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Improved results on perturbed T-S fuzzy systems with mixed delays using geometric sequence division related partitioning methods

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Abstract

This paper addresses improved stability results for T-S fuzzy systems with mixed delays and nonlinear perturbations. By introducing the geometric sequence division (GSD) method, the discrete delay interval can be separated into multiple subintervals with unequal lengths based on the common ratio α. Meanwhile integral partitioning method is applied to deal with the distributed delay. A Lyapunov-Krasovskii functional (LKF) is newly established with augmented factors and triple integral terms which are constructed by means of the length of every subintervals. In addition, in order to reduce the enlargement when we deal with the estimation of the LKF derivative, a free-matrix-based integral inequality, an extended reciprocal convex combination, and free weight matrices techniques are employed. A stability analysis of the delayed T-S fuzzy systems is presented with much less conservative criteria. At the end numerical examples are given to demonstrate the significant improvements of this proposed design.

Keywords: geometric sequence division; mixed delays; nonlinear perturbations; T-S fuzzy systems

1 Introduction

Because complex mathematical modeling in higher order commonly exists in engineering field, nonlinearity is frequently encountered in dynamic systems. The Takagi-Sugeno (T-S) fuzzy model introduced in [1], has recently attracted special attention for stability analysis and control design of complex dynamic systems due to its practical application. This T-S fuzzy based theory can be applied with expected approximation to the complex nonlinear systems. The T-S fuzzy system is commonly given by combining with various membership functions. In fact, the T-S fuzzy model is basically considered as a multimode method, where the sub-models can be directly consolidated to analyze the original nonlinear system behavior [2, 3]. Therefore, recently many efforts have been made as regards the stability analysis of the T-S fuzzy systems [4–6].

Time delay often exists in dynamic systems such as chemical reaction processes, communication networks, and biological systems; it is considered as the main source of poor performance and instability [7–9]. Stability analysis of time delayed T-S fuzzy systems has thus been paid special attention to; they are commonly classified into two categories: delay-independent and delay-dependent criteria [10, 11]. As much of information on the delay is concerned, the delay-dependent criteria are quite useful for providing less conservatism [6, 12, 13].

Delay partitioning, alternatively known as a fractionizing method, was first proposed in [14]. It was proved that delay partitioning method can improve the stability conditions significantly. Less conservative results can be obtained as soon as the partition segments get thinner [15, 16].

In the recent decades, distributed delay is researched as a different type of delays for stability analysis in nonlinear dynamic systems which commonly appears in some engineering applications, such as logistics, traffic communication, and biological systems. A number of research works have been carried out to investigate stability problems of systems with distributed delays [17–19]. In [17], Jensen's inequality and elimination methods were applied to deal with inequalities to improve stability conditions under the effects of distributed delay. Another work was proposed by using an analytic solution to handle the Lyapunov functionals in [18]. Then the necessary and sufficient condition for this distributed delay system is derived by considering a bounded constant delay. Later on, research work on the neutral system in the case of both discrete and distributed delays has been studied [19–21]. However, due to the absence of the distributed delay information in the weighting matrices of the Lyapunov functionals, the proposed work may lead to conservative results. An integral partitioning technique is therefore proposed in [22], which is applied to the construction of a new form of LKF to enhance the feasible region of the distributed delay-dependent conditions. Motivated by this approach, it is expected one may employ this integral partitioning method to solve the stability issues of perturbed T-S fuzzy systems with mixed delays.

In addition, because of the existence of process uncertainties and slowly varying parameters, nonlinear perturbations commonly occur in both current and delayed states [23]. Under this circumstance, previously developed approaches for deterministic systems are rarely employed for the stability analysis under the appearance of nonlinear perturbations.

In this research work, stability criteria are investigated by considering a nonlinear perturbed T-S fuzzy system with the appearance of mixed delays. The main contribution of this paper is described as follows.

(1) The discrete time delay interval is unequally separated into multiple segments with variable length which is based on the geometric sequence division (GSD) for a common ratio α. As a result, less conservatism is obtained efficiently. In addition, to deal with this distributed delay, integral partitioning method is introduced which is used to build a new form of LKF with double and triple integral forms.

(2) By employing the free-matrix-based integral inequality, an extended reciprocal convex combination and free weight matrices techniques, the less enlargement of bounding the derivative of the LKF is achieved. The implementation in the Matlab/Simulink is enhanced efficiently.

(3) Numerical results are given to show that the proposed stability conditions are much less conservative than some of the existing results. Significant stability conditions are derived in this proposed method in the case of T-S fuzzy systems with mixed delays and nonlinear perturbations.

Notations $\mathbb{R}^n$ is the n-dimensional Euclidean space. $P > (\geq) 0$ means that the matrix P is positive (semi-positive) definite. I_n (0_n) is the identity (zero) matrix with n-dimensions; A^{T} denotes the transpose, and $\mathrm{He}(A) = A + A^{\mathrm{T}}$. The symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. $\| \bullet \|$ is the Euclidean norm in $\mathbb{R}^n$.

2 Problem statements and preliminaries

Considering a nonlinear perturbed T-S fuzzy system with mixed delays, for each $l = 1, 2, \ldots, r$ (r is the number of the plant rules), the lth rule of this fuzzy model with r plant rules is described as follows.

Rule l: IF $z_1(t)$ is M_{l1} and $\cdots z_p(t)$ is M_{lp} THEN

$$\begin{aligned} \dot{x}(t) &= A_l x(t) + B_l x\bigl(t - \tau(t)\bigr) + C_l f\bigl(x(t), t\bigr) \\ &\quad + D_l f\bigl(x\bigl(t - \tau(t)\bigr), t\bigr) + E_l \int_{t-d}^{t} x(s)\,\mathrm{d}s, \quad t \geq 0, \\ x(t) &= \varphi(t), \quad t \in [-\phi, 0], \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $z_s(t)$, M_{ls} $(s = 1, 2, \ldots, p)$ are premise variables and the related fuzzy sets, respectively. A_l, B_l, C_l, D_l, E_l are the real constant matrices with appropriate dimensions. $\tau(t)$ is the time-varying delay. $f(x(t), t)$ and $f(x(t - \tau(t)), t)$ are unknown nonlinear perturbations with respect to the current state $x(t)$ and the delayed state $x(t - \tau(t))$. $\varphi(t) \in C([-\tau_q, 0], \mathbb{R}^n)$ is the initial function, and $\phi = \max\{\tau_q, d\}$.

Then the fuzzy model can be inferred to be

$$\begin{aligned} \dot{x}(t) &= \sum_{l=1}^{r} h_l(t) \biggl[A_l x(t) + B_l x\bigl(t - \tau(t)\bigr) + C_l f\bigl(x(t), t\bigr) \\ &\quad + D_l f\bigl(x\bigl(t - \tau(t)\bigr), t\bigr) + E_l \int_{t-d}^{t} x(s)\,\mathrm{d}s \biggr] \\ &= A(t)x(t) + B(t)x\bigl(t - \tau(t)\bigr) + C(t) f\bigl(x(t), t\bigr) \\ &\quad + D(t) f\bigl(x\bigl(t - \tau(t)\bigr), t\bigr) + E(t) \int_{t-d}^{t} x(s)\,\mathrm{d}s, \quad t \geq 0, \\ x(t) &= \varphi(t), \quad t \in [-\phi, 0], \end{aligned} \tag{2.2}$$

where r is the number of fuzzy implications, $h_l(t) = \frac{W_l(t)}{\sum_{l=1}^{r} W_l(t)}$, $W_l(t) = \prod_{s=1}^{p} M_{ls}(z_s(t))$ with $M_{ls}(z_s(t))$ is the grade of membership of $z_s(t)$ in M_{ls}. $A(t) = \sum_{l=1}^{r} h_l(t) A_l$, $B(t) = \sum_{l=1}^{r} h_l(t) B_l$, $C(t) = \sum_{l=1}^{r} h_l(t) C_l$, $D(t) = \sum_{l=1}^{r} h_l(t) D_l$, $E(t) = \sum_{l=1}^{r} h_l(t) E_l$. For $W_l(t) \geq 0$, $h_l(t) \geq 0$ and $\sum_{l=1}^{r} h_l(t) = 1$ thus holds.

The time-varying delay $\tau(t)$ is considered as

$$0 \leq \tau_0 \leq \tau(t) \leq \tau_q, \qquad \dot{\tau}(t) < \mu, \quad \forall t \geq 0, \tag{2.3}$$

where τ_0, τ_q, μ are constants.

Assumption 1 $f(0, t) \equiv 0$ and

$$f^{\mathrm{T}} f \leq x^{\mathrm{T}}(t) F^{\mathrm{T}} F x(t), \tag{2.4}$$

where F is for known constant matrices, $\forall x \in \mathbb{R}^n$, and f is short for the expressions of $f(x(t),t)$.

A few lemmas are used for stability analysis as follows.

Lemma 1 ([24, 25]) *For the $n \times n$ matrix $Q > 0$, the scalar $\tau > 0$, the vector-valued function $\dot{x} : [-\tau, 0] \longrightarrow \mathbb{R}^n$ such that the following integrations are well defined*:

$$-\tau \int_{t-\tau}^{t} \dot{x}^{\mathrm{T}}(s) Q \dot{x}(s)\,\mathrm{d}s \leq \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau) \end{bmatrix} \begin{bmatrix} -Q & Q \\ * & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \tag{2.5}$$

$$\begin{aligned} &-\frac{\tau^2}{2} \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Q \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad \leq \begin{bmatrix} \tau x^{\mathrm{T}}(t) & \int_{t-\tau}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s \end{bmatrix} \begin{bmatrix} -Q & Q \\ * & -Q \end{bmatrix} \begin{bmatrix} \tau x(t) \\ \int_{t-\tau}^{t} x(s)\,\mathrm{d}s \end{bmatrix}. \end{aligned} \tag{2.6}$$

Lemma 2 (Free-matrix-based integral inequality [26]) *Let x be a differentiable function*: $[a,b] \to \mathbb{R}^n$, $Z \in \mathbb{R}^{n\times n}$ *and* $W_1, W_3 \in \mathbb{R}^{3n\times 3n}$ *be symmetric matrices, and* $W_2 \in \mathbb{R}^{3n\times 3n}$, $N_1, N_2 \in \mathbb{R}^{3n\times n}$ *satisfying this condition*:

$$\begin{bmatrix} W_1 & W_2 & N_1 \\ * & W_3 & N_2 \\ * & * & Z \end{bmatrix} \geq 0,$$

we have

$$-\int_{a}^{b} \dot{x}^{\mathrm{T}}(s) Z \dot{x}(s)\,\mathrm{d}s \leq \varpi^{\mathrm{T}} \Omega \varpi, \tag{2.7}$$

where $\varpi = [x^{\mathrm{T}}(b)\ x^{\mathrm{T}}(a)\ \frac{1}{b-a}\int_a^b x^{\mathrm{T}}(s)\,\mathrm{d}s]^{\mathrm{T}}$, $\Omega = (b-a)(W_1 + \frac{1}{3}W_3) + \mathrm{He}(N_1\Lambda_1 + N_2\Lambda_2)$, $\Lambda_1 = \bar{e}_1 - \bar{e}_2$, $\Lambda_2 = 2\bar{e}_3 - \bar{e}_1 - \bar{e}_2$, $\bar{e}_1 = [I\ 0\ 0]$, $\bar{e}_2 = [0\ I\ 0]$, $\bar{e}_3 = [0\ 0\ I]$.

Lemma 3 (Extended reciprocal convex combination (RCC) [16]) *For any vectors $f_1, \ldots, f_N$ with appropriate dimensions, scalars $k_i(t) \in [0,1]$, $\sum_{i=1}^{N} k_i(t) = 1$, and matrices $R_i > 0$, there exists a matrix S_{ij} ($i = 1, \ldots, N-1$, $j = i+1, \ldots, N$) that satisfies*

$$\begin{bmatrix} R_i & S_{ij} \\ * & R_j \end{bmatrix} \geq 0;$$

then the following inequality holds:

$$-\sum_{i=1}^{N} \frac{1}{k_i(t)} f_i^{\mathrm{T}} R_i f_i \leq -\begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_1 & \cdots & S_{1,N} \\ * & \ddots & \vdots \\ * & * & R_N \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}.$$

Proof For $N = 2$, the following inequality always holds:

$$\begin{bmatrix} \sqrt{\frac{k_2(t)}{k_1(t)}} f_1 \\ -\sqrt{\frac{k_1(t)}{k_2(t)}} f_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_1 & S_{12} \\ * & R_2 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{k_2(t)}{k_1(t)}} f_1 \\ -\sqrt{\frac{k_1(t)}{k_2(t)}} f_2 \end{bmatrix} \geq 0.$$

Because $k_1(t) + k_2(t) = 1$, then it is deduced that

$$\begin{aligned}\frac{1}{k_1(t)}f_1^{\mathrm{T}}R_1f_1 + \frac{1}{k_2(t)}f_2^{\mathrm{T}}R_2f_2 &= \frac{1}{k_1(t)}f_1^{\mathrm{T}}\big(k_1(t)+k_2(t)\big)R_1f_1 + \frac{1}{k_2(t)}f_2^{\mathrm{T}}\big(k_1(t)+k_2(t)\big)R_2f_2 \\ &= f_1^{\mathrm{T}}R_1f_1 + \frac{k_2(t)}{k_1(t)}f_1^{\mathrm{T}}R_1f_1 + f_2^{\mathrm{T}}R_2f_2 + \frac{k_1(t)}{k_2(t)}f_2^{\mathrm{T}}R_2f_2 \\ &\geq f_1^{\mathrm{T}}R_1f_1 + f_2^{\mathrm{T}}R_2f_2 + f_1^{\mathrm{T}}S_{12}f_2 + f_2^{\mathrm{T}}S_{12}f_1 \\ &= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R_1 & S_{1,2} \\ * & R_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.\end{aligned}$$

In the case of $i = N$, the proof procedures are similar and omitted. □

Lemma 4 (Finsler's lemma [27]) *Let* $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, *and* $B \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank}(B) < n$. *The following statements are equivalent*:

(i) $\zeta^{\mathrm{T}}\Phi\zeta < 0$, $\forall B\zeta = 0$, $\zeta \neq 0$;

(ii) ${B^{\perp}}^{\mathrm{T}}\Phi B^{\perp} < 0$;

(iii) $\exists \mathcal{L} \in \mathbb{R}^{n\times m}$: $\Phi + \operatorname{He}(\mathcal{L}B) < 0$;

where $B^{\perp} \in \mathbb{R}^{n\times(n-\operatorname{rank}(B))}$ *is the right orthogonal complement of* B.

3 Main results

Based on the geometric sequence division method, a new delay partitioning approach is proposed in Figure 1. The stability criteria of T-S fuzzy system in the presences of mixed delays and nonlinear perturbations are analyzed in this section. For any integral $q \geq 1$, the delay interval $[\tau_0, \tau_q]$ is separated into q unequal geometric subintervals by

$$\begin{cases} \delta_i = \alpha^{q-i+1}, \\ \tau_i = \tau_0 + \sum_{a=1}^{i} \alpha^{q-a+1}, \quad i = 1, 2, \ldots, q, \end{cases} \tag{3.1}$$

where q is the number of segments of interval $[\tau_0, \tau_q]$. It is expressed as $[\tau_0, \tau_q] = [\tau_0, \tau_1] \bigcup_{i=2}^{q} (\tau_{i-1}, \tau_i] \triangleq I_1 \cup I_2 \cup \cdots \cup I_q$. α is a real positive number, and δ_i is the length of the ith subinterval which equals α^{q-i+1}.

In addition, using the integral partitioning method proposed in [22], the distributed delay $[0, d]$ is divided into L equivalent segments as $\rho = \frac{d}{L}$. ρ is the length of the subinterval of the distributed delay.

The following expressions are used for notational simplification:

$$e_j = \big[\underbrace{0_n, \ldots, 0_n}_{j-1}, \mathrm{I}_n, \underbrace{0_n, \ldots, 0_n}_{3q+L-j+7}\big]^{\mathrm{T}} \in \mathbb{R}^{(3q+L+7)n\times n}, \quad j = 1, 2, \ldots, 3q+L+7. \tag{3.2}$$

Figure 1 GSD based delay partitioning.

The augmented vector is defined by

$$\xi(t) = \Big[\dot{x}^{\mathrm{T}}(t), x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau_0), \sigma^{\mathrm{T}}(t), x^{\mathrm{T}}\big(t-\tau(t)\big), \sigma_1^{\mathrm{T}}(t), \sigma_2^{\mathrm{T}}(t), \sigma_3^{\mathrm{T}}(t),$$
$$\int_{t-(L+1)\rho}^{t-L\rho} x^{\mathrm{T}}(s)\,\mathrm{d}s, f^{\mathrm{T}}\big(x(t)\big), f^{\mathrm{T}}\big(x\big(t-\tau(t)\big)\big)\Big]^{\mathrm{T}}, \tag{3.3}$$

where

$$\begin{aligned}
\sigma(t) &= \big[x^{\mathrm{T}}(t-\tau_1), \ldots, x^{\mathrm{T}}(t-\tau_q)\big]^{\mathrm{T}}, \\
\sigma_1(t) &= \left[\int_{t-\tau_0}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s, \ldots, \int_{t-\tau_{q-1}}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\right]^{\mathrm{T}}, \\
\sigma_2(t) &= \left[\frac{1}{\delta_1}\int_{t-\tau_1}^{t-\tau_0} x^{\mathrm{T}}(s)\,\mathrm{d}s, \ldots, \frac{1}{\delta_q}\int_{t-\tau_q}^{t-\tau_{q-1}} x^{\mathrm{T}}(s)\,\mathrm{d}s\right]^{\mathrm{T}}, \\
\sigma_3(t) &= \left[\int_{t-\rho}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s, \int_{t-2\rho}^{t-\rho} x^{\mathrm{T}}(s)\,\mathrm{d}s, \ldots, \int_{t-L\rho}^{t-(L-1)\rho} x^{\mathrm{T}}(s)\,\mathrm{d}s\right]^{\mathrm{T}}.
\end{aligned}$$

Next, the new delay-dependent stability criteria are presented for the T-S fuzzy system in the presence of mixed delays and nonlinear perturbations described in (2.2).

Theorem 1 *Given a positive integer q, L, and $\delta_i = \alpha^{q-i+1}$. The system* (2.2) *is asymptotically stable if there exist symmetric positive definite matrices $Z_i, \mathcal{Z}_3, Q_i, R_{2i}, R_{3i}, R_4 \in \mathbb{R}^{n\times n}$ ($i = 1,2,\ldots,q$), $\widetilde{Q} \in \mathbb{R}^{2n\times 2n}$, $\mathcal{P} = [P_{ij}]_{(q+1)\times(q+1)} \in \mathbb{R}^{(q+1)n\times(q+1)n}$, $\mathcal{Q} = [\mathcal{Q}_{ij}]_{L\times L} \in \mathbb{R}^{Ln\times Ln}$, symmetric matrices $W_1, W_3 \in \mathbb{R}^{3n\times 3n}$, and $J \in \mathbb{R}^{n\times n}$, matrices $W_2 \in \mathbb{R}^{3n\times 3n}$, $N_1, N_2 \in \mathbb{R}^{3n\times n}$, $\widehat{N}_1, \widehat{N}_2$, $\widehat{N}_3 \in \mathbb{R}^{n\times n}$ and $\mathcal{Y} \in \mathbb{R}^{(3q+L+7)n\times n}$, such that the following LMIs hold*:

$$W_i = \begin{bmatrix} W_1 & W_2 & N_1 \\ * & W_3 & N_2 \\ * & * & Z_i \end{bmatrix} \geq 0, \tag{3.4}$$

$$\Xi_{k,l} + \mathrm{He}(\mathcal{Y}\Gamma_l) < 0, \quad l = 1,2,\ldots,r, k = 1,2,\ldots,q, \tag{3.5}$$

where

$$\Gamma_l = A_l e_2^{\mathrm{T}} + B_l e_{q+4}^{\mathrm{T}} + C_l e_{3q+L+6}^{\mathrm{T}} + D_l e_{3q+L+7}^{\mathrm{T}} + E_l \sum_{j=1}^{L} e_{3q+4+j}^{\mathrm{T}} - e_1^{\mathrm{T}},$$

$$\Xi_{k,l} = \Xi_1 + \Xi_2 + \Xi_{3k} + \Xi_4 + \Xi_{l,5} + \Xi_6 + e_1 \mathcal{Z} e_1^{\mathrm{T}},$$

$$\Xi_1 = \mathrm{He}\left\{\begin{bmatrix} e_2^{\mathrm{T}} \\ e_{2q+5}^{\mathrm{T}} \\ \vdots \\ e_{3q+4}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \mathcal{P} \begin{bmatrix} e_1^{\mathrm{T}} \\ \frac{1}{\delta_1}(e_3^{\mathrm{T}} - e_4^{\mathrm{T}}) \\ \vdots \\ \frac{1}{\delta_q}(e_{q+2}^{\mathrm{T}} - e_{q+3}^{\mathrm{T}}) \end{bmatrix}\right\},$$

$$\Xi_2 = \begin{bmatrix} e_2^{\mathrm{T}} \\ e_{3q+L+6}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \widetilde{Q} \begin{bmatrix} e_2^{\mathrm{T}} \\ e_{3q+L+6}^{\mathrm{T}} \end{bmatrix} - (1-\mu) \begin{bmatrix} e_{q+4}^{\mathrm{T}} \\ e_{3q+L+7}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \widetilde{Q} \begin{bmatrix} e_{q+4}^{\mathrm{T}} \\ e_{3q+L+7}^{\mathrm{T}} \end{bmatrix}$$

$$
\begin{aligned}
&+\sum_{i=1}^{q}\bigl(x^{\mathrm{T}}(t-\tau_{i-1})Q_i x(t-\tau_{i-1})-x^{\mathrm{T}}(t-\tau_i)Q_i x(t-\tau_i)\bigr)\\
&+\begin{bmatrix} e_{3q+5}^{\mathrm{T}}\\ e_{3q+6}^{\mathrm{T}}\\ \vdots\\ e_{3q+L+4}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\mathcal{Q}\begin{bmatrix} e_{3q+5}^{\mathrm{T}}\\ e_{3q+6}^{\mathrm{T}}\\ \vdots\\ e_{3q+L+4}^{\mathrm{T}}\end{bmatrix}-\begin{bmatrix} e_{3q+6}^{\mathrm{T}}\\ e_{3q+7}^{\mathrm{T}}\\ \vdots\\ e_{3q+L+5}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\mathcal{Q}\begin{bmatrix} e_{3q+6}^{\mathrm{T}}\\ e_{3q+7}^{\mathrm{T}}\\ \vdots\\ e_{3q+L+5}^{\mathrm{T}}\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{3k}={}&\sum_{i=1,i\neq k}^{q}\begin{bmatrix} e_{i+2}^{\mathrm{T}}\\ e_{i+3}^{\mathrm{T}}\\ e_{2q+4+i}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\Omega_3\begin{bmatrix} e_{i+2}^{\mathrm{T}}\\ e_{i+3}^{\mathrm{T}}\\ e_{2q+4+i}^{\mathrm{T}}\end{bmatrix}\\
&+\begin{bmatrix} e_{k+2}^{\mathrm{T}}\\ e_{q+4}^{\mathrm{T}}\\ e_{k+3}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -Z_k & Z_k-J & J\\ * & -2Z_k+J^{\mathrm{T}}+J & Z_k-J\\ * & * & -Z_k\end{bmatrix}\begin{bmatrix} e_{k+2}^{\mathrm{T}}\\ e_{q+4}^{\mathrm{T}}\\ e_{k+3}^{\mathrm{T}}\end{bmatrix}\\
&-e_{3q+5}\frac{1}{\rho}\mathcal{Z}_3 e_{3q+5}^{\mathrm{T}}+e_2\rho\mathcal{Z}_3 e_2^{\mathrm{T}},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_4={}&\sum_{i=1}^{q}\begin{bmatrix} \tau_{i-1}e_2^{\mathrm{T}}\\ e_{q+4+i}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -R_{2i} & R_{2i}\\ * & -R_{2i}\end{bmatrix}\begin{bmatrix} \tau_{i-1}e_2^{\mathrm{T}}\\ e_{q+4+i}^{\mathrm{T}}\end{bmatrix}\\
&+\sum_{i=1}^{q}\delta_i^2\begin{bmatrix} e_2^{\mathrm{T}}\\ e_{2q+4+i}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -R_{3i} & R_{3i}\\ * & -R_{3i}\end{bmatrix}\begin{bmatrix} e_2^{\mathrm{T}}\\ e_{2q+4+i}^{\mathrm{T}}\end{bmatrix}\\
&+\begin{bmatrix} \rho e_2^{\mathrm{T}}\\ e_{3q+5}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -R_4 & R_4\\ * & -R_4\end{bmatrix}\begin{bmatrix} \rho e_2^{\mathrm{T}}\\ e_{3q+5}^{\mathrm{T}}\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Xi_{l,5}={}&\begin{bmatrix} e_1^{\mathrm{T}}\\ e_2^{\mathrm{T}}\\ e_{q+4}^{\mathrm{T}}\\ e_{3q+L+6}^{\mathrm{T}}\\ e_{3q+L+7}^{\mathrm{T}}\\ \sum_{j=1}^{L}e_{3q+4+j}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\
&\times\begin{bmatrix} -\widehat{N}_1-\widehat{N}_1^{\mathrm{T}} & \widehat{N}_1A_l-\widehat{N}_2^{\mathrm{T}} & \widehat{N}_1B_l & \widehat{N}_1C_l-\widehat{N}_3^{\mathrm{T}} & \widehat{N}_1D_l & \widehat{N}_1E_l\\
* & \widehat{N}_2A_l+A_l^{\mathrm{T}}\widehat{N}_2^{\mathrm{T}} & \widehat{N}_2B_l & \widehat{N}_2C_l+B_l^{\mathrm{T}}\widehat{N}_3^{\mathrm{T}} & \widehat{N}_2D_l & \widehat{N}_2E_l\\
* & * & 0 & C_l^{\mathrm{T}}\widehat{N}_3^{\mathrm{T}} & 0 & 0\\
* & * & * & \widehat{N}_3C_l+C_l^{\mathrm{T}}\widehat{N}_3^{\mathrm{T}} & \widehat{N}_3D_l & \widehat{N}_3E_l\\
* & * & * & * & 0 & 0\\
* & * & * & * & * & 0\end{bmatrix}\\
&\times\begin{bmatrix} e_1^{\mathrm{T}}\\ e_2^{\mathrm{T}}\\ e_{q+4}^{\mathrm{T}}\\ e_{3q+L+6}^{\mathrm{T}}\\ e_{3q+L+7}^{\mathrm{T}}\\ \sum_{j=1}^{L}e_{3q+4+j}^{\mathrm{T}}\end{bmatrix},
\end{aligned}
$$

$$
\Xi_6=e_2\lambda F^{\mathrm{T}}Fe_2^{\mathrm{T}}-e_{3q+L+6}\lambda\mathrm{I}e_{3q+L+6}^{\mathrm{T}},
$$

with

$$\mathcal{Z} = \sum_{i=1}^{q} \delta_i \alpha^{q-i+1} Z_i + \sum_{i=1}^{q} \frac{1}{4}\tau_{i-1}^4 R_{2i} + \sum_{i=1}^{q} \frac{1}{4}\left(\tau_i^2 - \tau_{i-1}^2\right)^2 R_{3i} + \frac{\rho^4}{4} R_4,$$
$$\Omega_3 = \delta_i^2\left(W_1 + \frac{1}{3}W_3\right) + \delta_i \operatorname{He}(N_1\Lambda_1 + N_2\Lambda_2).$$

Proof For any $t \geq 0$, there should exist an integer $k \in \{1, 2, \dots, q\}$, such that $\tau(t) \in I_k$. The new Lyapunov-Krasovskii functional is as follows:

$$V(x_t, k)|_{\tau(t)\in I_k} = V_1(x_t) + V_2(x_t) + V_3(x_t, k) + V_4(x_t), \tag{3.6}$$

where

$$\begin{aligned}
V_1(x_t) &= \epsilon^{\mathrm{T}}(t)\mathcal{P}\epsilon(t),\\
V_2(x_t) &= \int_{t-\tau(t)}^{t} \begin{bmatrix} x(s)\\ f(x(s))\end{bmatrix}^{\mathrm{T}} \widetilde{Q} \begin{bmatrix} x(s)\\ f(x(s))\end{bmatrix} \mathrm{d}s\\
&\quad + \sum_{i=1}^{q} \int_{t-\tau_i}^{t-\tau_{i-1}} x^{\mathrm{T}}(t) Q_i x(t)\,\mathrm{d}s + \int_{t-\rho}^{t} \eta_3^{\mathrm{T}}(s)\mathcal{Q}\eta_3(s)\,\mathrm{d}s,\\
V_3(x_t, k) &= \sum_{i=1}^{q} \delta_i \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\beta}^{t} \dot{x}^{\mathrm{T}}(s) Z_i \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\beta + \int_{-\rho}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)\mathcal{Z}_3 x(s)\,\mathrm{d}s\,\mathrm{d}\theta,\\
V_4(x_t) &= \sum_{i=1}^{q} \frac{\tau_{i-1}^2}{2} \int_{-\tau_{i-1}}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) R_{2i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta\\
&\quad + \sum_{i=1}^{q} \frac{\tau_i^2 - \tau_{i-1}^2}{2} \int_{-\tau_i}^{-\tau_{i-1}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) R_{3i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta\\
&\quad + \frac{\rho^2}{2}\int_{-\rho}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) R_4 \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\lambda\,\mathrm{d}\theta
\end{aligned}$$

with $\epsilon(t) = [x^{\mathrm{T}}(t), \sigma_2^{\mathrm{T}}(t)]^{\mathrm{T}}$.

The derivative of the Lyapunov functional $V(x_t, k)|_{\tau(t)\in I_k}$ along the trajectory of the T-S fuzzy system shown in (2.2) is given by

$$\dot{V}(x_t, k)|_{\tau(t)\in I_k} = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t, k) + \dot{V}_4(x_t), \tag{3.7}$$

where

$$\dot{V}_1(x_t) = 2\epsilon^{\mathrm{T}}(t)\mathcal{P}\dot{\epsilon}(t) = \xi^{\mathrm{T}}(t)\Xi_1\xi(t), \tag{3.8}$$
$$\begin{aligned}
\dot{V}_2(x_t) &\leq \begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^{\mathrm{T}} \widetilde{Q} \begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix} - (1-\mu)\begin{bmatrix} x(t-\tau(t))\\ f(x(t-\tau(t)))\end{bmatrix}^{\mathrm{T}} \widetilde{Q} \begin{bmatrix} x(t-\tau(t))\\ f(x(t-\tau(t)))\end{bmatrix}\\
&\quad + \sum_{i=1}^{q}\left(x^{\mathrm{T}}(t-\tau_{i-1})Q_i x(t-\tau_{i-1}) - x^{\mathrm{T}}(t-\tau_i)Q_i x(t-\tau_i)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \eta_3^{\mathrm{T}}(t)\mathcal{Q}\eta_3(t) - \eta_3^{\mathrm{T}}(t-\rho)\mathcal{Q}\eta_3(t-\rho) \\
&= \xi^{\mathrm{T}}(t)\Xi_2\xi(t),
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\dot{V}_3(x_t) &= \dot{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{q}\delta_i\alpha^{q-i+1}Z_i\right)\dot{x}(t) - \sum_{i=1}^{q}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\,\mathrm{d}s \\
&\quad + x^{\mathrm{T}}(t)\rho\mathcal{Z}_3x(t) - \int_{t-\rho}^{t}x^{\mathrm{T}}(s)\mathcal{Z}_3x(s)\,\mathrm{d}s,
\end{aligned}
\tag{3.10}
$$

$\tau(t) \in I_k$ $(1 \le k \le q)$, the second term in (3.10) is derived as follows:

$$
\begin{aligned}
&-\sum_{i=1}^{q}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\,\mathrm{d}s \\
&= -\sum_{i=1,i\neq k}^{q}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\,\mathrm{d}s - \delta_k\int_{t-\tau_k}^{t-\tau_{k-1}}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)\,\mathrm{d}s.
\end{aligned}
\tag{3.11}
$$

Using Lemma 2 to deal with (3.11), we have

$$
-\sum_{i=1,i\neq k}^{q}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\,\mathrm{d}s \le \sum_{i=1,i\neq k}^{q}\varpi_{3i}^{\mathrm{T}}(t)\Omega_3\varpi_{3i}(t),
\tag{3.12}
$$

where $\varpi_{3i}(t) = [x^{\mathrm{T}}(t-\tau_{i-1})\ x^{\mathrm{T}}(t-\tau_i)\ \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}x^{\mathrm{T}}(s)\,\mathrm{d}s]^{\mathrm{T}}$.

In the case of $i = k$, applying Jensen's inequality and the extended RCC in Lemma 3, it is given by

$$
\begin{aligned}
&-(\tau_k-\tau_{k-1})\int_{t-\tau_k}^{t-\tau_{k-1}}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)\,\mathrm{d}s \\
&= -(\tau_k-\tau_{k-1})\left(\int_{t-\tau(t)}^{t-\tau_{k-1}}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)\,\mathrm{d}s + \int_{t-\tau_k}^{t-\tau(t)}\dot{x}^{\mathrm{T}}(s)Z_k\dot{x}(s)\,\mathrm{d}s\right) \\
&\le -\frac{(\tau_k-\tau_{k-1})}{(\tau(t)-\tau_{k-1})}\left(\int_{t-\tau(t)}^{t-\tau_{k-1}}\dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\right)Z_k\left(\int_{t-\tau(t)}^{t-\tau_{k-1}}\dot{x}(s)\,\mathrm{d}s\right) \\
&\quad -\frac{(\tau_k-\tau_{k-1})}{(\tau_k-\tau(t))}\left(\int_{t-\tau_k}^{t-\tau(t)}\dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\right)Z_k\left(\int_{t-\tau_k}^{t-\tau(t)}\dot{x}(s)\,\mathrm{d}s\right) \\
&= -\frac{(\tau_k-\tau_{k-1})}{(\tau(t)-\tau_{k-1})}\begin{bmatrix} x(t-\tau_{k-1}) \\ x(t-\tau(t)) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Z_k & -Z_k \\ * & Z_k \end{bmatrix}\begin{bmatrix} x(t-\tau_{k-1}) \\ x(t-\tau(t)) \end{bmatrix} \\
&\quad -\frac{(\tau_k-\tau_{k-1})}{(\tau_k-\tau(t))}\begin{bmatrix} x(t-\tau(t)) \\ x(t-\tau_k) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Z_k & -Z_k \\ * & Z_k \end{bmatrix}\begin{bmatrix} x(t-\tau(t)) \\ x(t-\tau_k) \end{bmatrix} \\
&\le -\eta_0^{\mathrm{T}}(t)\begin{bmatrix} Z_k & -Z_k+J & -J \\ * & 2Z_k-J^{\mathrm{T}}-J & -Z_k+J \\ * & * & Z_k \end{bmatrix}\eta_0(t),
\end{aligned}
\tag{3.13}
$$

where $\sigma_0(t) = [x^{\mathrm{T}}(t-\tau_{k-1})\ x^{\mathrm{T}}(t-\tau(t))\ x^{\mathrm{T}}(t-\tau_k)]^{\mathrm{T}}$.

Using Jensen's inequality, the last term of (3.10) is deduced:

$$-\int_{t-\rho}^{t} x^{\mathrm{T}}(s)\mathcal{Z}_3 x(s)\,\mathrm{d}s \le -\frac{1}{\rho}\left(\int_{t-\rho}^{t} x(s)^{\mathrm{T}}\,\mathrm{d}s\right)\mathcal{Z}_3\left(\int_{t-\rho}^{t} x(s)\,\mathrm{d}s\right). \tag{3.14}$$

It follows from (3.10)-(3.14) that

$$\begin{aligned}
&-\sum_{i=1}^{q}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\,\mathrm{d}s + x^{\mathrm{T}}(t)\rho\mathcal{Z}_3 x(t) - \int_{t-\rho}^{t} x^{\mathrm{T}}(s)\mathcal{Z}_3 x(s)\,\mathrm{d}s\\
&\quad\le \xi^{\mathrm{T}}(t)\Xi_{3k}\xi(t).
\end{aligned} \tag{3.15}$$

The derivative of $V_4(x_t)$ is presented as

$$\begin{aligned}
\dot{V}_4(x_t) ={}& \dot{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{q}\frac{1}{4}\tau_{i-1}^4 R_{2i} + \sum_{i=1}^{q}\frac{1}{4}\left(\tau_i^2-\tau_{i-1}^2\right)^2 R_{3i}\right)\dot{x}(t)\\
&-\sum_{i=1}^{q}\frac{\tau_{i-1}^2}{2}\int_{-\tau_{i-1}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{2i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&-\sum_{i=1}^{q}\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{3i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&-\frac{\rho^2}{2}\int_{-\rho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_4\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta.
\end{aligned} \tag{3.16}$$

By applying Lemma 1, the last three terms of (3.16) are derived as

$$\begin{aligned}
&-\sum_{i=1}^{q}\frac{\tau_{i-1}^2}{2}\int_{-\tau_{i-1}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{2i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad\le \sum_{i=1}^{q}\begin{bmatrix}\tau_{i-1}x(t)\\ \int_{t-\tau_{i-1}}^{t}x(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-R_{2i} & R_{2i}\\ * & -R_{2i}\end{bmatrix}\begin{bmatrix}\tau_{i-1}x(t)\\ \int_{t-\tau_{i-1}}^{t}x(s)\,\mathrm{d}s\end{bmatrix}\\
&\qquad-\sum_{i=1}^{q}\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_{3i}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad\le \sum_{i=1}^{q}\begin{bmatrix}(\tau_i-\tau_{i-1})x(t)\\ \int_{t-\tau_i}^{t-\tau_{i-1}}x(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-R_{3i} & R_{3i}\\ * & -R_{3i}\end{bmatrix}\begin{bmatrix}(\tau_i-\tau_{i-1})x(t)\\ \int_{t-\tau_i}^{t-\tau_{i-1}}x(s)\,\mathrm{d}s\end{bmatrix}\\
&\quad= \sum_{i=1}^{q}(\tau_i-\tau_{i-1})^2\begin{bmatrix}x(t)\\ \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}x(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-R_{3i} & R_{3i}\\ * & -R_{3i}\end{bmatrix}\begin{bmatrix}x(t)\\ \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}x(s)\,\mathrm{d}s\end{bmatrix}\\
&\qquad-\frac{\rho^2}{2}\int_{-\rho}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_4\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad\le \begin{bmatrix}\rho x(t)\\ \int_{t-\rho}^{t}x(s)\,\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-R_4 & R_4\\ * & -R_4\end{bmatrix}\begin{bmatrix}\rho x(t)\\ \int_{t-\rho}^{t}x(s)\,\mathrm{d}s\end{bmatrix}.
\end{aligned} \tag{3.17}$$

Thus (3.16) implies that

$$\dot{V}_4(x_t) \le \dot{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{q}\frac{1}{4}\tau_{i-1}^4 R_{2i} + \sum_{i=1}^{q}\frac{1}{4}\left(\tau_i^2-\tau_{i-1}^2\right)^2 R_{3i} + \frac{\rho^4}{4}R_4\right)\dot{x}(t) + \xi^{\mathrm{T}}(t)\Xi_4\xi(t). \tag{3.18}$$

According to the system in (2.1), with $\widehat{N}_1$, $\widehat{N}_2$ and $\widehat{N}_3$ are defined as $\widehat{N}_1 = \sum_{l=1}^{r} h_l(t)\widehat{N}_{1l}$, $\widehat{N}_2 = \sum_{l=1}^{r} h_l(t)\widehat{N}_{2l}$, and $\widehat{N}_3 = \sum_{l=1}^{r} h_l(t)\widehat{N}_{3l}$; $\widehat{N}_{1l}$, $\widehat{N}_{2l}$, $\widehat{N}_{3l}$ are constant matrices. Then it is given as

$$\begin{aligned} 0 &= 2\big[\dot{x}^{\mathrm{T}}(t)\widehat{N}_1 + x^{\mathrm{T}}(t)\widehat{N}_2 + f^{\mathrm{T}}\big(x(t),t\big)\widehat{N}_3\big] \\ &\quad \times \bigg[A_l x(t) + B_l x\big(t-\tau(t)\big) + C_l f\big(x(t),t\big) \\ &\quad + D_l f\big(x\big(t-\tau(t)\big),t\big) + E_l \int_{t-d}^{t} x(s)\,\mathrm{d}s - \dot{x}(t)\bigg] \\ &= \xi^{\mathrm{T}}(t)\Xi_{l,5}\xi(t). \end{aligned} \tag{3.19}$$

Referring to (2.4), for any scalars $\lambda \geq 0$, the nonlinear perturbations can be derived as

$$0 \leq \lambda\big(x^{\mathrm{T}}(t)F^{\mathrm{T}}Fx(t) - f^{\mathrm{T}}f\big) = \xi^{\mathrm{T}}(t)\Xi_6\xi(t). \tag{3.20}$$

Hence, the following inequality holds:

$$\dot{V}(x_t,k)|_{\tau(t)\in I_k} \leq \sum_{l=1}^{r} h_l(t)\xi^{\mathrm{T}}(t)\Xi_{k,l}\xi(t). \tag{3.21}$$

Using the augmented vector (3.3) with the simplification equation (3.2), the T-S fuzzy system (2.2) is represented as

$$0 = \sum_{l=1}^{r} h_l(t)\Gamma_l\xi(t), \tag{3.22}$$

where Γ_l is defined in Theorem 1.

Therefore, the asymptotic stability condition for the T-S fuzzy system (2.2) with mixed delays and nonlinear perturbations is expressed as

$$\begin{aligned} &\sum_{l=1}^{r} h_l(t)\xi^{\mathrm{T}}(t)\Xi_{k,l}\xi(t) < 0 \\ &\quad \text{subject to: } 0 = \sum_{i=1}^{r} h_l(t)\Gamma_l\xi(t). \end{aligned} \tag{3.23}$$

So, in terms of Lemma 4, there exists a matrix $\mathcal{Y}$ with appropriate dimensions such that (3.23) is equivalent to

$$\sum_{l=1}^{r} h_l(t)\xi^{\mathrm{T}}(t)\big[\Xi_{k,l} + \mathrm{He}(\mathcal{Y}\Gamma_l)\big]\xi(t) < 0. \tag{3.24}$$

As a result, the derivative of the proposed Lyapunov functionals is derived as $\dot{V}(x_t,k)|_{\tau(t)\in I_k} < 0$. It means $\dot{V}(x_t,k)|_{\tau(t)\in I_k} < \gamma\|x(t)\|^2$ for sufficiently small $\gamma > 0$. Therefore, the T-S fuzzy system in (2.2) is globally asymptotically stable. This completes the proof. □

Remark 1 For the absence of a perturbation, that is, $C(t) = 0$, $D(t) = 0$, and $E(t) = 0$, the T-S fuzzy system (2.2) is simplified as

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)x\bigl(t-\tau(t)\bigr), \quad t \geq 0, \\ &x(t) = \varphi(t), \quad t \in [-\tau_q, 0]. \end{aligned} \tag{3.25}$$

This system has been widely researched [15, 28, 29]. The stability conditions for the system are presented below.

Theorem 2 *Given a positive integer q, and $\delta_i = \alpha^{q-i+1}$. The system* (3.25) *is asymptotically stable if there exist symmetric positive definite matrices $Z_i, Q_i, R_{2i}, R_{3i}, \widetilde{Q} \in \mathbb{R}^{n\times n}$ $(i = 1, 2 \ldots, q)$, $\mathcal{P} = [P_{ij}]_{(q+1)\times(q+1)} \in \mathbb{R}^{(q+1)n\times(q+1)n}$, symmetric matrices $W_1, W_3 \in \mathbb{R}^{3n\times 3n}$, and $J \in \mathbb{R}^{n\times n}$, matrices $W_2 \in \mathbb{R}^{3n\times 3n}$, $N_1, N_2 \in \mathbb{R}^{3n\times n}$, $\widehat{N}_1, \widehat{N}_2 \in \mathbb{R}^{n\times n}$ and $\mathcal{Y} \in \mathbb{R}^{(3q+4)n\times n}$, such that the following LMIs hold*:

$$\begin{bmatrix} W_1 & W_2 & N_1 \\ * & W_3 & N_2 \\ * & * & Z_i \end{bmatrix} \geq 0, \tag{3.26}$$

$$\widetilde{\Xi}_{k,l} + \mathrm{He}(\mathcal{Y}\Gamma_l) < 0, \quad l = 1, 2, \ldots, r, k = 1, 2, \ldots, q, \tag{3.27}$$

where $\Gamma_l = A_l e_2^{\mathrm{T}} + B_l e_{q+4}^{\mathrm{T}} - e_1^{\mathrm{T}}$.

By ignoring all the distributed delay and nonlinear perturbation related elements in Theorem 1, *we find $\widetilde{\Xi}_{k,l} = \Xi_1 + \widetilde{\Xi}_2 + \widetilde{\Xi}_{3k} + \widetilde{\Xi}_4 + \widetilde{\Xi}_{l,5} + e_1\widetilde{\mathcal{Z}}e_1^{\mathrm{T}}$, where $\widetilde{\Xi}_2$, $\widetilde{\Xi}_{3k}$, $\widetilde{\Xi}_4$, $\widetilde{\Xi}_{l,5}$, and $\widetilde{\mathcal{Z}}$ are amended by removing all the distributed delay related elements based on the definition in Theorem* 1. *Also $\widetilde{\Xi}_{l,5}$ are deduced by removing the perturbed elements $C_l f(x(t), t)$ and $D_l f(x(t-\tau(t)), t)$ in* (3.19).

Proof The Lyapunov-Krasovskii functional (3.6) is modified for system (3.25) for stability analysis. The augment vector (3.3) is modified as

$$\widetilde{\xi}(t) = \bigl[\dot{x}^{\mathrm{T}}(t), x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau_0), \sigma^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau(t)), \sigma_1^{\mathrm{T}}(t), \sigma_2^{\mathrm{T}}(t)\bigr]^{\mathrm{T}}, \tag{3.28}$$

where $\sigma(t)$, $\sigma_1(t)$ and $\sigma_2(t)$ are defined in Theorem 1. Then following similar procedures to the proof of Theorem 1, the asymptotic stability condition for the T-S system (3.25) is equivalent to

$$\sum_{l=1}^{r} h_l(t)\widetilde{\xi}^{\mathrm{T}}(t)\bigl[\widetilde{\Xi}_k + \mathrm{He}(\mathcal{Y}\Gamma_l)\bigr]\widetilde{\xi}(t) < 0. \tag{3.29}$$

This completes the proof. □

Remark 2 Consider a fixed value of the common ratio α, *i.e.*, the lengths of the subintervals are equal to each other. The existing research results using the equivalent partition method [15, 30, 31] can be considered as a special case of this proposed approach. So, the developed partitioning method is more general.

4 Numerical example

Numerical examples are conducted in this section, to investigate the stability of the T-S fuzzy system by considering the mixed delays and nonlinear perturbations.

Example 4.1 Consider the nominal T-S fuzzy systems (3.25) with the fuzzy rules described in [4, 5, 32] as follows:

$$\begin{aligned}
&\text{Rule 1:}\quad \text{If } z_1(t) \text{ is } \pm\pi/2, \text{then } \dot{x}(t) = A_1x(t) + B_1x\bigl(t-\tau(t)\bigr),\\
&\text{Rule 2:}\quad \text{If } z_2(t) \text{ is } \pm 0, \text{then } \dot{x}(t) = A_2x(t) + B_2x\bigl(t-\tau(t)\bigr),
\end{aligned} \tag{4.1}$$

where the parameters widely discussed are given by

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -1.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}.$$

In Rules 1 and 2, the membership function are $h_1(z(t)) = \frac{1}{1+\exp(-2z(t))}$, $h_2(z(t)) = 1 - h_1(z(t))$.

Considering the lower bound $\tau_0 = 0$, different values of the delay derivative rate μ are selected to compare the upper bound of h_N with some previous results in Table 1.

In Table 1, considering different values of μ, the comparisons of the maximum upper bounds τ_q are given for $\tau_0 = 0$. Remarkable improvements of this proposed partitioning method have been illustrated.

By means of the simulation results in Table 1, selecting $\mu = 0$, $\tau_q = 5.75$ and $\mu = 0.5$, $\tau_q = 4.17$ the state response of the T-S fuzzy system (4.1) is shown in Figure 2.

Figure 2 displays the state response performance under the maximum tolerant delay τ_q shown in Table 1; the nominal T-S fuzzy system (4.1) is asymptotically stable.

Regarding the results of Example 4.1, by comparing with recent results in [5, 33, 34], one illustrates that the derived stability condition can increase the upper bound of the interval time-varying delay in T-S fuzzy system as given in Table 1. Figure 2 is presented to show that the system state response performs well based on the obtained results.

Example 4.2 Consider the T-S fuzzy systems (2.2) with mixed delays in the presence of nonlinear perturbations with the fuzzy rules as follows:

$$\begin{aligned}
&\text{Rule 1:}\quad \text{If } z_1(t) \text{ is } \pm\pi/2,\\
&\qquad\qquad \text{then } \dot{x}(t) = A_1x(t) + B_1x\bigl(t-\tau(t)\bigr) + C_1f\bigl(x(t),t\bigr)\\
&\qquad\qquad\qquad + D_1f\bigl(x\bigl(t-\tau(t)\bigr),t\bigr) + E_1\int_{t-d}^{t} x(s)\,\mathrm{d}s,
\end{aligned} \tag{4.2}$$

Table 1 Upper bounds of τ_q for $\tau_0 = 0$ and different values of μ

Methods	$\mu = 0$	$\mu = 0.1$	$\mu = 0.5$
Liu *et al.* [33]	3.30	2.65	1.50
Zeng *et al.* [5] ($q = 3$)	4.37	3.41	1.95
Lian *et al.* [34]	4.35	3.55	2.32
Theorem 2 ($q = 4$)	5.75	5.11	4.17

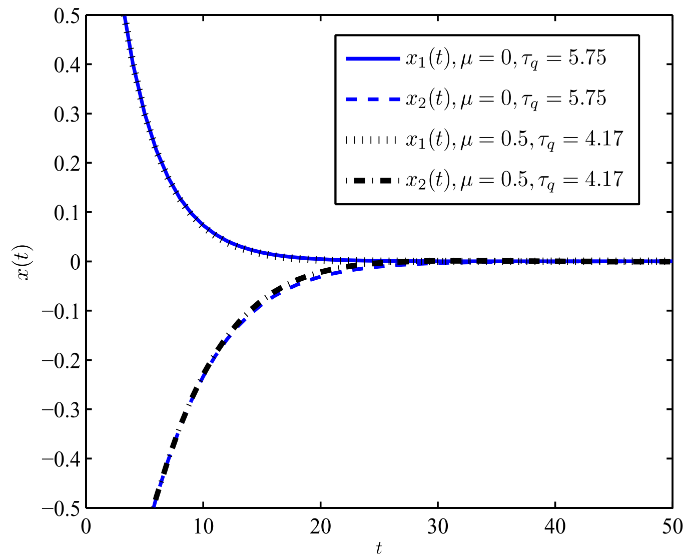

Figure 2 The state response of system (4.1).

Rule 2: If $z_2(t)$ is ± 0,

$$\text{then } \dot{x}(t) = A_2x(t) + B_2x\bigl(t-\tau(t)\bigr) + C_2f\bigl(x(t),t\bigr) + D_2f\bigl(x\bigl(t-\tau(t)\bigr),t\bigr) + E_2\int_{t-d}^{t} x(s)\,\mathrm{d}s.$$

Referring to Assumption 1, the system parameters are given by

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -1.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix},$$

$$C_1 = C_2 = D_1 = D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0.5 & -0.3 \\ 0.2 & 1.2 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.1 & 0.4 \\ -0.3 & -0.1 \end{bmatrix}, \qquad F_1 = F_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

In Rules 1 and 2, the membership function are $h_1(z(t)) = \frac{1}{1+\exp(-2z(t))}$, $h_2(z(t)) = 1 - h_1(z(t))$.

For the delay derivative $\mu = 0.1$, by considering the integral partitioning number $L = 3$ and the GSD partitioning number $q = 3$ and $q = 4$, respectively, the lower bounds are selected as $\tau_0 = 0$ and $\tau_0 = 0.8$ with the distributed delay $d = 0.1$ and $d = 0.5$ to find the upper bound of τ_q in Table 2.

Considering nonlinear perturbations with different values of d and τ_0 and $\mu = 0.1$, the upper bound of delays are conducted. It is shown that the proposed method works well in the perturbed T-S fuzzy system (2.2). For $\mu = 0.1$, selecting $\tau_0 = 0.8$, $\tau_q = 1.68$, $d = 0.5$, the state responses of the T-S system (4.2) are presented in Figure 3.

Table 2 Upper bounds of τ_q for $\mu = 0.1$ with different values of d and τ_0

$\tau_0 \setminus d$	$d = 0.1$ ($q = 3$)	$d = 0.5$ ($q = 3$)	$d = 0.1$ ($q = 4$)	$d = 0.5$ ($q = 4$)
$\tau_0 = 0$	3.57	1.29	4.56	2.07
$\tau_0 = 0.8$	4.17	1.46	5.13	1.68

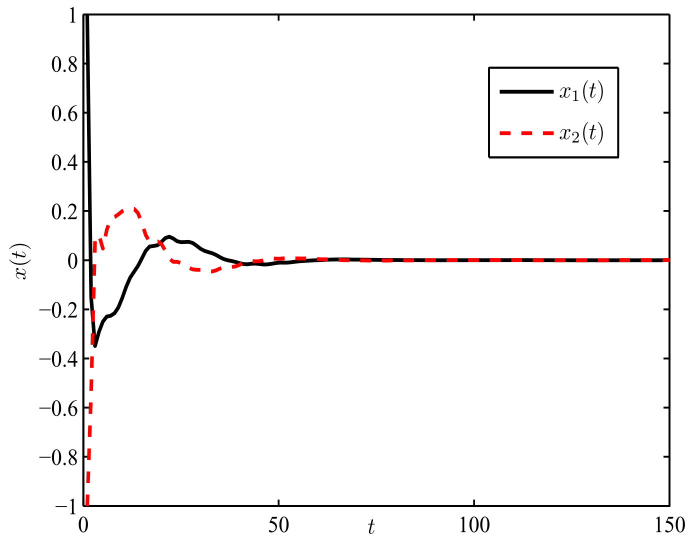

Figure 3 The state response of system (4.1) with $\mu = 0.1$, $\tau_0 = 0.8$, $\tau_q = 1.68$, $d = 0.5$.

Table 3 Upper bounds of τ_q for different values of τ_0, d, and μ

Methods $L = 3$, $q = 3$	$\tau_0 = 0$, $d = 0.1$	$\tau_0 = 0$, $d = 0.5$	$\tau_0 = 0.8$, $d = 0.1$	$\tau_0 = 0.8$, $d = 0.5$
$\mu = 0$	3.77	3.00	4.37	1.56
$\mu = 0.1$	3.57	1.29	4.17	1.46

For different values of τ_0 and d, and $L = 3$, $q = 3$, the upper bound of τ_q of this proposed work is evaluated for $\mu = 0$ and $\mu = 0.1$ in Table 3.

Table 3 shows the upper bound of the time-varying discrete delay with different values of μ. From the simulation results in Table 3, for $\mu = 0$, $\tau_0 = 0.8$, $\tau_q = 1.56$, $d = 0.5$ the state responses are given in Figure 4.

Remark 3 Selecting different values of τ_0 and μ, simulations are given to show the remarkable improvements of the proposed method. Based on the numerical results, it is noticed that the upper bound of the time-varying delay is reduced when the derivative μ increasing. Conservatism can be dramatically reduced for the nominal T-S fuzzy system by comparing with the results in [5, 33, 34].

Remark 4 Based on the geometric progression method, excellent stability criteria are presented in the T-S fuzzy systems with nonlinear perturbations by splitting the delay interval into unequal subintervals. Table 2 shows less conservativeness can be provided by increasing the partitioning number from $q = 3$ to $q = 4$. However, a system with a high number of dimensions will increase the computation burden so one can hardly find the feasible solutions. Commonly, systems with a lower number of dimensions are employed for stability

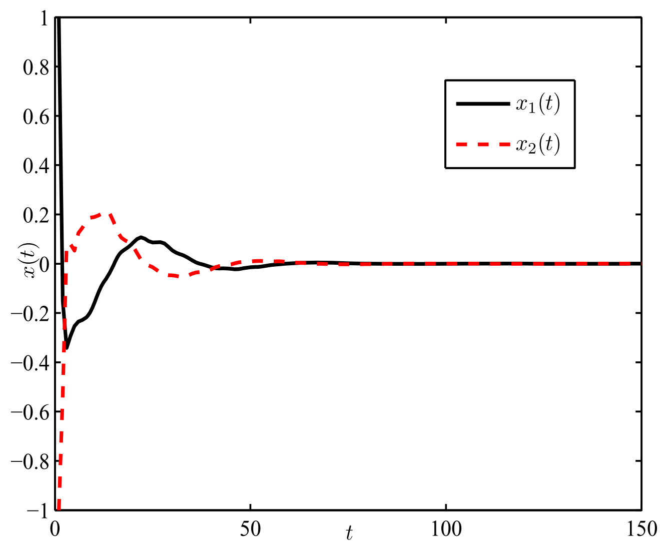

Figure 4 The state response of system (4.1) with $\mu = 0$, $\tau_0 = 0.8$, $\tau_q = 1.56$, $d = 0.5$.

analysis. In addition, the cost time becomes longer as soon as the delay partitioning number is increased. We will focus on reducing the number of dimensions of the researched system and find a better solution for compromising the computation cost in the future.

5 Conclusions

In this paper, by introducing the geometric sequence division and integral delay partitioning approaches, stability problems of the perturbed T-S fuzzy systems with mixed delays are investigated. Comparing with some existing work, in a nominal T-S fuzzy system, the maximum upper bound τ_q is conducted to show the improvements of the proposed method with less conservative results. Numerical results are obtained in LMI toolbox of Matlab/Simulink to demonstrate that remarkable stability criteria are provided in the case of T-S fuzzy systems with mixed delays and nonlinear perturbations. Recently, the control design of T-S systems has attracted special attention. Future work will focus on the implementation of H_∞ control with stochastic disturbances and uncertainties.

Competing interests

The author declares that they have no competing interests.

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