifurcation theory

1 Introduction

Recently, many researchers pay more attention on the studies of reaction-diffusion equations; we refer to, for example, [1–9]. Ecologically, positive solutions correspond to the existence of steady states of species. It is well known that the set of positive solutions may contain crucial clues for the stationary patterns. From the mathematical viewpoint, it is important to derive some information about the set of positive solutions by means of the coefficients such as the growth rate of the species. Especially, most of the references concentrated on diffusive models with a single population (see, *e.g.*, [2, 4, 8]). One of the most classical diffusive logistic equations is

$$\begin{cases} u_t - \Delta u = \lambda u(1 - K(x)u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

which was regarded as a logistic system of individual species in the ecological studies. Here, $u(x)$ is the population density at location $x \in \Omega$, $\lambda \in \mathbb{R}^+$ is the growth rate of the species and is usually deemed to be a variable, K is a positive function denoting the carrying capacity, and $p > 0$. In (1.1), we assume that Ω is surrounded by inhospitable areas, subjected to the homogeneous Dirichlet boundary conditions.

Later, many scientists found that the movement of an individual species is sometimes determined by surrounding conditions around the point where the species stays. For example, we consider movements of animals, where each individual species mutually interacts by seeing, hearing, and smelling around themselves. That is why interaction by chemical means may take place under certain circumstances. Hence, it seems more realistic to take account of nonlocal effects in the study of species dynamics; see [1, 4, 5, 7, 10, 11]. Usually, this nonlocal effect depends on the value of the population around x, that is, the crowding effect depends on the series of values of u. In some special cases, this nonlocal effect also depends on the value in a neighborhood $B_r(x)$ of x, where $B_r(x)$ represents the ball centered at x of radius $r > 0$. Along these reasons, system (1.1) is replaced by the following more general diffusive logistic population models with nonlocal effect:

$$\begin{cases} u_t - \Delta u = \lambda u(1 - \int_\Omega K(x,y)u^p(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u_t - \Delta u = u(\lambda - \int_{\Omega\cap B_r(x)} \mathcal{K}(y)u^p(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p > 0$, and $K : \Omega \times \Omega \to \mathbb{R}$ and $\mathcal{K} : \Omega \to \mathbb{R}$ are nonnegative and nontrivial continuous functions. Chen and Shi [12] considered the dynamical behavior of system (1.2) when $p = 1$ and the kernel function $K(x,y)$ is a continuous and nonnegative function on $\Omega \times \Omega$ satisfying $\int_\Omega K(x,y)u(y)\,\mathrm{d}y > 0$ for all positive continuous functions u on Ω. Applying the implicit function theorem, Chen and Shi [12] obtained the existence and uniqueness of a positive steady-state solution of system (1.2) when $0 < \lambda - \lambda_1 \ll 1$, where λ_1 denotes the first eigenvalue of the minus Laplacian operator under homogeneous Dirichlet boundary conditions. Some researchers [1, 5, 13, 14] also realized that the kernel function $K(x,y)$ may have no direct connection with the growth rate λ of the species. For example, Allegretto and Nistri [1] studied the following model:

$$\begin{cases} -\Delta u = u(\lambda - \int_\Omega K(x,y)u^p(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{1.4}$$

where $K(x,y)$ vanishes away from the diagonal domain of $\mathbb{R}^N \times \mathbb{R}^N$. Allegretto and Nistri [1] found that (1.4) possesses a unique positive solution when $\lambda > \lambda_1$ if $K(x,y) = K_\delta(|x-y|)$ is a mollifier in $\mathbb{R}^N$, that is, $K_\delta(|x-y|) \in C_0^\infty$, $\int_{\mathbb{R}^N} K_\delta(|x-y|)\,\mathrm{d}y = 1$ for any x with

$$K_\delta\bigl(|x-y|\bigr) = 0 \quad \text{when } |x-y| \ge \delta,$$

and $K_\delta(|x-y|)$ is bounded away from zero when $|x-y| < \mu < \delta$. Later, Corrêa *et al.* [5] proved that (1.4) possesses a unique positive solution if $K(x,y)$ is a separable variable, that is, $K(x,y) = g(x)h(y)$, where $h \ge 0$, $h \not\equiv 0$, and $g(x) > 0$ in Ω. Sun *et al.* [14] investigated the existence of positive solutions of system (1.4) with $K(x,y) = K_1(|x-y|)$ and $\Omega = (-1,1)$, where $K_1 : [0,2] \to (0,\infty)$ is a nondecreasing and piecewise continuous function satisfying $\int_\Omega K_1(y)\,\mathrm{d}y > 0$. Besides, Alves *et al.* [13] also studied the existence of a positive solution of system (1.2).

In the aforementioned literature, the authors only concentrated on the single species. For the model with two populations, in particular, for those diffusive Lotka-Volterra systems without nonlocal terms, the questions posed have been extensively studied in [3, 6, 9] and references therein. However, the discussion of the dynamical behavior of two interacting species in the presence of nonlocal term effects is more difficult than those models without nonlocal term effects. Lately, Guo and Yan [15] employ Lyapunov-Schmidt reduction to investigate the existence of the positive solution of the following model:

$$\begin{cases} -\Delta u = \lambda u(1 - \int_\Omega A_{11}(x,y)u(y)\,\mathrm{d}y - \int_\Omega A_{12}(x,y)v(y)\,\mathrm{d}y) & \text{in } \Omega, \\ -\Delta v = \lambda v(1 - \int_\Omega A_{21}(x,y)u(y)\,\mathrm{d}y - \int_\Omega A_{22}(x,y)v(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u = v = 0 & \text{on } \Omega, \end{cases} \tag{1.5}$$

where $u(x)$ and $v(x)$ are the population densities at location x, $\lambda > 0$ is a scaling constant, and Ω is a connected bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$. The kernel functions A_{ij} $(i,j = 1,2)$ describe the dispersal behaviors of the populations.

A natural problem is whether (1.5) has a positive solution for λ not only near to but also far away from λ_1. Moreover, it is very interesting to investigate the following more general population model with nonlocal delay effect:

$$\begin{cases} -\Delta u = u(\lambda - \int_\Omega A_{11}(x,y)u^p(y)\,\mathrm{d}y - \int_\Omega A_{12}(x,y)v^q(y)\,\mathrm{d}y) & \text{in } \Omega, \\ -\Delta v = v(\lambda - \int_\Omega A_{21}(x,y)u^p(y)\,\mathrm{d}y - \int_\Omega A_{22}(x,y)v^q(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u = v = 0 & \text{on } \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, p, q are positive constants, and $A_{ij} \in L^\infty(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$, $i,j = 1,2$. Here $u(x)$ and $v(x)$ can be interpreted as the densities of prey and predator populations at a spatial position $x \in \Omega$, and the parameter λ is a positive real number representing the growth rate of the prey and predator.

The purpose of this paper is to find sufficient conditions ensuring the existence of a positive solution for all $\lambda > \lambda_1$. Our main approach is global bifurcation theory, which is different from the method adopted in [15]. Moreover, we also obtain the stability of the positive solution by analyzing the distribution of the eigenvalues, which was not considered by Alves *et al.* [13]. Throughout this paper, we impose the following assumptions on the dispersal kernel functions $A_{ij}(x,y)$, $i,j = 1,2$.

(C1) T_1 and T_2 are positive on the space $C^+(\Omega) \times C^+(\Omega)$ in the sense that $T_i(C^+(\Omega) \times C^+(\Omega)) \subset C^+(\Omega) \times C^+(\Omega) \setminus \{(0,0)\}$, $i = 1,2$, where $C^+(\Omega)$ represents the space of positive continuous functions, and

$$T_j(u,v) = \int_\Omega \bigl[A_{j1}(\cdot,y)u^p(y) + A_{j2}(\cdot,y)v^q(y)\bigr]\,\mathrm{d}y, \quad j = 1,2.$$

(C2) If u, v are measurable and satisfy

$$\begin{cases} \int_{\Omega\times\Omega}[A_{11}(x,y)|\frac{u(y)}{\|u\|}|^p + A_{12}(x,y)\frac{|v(y)|^q}{\|u\|^p}](\frac{u(x)}{\|u\|})^2\,\mathrm{d}y\,\mathrm{d}x = 0, \\ \int_{\Omega\times\Omega}[A_{21}(x,y)\frac{|u(y)|^p}{\|v\|^q} + A_{12}(x,y)|\frac{v(y)}{\|v\|}|^q](\frac{v(x)}{\|v\|})^2\,\mathrm{d}y\,\mathrm{d}x = 0, \end{cases}$$

then $u = v = 0$ a.e. in Ω. Here we set $\frac{u}{\|u\|} = 0$ for $u = 0$ and denote by $\|\cdot\|$ the usual norm in $H_0^1(\Omega)$, that is,

$$\|u\|^2 = \|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2\,\mathrm{d}x,$$

where the space $H_0^1(\Omega) = \{U \in H^1(\Omega) \mid U(x) = 0, \forall x \in \partial\Omega\}$ and $H^k(\Omega)$ $(k \geq 0)$ is the Sobolev space of L^2-functions f on Ω with derivatives $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$ $(n = 1, 2, \dots, k)$ belonging to $L^2(\Omega)$.

Our main results are stated as follows.

Theorem 1.1 *Suppose that* A_{ij} $(i,j = 1,2)$ *satisfy* (C1) *and* (C2). *Then problem* (1.6) *has a positive solution if and only if* $\lambda > \lambda_1$.

In view of Theorem 1.1, we see that system (1.6) with $A_{ij}(x,y) = \mathcal{A}_{ij}(x,y)\chi_{B_r(x)}$ $(i,j = 1,2)$ has a positive solution if and only if $\lambda > \lambda_1$, where $\mathcal{A}_{ij}(x,y)$ $(i,j = 1,2)$ are positive functions on $\Omega \times \Omega$. If A_{ij} $(i,j = 1,2)$ do not satisfy assumption (C2), then A_{ij} $(i,j = 1,2)$ may vanish in some neighborhood of the diagonal of $\Omega \times \Omega$ (see Section 3). In this case, Theorem 1.1 is inapplicable. However, we are also able to investigate the existence and nonexistence of positive solutions for some value of λ under the following assumptions on the dispersal kernel functions $A_{ij}(x,y)$, $i,j = 1,2$.

(C3) There are $r > 0$ and m connected open sets $\Omega_1, \Omega_2, \dots, \Omega_m \subset \Omega$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$, $i \neq j$, and $A_{ij}(x,y) > 0$ for all $(x,y) \in \Omega \times \Omega$ satisfying $x \notin \bigcup_{j=1}^m \Omega_j$ and $|x - y| < r$, $i,j = 1,2$.

In view of [16], we know that if $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$. Moreover, the inequality is strict as soon as $\Omega_2 \setminus \Omega_1$ contains a set of positive capacity (since the first eigenfunction cannot vanish on such a set). Hence, we have the following result.

Theorem 1.2 *Suppose that* A_{ij} $(i,j = 1,2)$ *satisfy* (C1) *and* (C3). *Then, problem* (1.6) *has a positive solution when* $\lambda_1 < \lambda < \min\{\lambda_1(\Omega_1), \dots, \lambda_1(\Omega_m)\}$, *where* $\lambda_1(\Omega_i)$ *denotes the principal eigenvalue of the minus Laplacian operator in* Ω_i *under homogeneous Dirichlet boundary conditions*, $i = 1, 2, \dots, m$. *Moreover, the solution is stable*.

The remaining parts of the paper are structured in the following way. In Section 2, we employ the global bifurcation theory to obtain the existence and stability of positive solutions of (1.6) under conditions (C1) and (C2). Section 3 is devoted to the case where A_{ij} $(i,j = 1,2)$ satisfy condition (C3). Section 4 is devoted to the application of our theoretical results to some one-dimensional models.

2 Proof of Theorem 1.1

In this section, we introduce some basic results. First, consider the functions $\phi_{p,\omega}^{ij} : \Omega \to \mathbb{R}$ $(i,j = 1,2)$ given by

$$\phi_{p,\omega}^{ij}(x) = \int_\Omega A_{ij}(x,y)\left|\omega(y)\right|^p \mathrm{d}y, \quad x \in \Omega.$$

If A_{ij} $(i,j = 1,2)$ and ω are bounded, then $\phi_{p,\omega}^{ij}$ $(i,j = 1,2)$ are well defined. Moreover, we have the following observations:

$$\left\|\phi_{p,\omega}^{ij}\right\|_\infty \leq \|A_{ij}\|_\infty |\Omega| \|\omega\|_\infty^p \quad \text{for all } \omega \in L^\infty(\Omega), \tag{2.1}$$

$$\left\|\phi_{p,\omega}^{ij} - \phi_{p,\nu}^{ij}\right\|_\infty \leq \|A_{ij}\|_\infty |\Omega| \left\| |\omega|^p - |\nu|^p \right\|_\infty \quad \text{for all } \omega, \nu \in L^\infty(\Omega), \tag{2.2}$$

and

$$\begin{matrix} \phi_p^{ij} : L^\infty(\Omega) \to L^\infty(\Omega), \\ \phi_p^{ij}(u) = \phi_{p,u}^{ij} \end{matrix} \quad (i,j = 1,2) \text{ are uniformly continuous in } L^\infty(\Omega). \tag{2.3}$$

Using these notations, it is easy to observe that (u,v) is a positive solution of (1.6) if and only if (u,v) is a positive solution of

$$\begin{cases} -\Delta u = u(\lambda - \phi_{p,u}^{11} - \phi_{q,v}^{12}) & \text{in } \Omega, \\ -\Delta v = v(\lambda - \phi_{p,u}^{21} - \phi_{q,v}^{22}) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

First, we show the nonexistence of a positive solution of (1.6) for small λ.

Lemma 2.1 *Suppose that A_{ij} $(i,j = 1,2)$ satisfy* (C1) *and* (C2). *Then system* (1.6) *with $\lambda < \lambda_1$ has no positive solutions.*

Proof We prove this lemma by contradiction. Assume that (1.6) with $\lambda \le \lambda_1$ has a positive solution (u_*, v_*). Then we have

$$\begin{cases} -\Delta u_* = u_*(\lambda - \int_\Omega A_{11}(x,y)u_*^p(y)\,\mathrm{d}y - \int_\Omega A_{12}(x,y)v_*^q(y)\,\mathrm{d}y) & \text{in } \Omega, \\ -\Delta v_* = v_*(\lambda - \int_\Omega A_{21}(x,y)u_*^p(y)\,\mathrm{d}y - \int_\Omega A_{22}(x,y)v_*^q(y)\,\mathrm{d}y) & \text{in } \Omega, \\ u_* = v_* = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let (λ_1, ψ_1) with $\psi_1 > 0$ be the principle eigenpair of the eigenvalue problem

$$\begin{cases} -\Delta \psi = \mu \psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Multiplying (2.5) by ψ_1 and then integrating it on Ω, we have

$$\begin{cases} \int_\Omega (\lambda - \lambda_1)\psi_1(x)u_*(x)\,\mathrm{d}x \\ \quad = \int_{\Omega\times\Omega} [A_{11}(x,y)u_*^p(y) + A_{12}(x,y)v_*^q(y)]u_*(x)\psi_1(x)\,\mathrm{d}y\,\mathrm{d}x, \\ \int_\Omega (\lambda - \lambda_1)\psi_1(x)v_*(x)\,\mathrm{d}x \\ \quad = \int_{\Omega\times\Omega} [A_{21}(x,y)u_*^p(y) + A_{22}(x,y)v_*^q(y)]v_*(x)\psi_1(x)\,\mathrm{d}y\,\mathrm{d}x. \end{cases} \tag{2.7}$$

Since $\psi_1 > 0$, $u_* > 0$, $v_* > 0$, and $\lambda \le \lambda_1$, we find that each of the left-hand sides of the two equations of (2.7) is less than 0 and that each of the right-hand sides of the two equations is greater than 0, which is a contradiction. So system (1.6) has no positive solution for $\lambda \le \lambda_1$. ☐

Proposition 1 ([5]) *Assume that there exists a pair of positive functions $\overline{u}, \overline{v} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$, $\delta \in (0,1)$, such that*

$$\begin{cases} -\Delta\overline{u} - \overline{u}[\lambda - \int_\Omega A_{11}(x,y)u_0^p(y)\,\mathrm{d}y - \int_\Omega A_{12}(x,y)v_0^q(y)\,\mathrm{d}y] \\ \quad + u_0 \int_\Omega [pA_{11}(x,y)u_0^{p-1}(y)\overline{u}(y) + qA_{12}(x,y)v_0^{q-1}(y)\overline{v}(y)]\,\mathrm{d}y > 0, \\ -\Delta\overline{v} - \overline{v}[\lambda - \int_\Omega A_{21}(x,y)u_0^p(y)\,\mathrm{d}y - \int_\Omega A_{22}(x,y)v_0^q(y)\,\mathrm{d}y] \\ \quad + v_0 \int_\Omega [pA_{21}(x,y)u_0^{p-1}(y)\overline{u}(y) + qA_{22}(x,y)v_0^{q-1}(y)\overline{v}(y)]\,\mathrm{d}y > 0. \end{cases}$$

Then the principle eigenvalue of the eigenvalue problem (1.6) *is positive.*

In the following section, we intend to prove the existence of a positive solution for (1.6) by using the classical bifurcation result of Rabinowitz [17]. To this end, we recall that there exists $c_\infty = c_\infty(\Omega) > 0$ such that, for each $f \in L^\infty(\Omega)$, there exists a unique $\omega \in C^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta\omega = f(x), & x \in \Omega, \\ \omega = 0, & x \in \partial\Omega, \end{cases} \tag{2.8}$$

and $\|\omega\|_{C^1(\overline{\Omega})} \le c_\infty \|f\|_\infty$. Thus, the solution operator $S : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ can be given by

$$SU = \omega_1 \quad \Leftrightarrow \quad \begin{cases} -\Delta\omega_1 = U, & x \in \Omega, \\ \omega_1 = 0, & x \in \partial\Omega. \end{cases}$$

Obviously, S is well defined, linear, and satisfies

$$\|SU\|_{C^1(\overline{\Omega})} \le c_\infty \|U\|_{C^0(\overline{\Omega})}, \quad \forall U \in C^0(\overline{\Omega}).$$

Moreover, by the Schauder imbedding theorem, $S : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is a compact operator. In view of the spectrum of S, it is easy to see that

$$\sigma(S) = \{\lambda_j^{-1} \mid \lambda_j \text{ is an eigenvalue of the minus Laplacian operator}\}.$$

On the other hand, define the nonlinear operator $F : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ as

$$FV = \omega_2 \quad \Leftrightarrow \quad \begin{cases} -\Delta\omega_2 + \Phi_V V = 0, & x \in \Omega, \\ \omega_2 = 0, & x \in \partial\Omega, \end{cases}$$

where $V = (u, v)^{\mathrm{T}} \in C^0(\Omega)$ and

$$\Phi_V = \begin{pmatrix} \phi_{p,u}^{11} + \phi_{q,v}^{12} \\ \phi_{p,u}^{21} + \phi_{q,v}^{22} \end{pmatrix}.$$

Obviously, F is continuous and satisfies

$$\|FV\|_{C^1(\overline{\Omega})} \le c_\infty \|\Phi_V\|_\infty \|V\|_{C^0(\overline{\Omega})}, \quad \forall V \in C^0(\overline{\Omega}),$$

Using again the Schauder imbedding theorem, we see that $F : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is compact. Furthermore, note that

$$\|FV\|_{C^0(\overline{\Omega})} \le \|FV\|_{C^1(\overline{\Omega})}.$$

Then we have

$$\left\| \frac{FV}{\|V\|_{C^0(\overline{\Omega})}} \right\|_{C^0(\overline{\Omega})} \le \frac{\|FV\|_{C^1(\overline{\Omega})}}{\|V\|_{C^0(\overline{\Omega})}} \le c_\infty \|\Phi_V\|_\infty,$$

from which it follows that

$$\lim_{V\to 0}\frac{FV}{\|V\|_{C^0(\overline{\Omega})}}=0,\quad i.e.,\quad FV=o\big(\|V\|_{C^0(\overline{\Omega})}\big). \tag{2.9}$$

Obviously, $V=(u,v)^{\mathrm{T}}$ solves (1.6) if and only if

$$V=G(\lambda,V)\triangleq\lambda SV+FV.$$

In view of [17], considering $E=C^0(\overline{\Omega})$, we have the following result.

Theorem 2.1 *Let E be a Banach space. Suppose that F satisfies* (2.9), *S is a compact linear operator, and $\lambda^{-1}\in\sigma(S)$ with odd algebraic multiplicity. Let*

$$\Gamma=\overline{\big\{(\lambda,V)\in\mathbb{R}\times E: V=\lambda SV+FV, V\neq 0\big\}},$$

and let $\mathcal{C}$ be a closed connected component of Γ that contains $(\lambda,0)$. Then either

(1) *$\mathcal{C}$ is unbounded in $\mathbb{R}\times E$, or else*

(2) *there exists $\tilde{\lambda}\neq\lambda$ such that $(\tilde{\lambda},0)\in\mathcal{C}$ and $\tilde{\lambda}^{-1}\in\sigma(S)$.*

Remark 2.1 Because λ_1 is the principle eigenvalue of the eigenvalue problem (2.6) with the associated eigenfunction $\psi_1>0$ on Ω and its multiplicity is simple, by the global bifurcation theorem there exists a closed connected component $\mathcal{C}$ containing $(\lambda_1,0)$ and satisfying (1) or (2) for solutions to (1.6).

In order to prove the existence of positive solutions of (1.6) with $\lambda>\lambda_1$, it follows from Lemma 2.1 and Theorem 2.1 that it only suffices to prove that conclusion (1) of Theorem 2.1 holds and that V is bounded when $\lambda>\lambda_1$.

Lemma 2.2 *There exists $\varepsilon>0$ such that if $(\lambda,V)=(\lambda,u,v)\in\mathcal{C}$ with $\lambda-\lambda_1<\varepsilon$ and $\|V\|_{C^0(\overline{\Omega})}<\varepsilon$ where $u\neq 0$ and $v\neq 0$, then u and v have definite signals, that is,*

(i) *$u(x)>0$ and $v(x)>0$ for all $x\in\Omega$, or*

(ii) *$u(x)>0$ and $v(x)<0$ for all $x\in\Omega$, or*

(iii) *$u(x)<0$ and $v(x)>0$ for all $x\in\Omega$, or*

(iv) *$u(x)<0$ and $v(x)<0$ for all $x\in\Omega$.*

Proof Take $V_n=(u_n,v_n)\in C^0(\overline{\Omega})$ and $\lambda_n\to\lambda_1$ as $n\to\infty$ such that $\|V_n\|_{C^0(\overline{\Omega}\times\overline{\Omega})}\to 0$ as $n\to\infty$ and

$$V_n=G(\lambda_n,V_n).$$

Let $w_n^1=\frac{u_n}{\|u_n\|_{C^0(\overline{\Omega})}}$ and $w_n^2=\frac{v_n}{\|v_n\|_{C^0(\overline{\Omega})}}$. Then we have

$$\begin{cases}-\Delta w_n^1+\phi_{p,u_n}^{11}w_n^1+\phi_{q,v_n}^{12}w_n^1=\lambda_n w_n^1 & \text{in }\Omega,\\ -\Delta w_n^2+\phi_{p,u_n}^{21}w_n^2+\phi_{q,v_n}^{22}w_n^2=\lambda_n w_n^2 & \text{in }\Omega,\\ w_n^1=w_n^2=0 & \text{on }\partial\Omega.\end{cases} \tag{2.10}$$

It follows from (2.1) that $\|\phi^{11}_{p,u_n}\|_\infty$, $\|\phi^{12}_{q,v_n}\|_\infty$, $\|\phi^{21}_{p,u_n}\|_\infty$, and $\|\phi^{22}_{q,v_n}\|_\infty$ are bounded if both u_n and v_n are bounded in $C^0(\overline{\Omega})$. Thus, it is easy to see that

$$\begin{aligned}
&\|w^1_n\|_{C^1(\overline{\Omega})} \le c_\infty\big[\lambda_n + \|\phi^{11}_{p,u_n}\|_\infty + \|\phi^{12}_{q,v_n}\|_\infty\big]\|w^1_n\|_{C^0(\overline{\Omega})}, \quad \forall n\in\mathbb{N},\\
&\|w^2_n\|_{C^1(\overline{\Omega})} \le c_\infty\big[\lambda_n + \|\phi^{21}_{p,u_n}\|_\infty + \|\phi^{22}_{q,v_n}\|_\infty\big]\|w^2_n\|_{C^0(\overline{\Omega})}, \quad \forall n\in\mathbb{N}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|w^1_n - w^1_m\|_{C^1(\overline{\Omega})} \le{}& c_\infty\big[\|\lambda_n w^1_n - \lambda_m w^1_m\|_{C^0(\overline{\Omega})} + \big(\|\phi^{11}_{p,u_n}\|_\infty + \|\phi^{12}_{q,v_n}\|_\infty\big)\|w^1_n - w^1_m\|_{C^0(\overline{\Omega})}\\
&+ \|\phi^{11}_{p,u_n} - \phi^{11}_{p,u_m}\|_\infty + \|\phi^{12}_{q,v_n} - \phi^{12}_{q,v_m}\|_\infty\big], \quad \forall n\in\mathbb{N},\\
\|w^2_n - w^2_m\|_{C^1(\overline{\Omega})} \le{}& c_\infty\big[\|\lambda_n w^2_n - \lambda_m w^2_m\|_{C^0(\overline{\Omega})} + \big(\|\phi^{21}_{p,u_n}\|_\infty + \|\phi^{22}_{q,v_n}\|_\infty\big)\|w^2_n - w^2_m\|_{C^0(\overline{\Omega})}\\
&+ \|\phi^{21}_{p,u_n} - \phi^{21}_{p,u_m}\|_\infty + \|\phi^{22}_{q,v_n} - \phi^{22}_{q,v_m}\|_\infty\big], \quad \forall n\in\mathbb{N}.
\end{aligned}$$

Then, using the Arzelà-Ascoli theorem, we see that, for each fixed $i\in\{1,2\}$, w^i_n converge to some $w^i\in C^1(\overline{\Omega})$ uniformly in $\overline{\Omega}$, and hence there exists a convergent subsequence. By the definition of w^i_n, $\|w^i\|_{C^0(\overline{\Omega})} = 1$ implies $w^i \not\equiv 0$, $i = 1,2$. Multiplying (2.10) by v and integrating on Ω, we have

$$\begin{aligned}
&\int_\Omega \nabla w^1_n \nabla v\,\mathrm{d}x + \int_\Omega \big(\phi^{11}_{p,u_n} + \phi^{12}_{q,v_n}\big) w^1_n v\,\mathrm{d}x = \lambda_n \int_\Omega w^1_n v\,\mathrm{d}x,\\
&\int_\Omega \nabla w^2_n \nabla v\,\mathrm{d}x + \int_\Omega \big(\phi^{21}_{p,u_n} + \phi^{22}_{q,v_n}\big) w^2_n v\,\mathrm{d}x = \lambda_n \int_\Omega w^2_n v\,\mathrm{d}x.
\end{aligned}$$

In view of (2.1), we have $\phi^{11}_{p,u_n} w^1_n, \phi^{12}_{q,v_n} w^1_n, \phi^{21}_{p,u_n} w^2_n, \phi^{22}_{q,v_n} w^2_n \to 0$ as $n\to\infty$ in $C^0(\overline{\Omega})$. Then

$$\begin{cases}
-\Delta w^1 = \lambda_1 w^1 & \text{in } \Omega,\\
-\Delta w^2 = \lambda_1 w^2 & \text{in } \Omega,\\
w^1 = w^2 = 0 & \text{on } \partial\Omega.
\end{cases}$$

Since $w^1 w^2 \not\equiv 0$, it follows from the spectral and limit theory that, for each fixed $i\in\{1,2\}$,

$$w^i(x) > 0 \quad \text{or} \quad w^i(x) < 0$$

for all $x\in\Omega$. In what follows, we only consider the case where $w^1(x) > 0$ and $w^2(x) > 0$ for all $x\in\Omega$ because the other three cases can be discussed analogously. Note that w^1 and w^2 are the $C^1(\overline{\Omega})$-limits of w^1_n and w^2_n, respectively. Then, on Ω, $w^i_n > 0$ and $w^i_n > 0$ for n large enough. Thus, the signs of u_n and v_n are the same as those of w^1_n and w^2_n for n large enough. This completes the proof. □

It is easy to check that if $(\lambda,u,v)\in\Gamma$, then the pairs $(\lambda,-u,v)$, $(\lambda,u,-v)$, and $(\lambda,-u,-v)$ are also in Γ. In what follows, we decompose $\mathcal{C}$ into $\mathcal{C} = \mathcal{C}^{+,+}\cup\mathcal{C}^{+,-}\cup\mathcal{C}^{-,+}\cup\mathcal{C}^{-,-}$, where

$$\mathcal{C}^{+,+} = \big\{(\lambda,u,v)\in\mathcal{C} : u(x)\ge 0, v(x)\ge 0, \forall x\in\Omega\big\};$$

$$\mathcal{C}^{+,-} = \big\{(\lambda,u,v)\in\mathcal{C} : u(x)\ge 0, v(x)\le 0, \forall x\in\Omega\big\};$$

$$\mathcal{C}^{-,+} = \{(\lambda, u, v) \in \mathcal{C} : u(x) \leq 0, v(x) \geq 0, \forall x \in \Omega\};$$
$$\mathcal{C}^{-,-} = \{(\lambda, u, v) \in \mathcal{C} : u(x) \leq 0, v(x) \leq 0, \forall x \in \Omega\}.$$

The following lemma tells the fact that system (1.6) satisfies Theorem 2.1(1).

Lemma 2.3 *Each of* $\mathcal{C}^{+,+}$, $\mathcal{C}^{-,+}$, $\mathcal{C}^{-,+}$, *and* $\mathcal{C}^{-,-}$ *is unbounded.*

Proof It is easy to see that if one of $\mathcal{C}^{+,+}$, $\mathcal{C}^{-,+}$, $\mathcal{C}^{-,+}$, and $\mathcal{C}^{-,-}$ is unbounded then the others are also unbounded. Therefore, it suffices to show that $\mathcal{C}^{+,+}$ is unbounded. Suppose that $\mathcal{C}^{+,+}$ is bounded. Then $\mathcal{C}$ is also bounded. In view of the global bifurcation theorem (Theorem 2.1), $\mathcal{C}$ satisfies conclusion (2) of Theorem 2.1, that is, $\mathcal{C}$ contains $(\tilde{\lambda}, 0, 0)$, where $\tilde{\lambda} \neq \lambda_1$ and $\tilde{\lambda}^{-1} \in \sigma(S)$.

We take $\{(\lambda_n, u_n, v_n)\}_{n=1}^{\infty} \subseteq \mathcal{C}^{+,+}$ such that $u_n v_n \not\equiv 0$ and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$ and such that $\lambda_n \to \tilde{\lambda}$, $\|u_n\|_{C^0(\overline{\Omega})} \to 0$, and $\|u_n\|_{C^0(\overline{\Omega})} \to 0$ as $n \to \infty$. Letting

$$w_n^1 = \frac{u_n}{\|u_n\|_{C^0(\overline{\Omega})}}, \qquad w_n^2 = \frac{v_n}{\|v_n\|_{C^0(\overline{\Omega})}}$$

and using similar arguments as in the proof of Lemma 2.2, we get that (w_n^1, w_n^2) converges to (w^1, w^2) in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ as $n \to \infty$, which is a nonzero solution pair of the eigenvalue problem

$$\begin{cases} -\Delta w^1 = \tilde{\lambda} w^1 & \text{in } \Omega, \\ -\Delta w^2 = \tilde{\lambda} w^2 & \text{in } \Omega, \\ w^1 = w^2 = 0 & \text{on } \partial\Omega, \end{cases}$$

which implies that both w^1 and w^2 are eigenfunctions associated with $\tilde{\lambda}$. Since $\tilde{\lambda} \neq \lambda_1$, both w^1 and w^2 change signs in Ω. Thus, for n large enough, w_n^1 and w_n^2 change signs, and hence the same results hold for $u_n = \|u_n\|_{C^0(\overline{\Omega})} w_n^1$ and $v_n = \|v_n\|_{C^0(\overline{\Omega})} w_n^2$. However, this contradicts the assumption that $(\lambda_n, u_n, v_n) \in \mathcal{C}^{+,+}$. This completes the proof. □

Now, we shall prove that system (1.6) satisfies Theorem 2.1(1). It suffices to show that the connected component $\mathcal{C}^{+,+}$ intersects any set of the form $\{\lambda\} \times H_0^1(\Omega) \times H_0^1(\Omega)$ for $\lambda > \lambda_1$.

Lemma 2.4 *Suppose that* A_{ij} $(i,j = 1, 2)$ *satisfy conditions* (C1) *and* (C2). *For any* $\Lambda > 0$, *there exists a constant* $r > 0$ *such that* $\|u\|_{C^0(\overline{\Omega})} \leq r$ *and* $\|v\|_{C^0(\overline{\Omega})} \leq r$ *whenever* $(\lambda, u, v) \in \mathcal{C}^{+,+}$ *and* $\lambda \leq \Lambda$.

Proof First, we denote by $\|\cdot\|$ the usual norm in $H_0^1(\Omega)$, that is,

$$\|u\|^2 = \|u\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Indeed, if this were not true, there would exist $\{(\lambda_n, u_n, v_n)\}_{n=1}^{\infty} \subset [0, \Lambda] \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that

Case 1. $\|u_n\| \to \infty$, $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, $\|v_n\| \leq r$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$;
Case 2. $\|v_n\| \to \infty$, $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, $\|u_n\| \leq r$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$;

Case 3. $\|u_n\| \to \infty$, $\|v_n\| \to \infty$ and $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$. We only discuss Case 3 because the other cases can be dealt with analogously. Let

$$w_n^1 = \frac{u_n}{\|u_n\|_{C^0(\overline{\Omega})}}, \qquad w_n^2 = \frac{v_n}{\|v_n\|_{C^0(\overline{\Omega})}}.$$

Then we have

$$\begin{cases} \int_\Omega \nabla w_n^1 \nabla \theta_1 \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{11} + \phi_{q,v_n}^{12}) w_n^1 \theta_1 \,\mathrm{d}x = \lambda_n \int_\Omega w_n^1 \theta_1 \,\mathrm{d}x, \\ \int_\Omega \nabla w_n^2 \nabla \theta_2 \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{21} + \phi_{q,v_n}^{22}) w_n^2 \theta_2 \,\mathrm{d}x = \lambda_n \int_\Omega w_n^2 \theta_2 \,\mathrm{d}x. \end{cases} \tag{2.11}$$

Note that $\{(w_n^1, w_n^2)\}_{n=1}^\infty$ are bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. Then, without loss of generality, we suppose that there is $w = (w^1, w^2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} &w_n^i \rightharpoonup w^i \quad \text{as } n \to \infty \text{ in } H_0^1(\Omega), i = 1, 2, \\ &w_n^i \to w^i \quad \text{as } n \to \infty \text{ in } L^2(\Omega), i = 1, 2, \end{aligned}$$

and $w_n^i(x) \to w^i(x)$ as $n \to \infty$ a.e. in Ω, $i = 1, 2$. Taking $\theta_1 = \frac{u_n}{\|u_n\|^{p+1}}$ as a test function in the first equation of (2.11) and $\theta_2 = \frac{v_n}{\|v_n\|^{q+1}}$ as a test function in the second equation of (2.11), respectively, equations (2.11) reduce to

$$\begin{cases} \frac{1}{\|u_n\|^p} + \int_\Omega (\phi_{p,w_n^1}^{11} + \phi_{q,\frac{v_n}{\|u_n\|^{p/q}}}^{12})(w_n^1)^2 \,\mathrm{d}x = \frac{\lambda_n}{\|u_n\|^p} \int_\Omega (w_n^1)^2 \,\mathrm{d}x, \\ \frac{1}{\|v_n\|^q} + \int_\Omega (\phi_{p,\frac{u_n}{\|v_n\|^{q/p}}}^{21} + \phi_{q,w_n^2}^{22})(w_n^2)^2 \,\mathrm{d}x = \frac{\lambda_n}{\|v_n\|^q} \int_\Omega (w_n^2)^2 \,\mathrm{d}x. \end{cases}$$

Passing to the limit in these equalities and using the Fatou lemma, we have

$$\begin{cases} 0 \le \int_\Omega (\phi_{p,w^1}^{11} + \phi_{q,\frac{v}{\|u\|^{p/q}}}^{12})(w^1)^2 \,\mathrm{d}x \\ \quad \le \lim_{n\to\infty} \int_\Omega (\phi_{p,w_n^1}^{11} + \phi_{q,\frac{v_n}{\|u_n\|^{p/q}}}^{12})(w_n^1)^2 \,\mathrm{d}x = 0, \\ 0 \le \int_\Omega (\phi_{p,\frac{u}{\|v\|^{q/p}}}^{21} + \phi_{q,w^2}^{12})(w^2)^2 \,\mathrm{d}x \\ \quad \le \lim_{n\to\infty} \int_\Omega (\phi_{p,\frac{u_n}{\|v_n\|^{q/p}}}^{21} + \phi_{q,w_n^2}^{22})(w_n^2)^2 \,\mathrm{d}x = 0. \end{cases}$$

Take into account condition (C2), we have $w^1 = w^2 = 0$. Namely, w_n^1 and w_n^2 converge to 0 in $L^2(\Omega)$. On the other hand, taking $\theta_1 = w_n^1$ and $\theta_2 = w_n^2$ as test functions in (2.11), we see that

$$\begin{cases} \int_\Omega |\nabla w_n^1|^2 \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{11} + \phi_{q,v_n}^{12})(w_n^1)^2 \,\mathrm{d}x = \lambda_n \int_\Omega (w_n^1)^2 \,\mathrm{d}x, \\ \int_\Omega |\nabla w_n^2|^2 \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{21} + \phi_{q,v_n}^{22})(w_n^2)^2 \,\mathrm{d}x = \lambda_n \int_\Omega (w_n^2)^2 \,\mathrm{d}x. \end{cases}$$

Note that $\{\lambda_n\}_{n=1}^\infty$ is bounded from above by Λ and

$$\begin{aligned} &\int_\Omega (\phi_{p,u_n}^{11} + \phi_{q,v_n}^{12})(w_n^1)^2 \,\mathrm{d}x \ge 0, \\ &\int_\Omega (\phi_{p,u_n}^{21} + \phi_{q,v_n}^{22})(w_n^2)^2 \,\mathrm{d}x \ge 0. \end{aligned}$$

Then we have

$$\begin{cases} \int_\Omega |\nabla w_n^1|^2\,\mathrm{d}x \le \lambda_n \int_\Omega (w_n^1)^2\,\mathrm{d}x, \\ \int_\Omega |\nabla w_n^2|^2\,\mathrm{d}x \le \lambda_n \int_\Omega (w_n^2)^2\,\mathrm{d}x. \end{cases}$$

Taking the limit, we conclude that $\|w_n^1\| \to 0$ and $\|w_n^2\| \to 0$ as $n \to \infty$, which is absurd because $\|w_n^1\| = \|w_n^2\| = 1$ for all n. This completes the proof. □

3 Proof of Theorem 2.1

We observe that if A_{ij} $(i,j = 1,2)$ do not satisfy condition (C2), that is:

(C4) There exists a measurable function $\varphi = (\varphi_1,\varphi_2) : \Omega \to \mathbb{R}^2 \setminus \{(0,0)\}$ such that

$$\begin{aligned} &\int_\Omega \Bigl(\phi^{11}_{p,\frac{\varphi_1}{\|\varphi_1\|}} + \phi^{12}_{q,\frac{\varphi_2}{\|\varphi_1\|^{p/q}}}\Bigr)\Bigl(\frac{\varphi_1}{\|\varphi_1\|}\Bigr)^2\,\mathrm{d}x = 0 \quad \text{or} \\ &\int_\Omega \Bigl(\phi^{21}_{p,\frac{\varphi_1}{\|\varphi_2\|^{q/p}}} + \phi^{12}_{q,\frac{\varphi_2}{\|\varphi_2\|}}\Bigr)\Bigl(\frac{\varphi_2}{\|\varphi_2\|}\Bigr)^2\,\mathrm{d}x = 0. \end{aligned} \tag{3.1}$$

Lemma 3.1 *If* A_{ij} $(i,j = 1,2)$ *satisfy condition* (C3), *then* (3.1) *implies that* $\varphi_1 = \varphi_2 = 0$ *a.e. in* $\Omega \setminus U$.

Proof In view of equation (3.1), we have

$$\Bigl(\phi^{11}_{p,\frac{\varphi_1}{\|\varphi_1\|}} + \phi^{12}_{q,\frac{\varphi_2}{\|\varphi_1\|^{p/q}}}\Bigr)\Bigl(\frac{\varphi_1}{\|\varphi_1\|}\Bigr)^2 = 0 \quad \text{a.e. in } \Omega,$$

or

$$\Bigl(\phi^{21}_{p,\frac{\varphi_1}{\|\varphi_2\|^{q/p}}} + \phi^{12}_{q,\frac{\varphi_2}{\|\varphi_2\|}}\Bigr)\Bigl(\frac{\varphi_2}{\|\varphi_2\|}\Bigr)^2 = 0 \quad \text{a.e. in } \Omega.$$

Fixing $\varepsilon > 0$ and $D_\varepsilon = \{x \in \Omega \setminus U : |\varphi_1| \ge \varepsilon \text{ or } |\varphi_2| \ge \varepsilon\}$, it follows that

$$\begin{aligned} 0 &= \Bigl(\phi^{11}_{p,\frac{\varphi_1}{\|\varphi_1\|}} + \phi^{12}_{q,\frac{\varphi_2}{\|\varphi_1\|^{p/q}}}\Bigr)\Bigl(\frac{\varphi_1}{\|\varphi_1\|}\Bigr)^2 \\ &= \int_\Omega \biggl[A_{11}(x,y)\biggl|\frac{\varphi_1(y)}{\|\varphi_1\|}\biggr|^p + A_{12}(x,y)\frac{|\varphi_2(y)|^q}{\|\varphi_1\|^p}\biggr]\,\mathrm{d}y \\ &\ge \min\biggl\{\int_{D_\varepsilon\cap B_r} A_{11}(x,y)\Bigl(\frac{\varepsilon}{\|\varphi_1\|}\Bigr)^p\,\mathrm{d}y, \int_{D_\varepsilon\cap B_r} \frac{\varepsilon^q}{\|\varphi_1\|^p}A_{12}(x,y)\,\mathrm{d}y\biggr\}, \end{aligned}$$

or

$$\begin{aligned} 0 &= \Bigl(\phi^{21}_{p,\frac{\varphi_1}{\|\varphi_2\|^{q/p}}} + \phi^{22}_{q,\frac{\varphi_2}{\|\varphi_2\|}}\Bigr)\Bigl(\frac{\varphi_2}{\|\varphi_2\|}\Bigr)^2 \\ &= \int_\Omega \biggl[A_{21}(x,y)\frac{|\varphi_1(y)|^p}{\|\varphi_2\|^q} + A_{22}(x,y)\biggl|\frac{\varphi_2(y)}{\|\varphi_2\|}\biggr|^q\biggr]\,\mathrm{d}y \\ &\ge \min\biggl\{\int_{D_\varepsilon\cap B_r} A_{21}(x,y)\frac{\varepsilon^p}{\|\varphi_2\|^p}\,\mathrm{d}y, \int_{D_\varepsilon\cap B_r} A_{22}(x,y)\Bigl(\frac{\varepsilon}{\|\varphi_2\|}\Bigr)^q\,\mathrm{d}y\biggr\}. \end{aligned}$$

We conclude that $|D_\varepsilon \cap B_r| = 0$ for all $\varepsilon > 0$ and $r > 0$, where $B_r(x) \triangleq B_r$ is the ball centered at x of radius $r > 0$. Hence, $|D_\varepsilon| = 0$ for all $\varepsilon > 0$. This completes the proof. □

Hereafter, we proceed as in the proof of Lemma 2.4. Our goal is also to derive an estimate of *a priori* bounds for $(\lambda, u, v) \in \mathcal{C}^{+,+}$, where $\lambda \in \tilde{\Lambda} = [\lambda_1, \tilde{\lambda}]$ with $\tilde{\lambda} < \min\{\lambda_1(\Omega_1), \lambda_1(\Omega_2), \ldots, \lambda_1(\Omega_n)\}$.

Lemma 3.2 *Suppose that* A_{ij} $(i,j = 1,2)$ *satisfy conditions* (C1), (C3), *and* (C4). *Then there exists a constant* $l > 0$ *such that* $\|u\|_{C^0(\overline{\Omega})} \le l$ *and* $\|v\|_{C^0(\overline{\Omega})} \le l$ *whenever* $(\lambda, u, v) \in \mathcal{C}^{+,+}$ *and* $\lambda \in \tilde{\Lambda}$.

Proof Indeed, arguing by contradiction, if this were not true, then there would exist $\{(\lambda_n, u_n, v_n)\}_{n=1}^\infty \subset \tilde{\Lambda} \times H_0^1(\Omega) \times H_0^1(\Omega)$ such that one of the following three cases holds:

1. $\|u_n\| \to \infty$, $\lambda_n \to \lambda_0 \in \tilde{\Lambda}$ as $n \to \infty$, $\|v_n\| \le l$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$;
2. $\|v_n\| \to \infty$, $\lambda_n \to \lambda_0 \in \tilde{\Lambda}$ as $n \to \infty$, $\|u_n\| \le l$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$;
3. $\|u_n\| \to \infty$, $\|v_n\| \to \infty$, $\lambda_n \to \lambda_0 \in \tilde{\Lambda}$ as $n \to \infty$, and $(u_n, v_n) = G(\lambda_n, u_n, v_n)$.

We just discuss the case 3 because the other two cases can be dealt with analogously. Let

$$\varphi_1^n = \frac{u_n}{\|u_n\|_{C^0(\overline{\Omega})}}, \qquad \varphi_2^n = \frac{v_n}{\|v_n\|_{C^0(\overline{\Omega})}}.$$

Then it follows that

$$\begin{cases} \int_\Omega \nabla\varphi_1^n \nabla\rho \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{11} + \phi_{q,v_n}^{12})\varphi_n^1 \rho \,\mathrm{d}x = \lambda_n \int_\Omega \varphi_1^n \rho \,\mathrm{d}x, \\ \int_\Omega \nabla\varphi_2^n \nabla\rho \,\mathrm{d}x + \int_\Omega (\phi_{p,u_n}^{21} + \phi_{q,v_n}^{22}) w_n^2 \rho \,\mathrm{d}x = \lambda_n \int_\Omega \varphi_2^n \rho \,\mathrm{d}x. \end{cases} \tag{3.2}$$

Note that $\{(\varphi_1^n, \varphi_2^n)\}_{n=1}^\infty$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$. Then, without loss of generality, we suppose that there is $\varphi = (\varphi_1, \varphi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} &\varphi_i^n \rightharpoonup \varphi_i \quad \text{as } n \to \infty \text{ in } H_0^1(\Omega), i = 1,2, \\ &\varphi_i^n \to \varphi_i \quad \text{as } n \to \infty \text{ in } L^2(\Omega), i = 1,2, \end{aligned}$$

and $\varphi_i^n \to \varphi_i$ $(i = 1,2)$ as $n \to \infty$ a.e. in Ω.

Using similar arguments as in the proof of Lemma 2.4, we have $\varphi \not\equiv 0$, which implies that at least one of φ_1 and φ_2 is not equal to 0. Without loss of generality, assume that $\varphi_1 \not\equiv 0$. Then by Lemma 3.2 there exists some $j \in \{1, 2, \ldots, m\}$ such that $\varphi_1|_{\Omega_j} \not\equiv 0$. In view of the first equation of (3.2), for any positive $\rho \in H_0^1(\Omega_j)$, we have

$$\begin{aligned} \int_{\Omega_j} \nabla\varphi_1^n \nabla\rho \,\mathrm{d}x &\le \int_{\Omega_j} \nabla\varphi_1^n \nabla\rho \,\mathrm{d}x + \int_{\Omega_j} \bigl(\phi_{p,u_n}^{11} + \phi_{q,v_n}^{12}\bigr)\varphi_n^1 \rho \,\mathrm{d}x \\ &= \lambda_n \int_{\Omega_j} \varphi_1^n \rho \,\mathrm{d}x \le \tilde{\lambda} \int_{\Omega_j} \varphi_1^n \rho \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

Taking the limit in equation (3.3) and taking $\rho = \psi_1$, where ψ_1 is the first eigenfunction associated $\lambda_1(\Omega_j)$, we have

$$\lambda_1(\Omega_j) \int_{\Omega_j} \varphi_1 \psi_1 \,\mathrm{d}x \le \tilde{\lambda} \int_{\Omega_j} \varphi_1 \psi_1 \,\mathrm{d}x,$$

which is a contradiction. □

4 Examples

In this section, we consider system (1.6) with $\Omega = (0,\pi)$ to check the validity of the main results obtained in Sections 1 and 2. Notice that $\lambda \triangleq \sigma_n = n^2$ ($n \in \mathbb{N}$) are the eigenvalues of the linear eigenvalue problem $u'' + \lambda u = 0$ with $u(0) = u(\pi) = 0$ and $\sin nx$ is the eigenfunction associated with the eigenvalue n^2, $n \in \mathbb{N}$. In particular, $\sigma_1 = 1$ and $\sin x > 0$ on $(0,\pi)$. In what follows, we consider the following example:

$$A_{i,j}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [(0,\frac{\pi}{2}) \times (\frac{\pi}{2},\pi)] \cup [(\frac{\pi}{2},\pi) \times (0,\frac{\pi}{2})], \\ 0 & \text{otherwise} \end{cases}$$

for all $i,j = 1,2$. Obviously, A_{ij} $(i,j = 1,2)$ vanish on the diagonal. It follows from Theorem 1.1 that there exists a positive solution for all $\lambda > 1$. In fact, we see that system (1.6) with $\Omega = (0,\pi)$ and given A_{ij} $(i,j = 1,2)$ has a positive solution $u = \chi_1 \sin x$ and $v = \chi_2 \sin x$, where χ_1 and χ_2 are positive constants satisfying

$$\lambda = (\chi_1)^p \int_0^{\frac{\pi}{2}} \sin^p x \,\mathrm{d}x + (\chi_2)^q \int_0^{\frac{\pi}{2}} \sin^q x \,\mathrm{d}x + 1.$$

Competing interests
The authors declare that they have no competing interests regarding the publication of this paper.

Authors' contributions
Both authors, XHX and LM, contributed substantially to this paper, participated in drafting and checking the manuscript, and have approved the version to be published.

Acknowledgements
Special thanks to the anonymous referees for very useful suggestions. The research has been supported by the Natural Science Foundation of China (11361004). Xian-Hua Xie is supported by the Bidding Project of Gannan Normal University (16zb01). Both authors are supported by the Key Laboratory of Jiangxi Province for Numerical Simulation and Emulation Techniques of Gannan Normal University.

Received: 25 November 2016 Accepted: 26 December 2016 Published online: 10 January 2017

References

1. Allegretto, W, Nistri, P: On a class of nonlocal problems with applications to mathematical biology. In: Differential Equations with Applications to Biology (Halifax, NS, 1997), pp. 1-14. Am. Math. Soc., Providence (1999)
2. Busenberg, S, Huang, W: Stability and Hopf bifurcation for a population delay model with diffusion effects. J. Differ. Equ. **124**(1), 80-107 (1996)
3. Cantrell, RS, Cosner, C: Spatial Ecology via Reaction-Diffusion Equations. Wiley, New York (2004)
4. Chipot, M: Remarks on some class of nonlocal elliptic problems. In: Recent Advances of Elliptic and Parabolic Issues, pp. 79-102. World Scientific, Singapore (2006)
5. Corrêa, FJSA, Delgado, M, Suárez, A: Some nonlinear heterogeneous problems with nonlocal reaction term. Adv. Differ. Equ. **16**(7/8), 623-641 (2011)
6. Du, Y, Shi, J: Some recent results on diffusive predator-prey models in spatially heterogeneous environment. In: Nonlinear Dynamics and Evolution Equations. Fields Inst. Commun., vol. 48, pp. 95-135. Am. Math. Soc., Providence (2006)
7. Fiedler, B, Poláčik, P: Complicated dynamics of scalar reaction diffusion equations with a nonlocal term. Proc. R. Soc. Edinb., Sect. A, Math. **115**(1-2), 167-192 (1990)
8. Guo, S: Stability and bifurcation in a reaction-diffusion model with nonlocal delay effect. J. Differ. Equ. **259**(4), 1409-1448 (2015)
9. Guo, S, Ma, L: Stability and bifurcation in a delayed reaction-diffusion equation with Dirichlet boundary condition. J. Nonlinear Sci. **26**(2), 545-580 (2016)
10. Allegretto, W, Barabanova, A: Existence of positive solutions of semilinear elliptic equations with nonlocal terms. Funkc. Ekvacioj **40**(3), 395-410 (1997)
11. Davidson, FA, Dodds, N: Existence of positive solutions due to non-local interactions in a class of nonlinear boundary value problems. Methods Appl. Anal. **14**(1), 15-28 (2007)
12. Chen, S, Shi, J: Stability and Hopf bifurcation in a diffusive logistic population model with nonlocal delay effect. J. Differ. Equ. **253**(12), 3440-3470 (2012)
13. Alves, CO, Delgado, M, Souto, MAS, Suárez, A: Existence of positive solution of a nonlocal logistic population model. Z. Angew. Math. Phys. **66**, 943-953 (2015)
14. Sun, L, Shi, J, Wang, Y: Existence and uniqueness of steady state solutions of a nonlocal diffusive logistic equation. Z. Angew. Math. Phys. **64**(4), 1267-1278 (2013)

15. Guo, S, Yan, S: Hopf bifurcation in a diffusive Lotka-Volterra type system with nonlocal delay effect. J. Differ. Equ. **260**(1), 781-817 (2016)
16. Henrot, A: Extremum Problems for Eigenvalues of Elliptic Operators. Springer, Berlin (2006)
17. Rabinowitz, PH: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. **7**(3), 487-513 (1971)