Zhao and Wang *Advances in Difference Equations* (2016) 2016:284
DOI 10.1186/s13662-016-1012-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions for the delayed nonlinear fractional functional differential equations with three-point integral boundary value conditions

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Abstract

This paper is concerned with the three-point integral boundary value problems of time-delay nonlinear fractional functional differential equations involving Caputo fractional derivatives of order $\alpha \in (2,3)$. By employing the Schauder fixed point theorem, the Banach contraction principle, and a nonlinear alternative of Leray-Schauder type, some sufficient criteria are established to guarantee the existence of solutions. Our study improves and extends the previous results in the literature. As applications, some examples are provided to illustrate our main results.

MSC: 34B10; 34B15; 34B27

Keywords: fractional functional differential equations; three-point integral boundary value problems; fixed point theorem; time delays

1 Introduction

This paper is considered with the existence and uniqueness of solutions to the integral boundary value problems (short for BVP) for the nonlinear fractional differential equations (1.1)-(1.3):

$$ {}^cD^{\alpha}_{0+}u(t) + f\bigl(t, u_t, u'(t), {}^cD^{\beta}_{0+}u(t)\bigr) = 0, \quad t \in J = (0,1], \tag{1.1}$$

subject to time-delay conditions

$$u(s) = \varphi(s) \in C\bigl([-r,0]\bigr), \quad s \in [-r,0], \tag{1.2}$$

and the integral boundary conditions

$$\begin{cases} u(0) - \gamma_1 u(\eta) = \delta_1 \int_0^{\eta} u(s)\,ds, \\ u(1) - \gamma_2 u(\eta) = \delta_2 \int_0^{\eta} u(s)\,ds, \qquad u''(0) = 0, \end{cases} \tag{1.3}$$

where $0 < \eta < 1$ and γ_1, γ_2, δ_1, δ_2 are nonnegative constants, ${}^cD^{\alpha}_{0+}$, ${}^cD^{\beta}_{0+}$ are Caputo fractional derivatives of order $2 < \alpha < 3$, $0 < \beta < 1$, $f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Here $u_t(\cdot)$ represents the properitoneal state from time $-r$ up to time t, which is defined by $u_t \triangleq u_t(\theta) = u(t+\theta)$, $-r \le \theta \le 0$.

Fractional differential equations have played a significant role in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. As a consequence, the subject of fractional differential equations is gaining much importance and attention. Especially, the boundary value problems of fractional differential equations have been one of the aspects drawing closest attention. There have been many papers focused on boundary value problems of fractional differential equations (see [1–27]). Recently, the integral boundary value problems of fractional-order differential equation arise in a variety of different areas of applied mathematics and physics such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so on. Therefore, some scholars begin with studying these problems (see [7, 8, 12, 16, 17, 19, 21–23]).

In the real world, the time-delay phenomenon exists commonly and is inevitable. Many changes and processes not only depend on the present status but also on the past status. Therefore, it is necessary to consider the time-delay effect in the mathematical modeling of fractional differential equations. However, there are relatively scarce results dealing with the boundary value problems of fractional functional differential equations with time delays. The aim of this paper is to study the existence of solutions for triple-point boundary value problems of fractional functional differential equations with time delays and integer boundary value conditions.

In addition, the inspiration of this paper comes from the following three systems (see [2, 6, 14]). In [6], Cabada *et al.* investigated the existence of positive solutions of the following nonlinear fractional differential equations with integral boundary-value conditions:

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \lambda\int_0^1 u(s)\,ds, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

In [14], Rehman *et al.* studied the existence and uniqueness of solutions to nonlinear three-point boundary value problems for the following fractional differential equation:

$$\begin{cases} {}^cD_{0+}^{\delta}u(t) = f(t,u(t),{}^cD_{0+}^{\sigma}u(t)), & 0 \le t \le 1, \\ u(0) = \alpha u(\eta), & u(1) = \beta u(\eta), \end{cases}$$

where $1 < \delta < 2$, $0 < \sigma < 1$, $\alpha, \beta \in \mathbb{R}$, $\alpha\eta(1-\beta) + (1-\alpha)(1-\beta\eta) \neq 0$ and ${}^cD_{0^+}^{\delta}$, ${}^cD_{0^+}^{\sigma}$ denote Caputo fractional derivatives. By the Banach contraction principle and the Schauder fixed point theorem, they obtained some new results as regards existence and uniqueness.

By using standard fixed point theorems and Leray-Schauder degree theory, Ahmad *et al.* [2] investigated the existence and uniqueness of solutions of boundary value problem for the following nonlinear fractional differential equations:

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t,x(t)), & 0 < t < 1, 1 < q \le 2, \\ x(0) = 0, \quad x(1) = \alpha \int_0^\eta x(s)\,ds, & 0 < \eta < 1, \end{cases}$$

where ${}^cD^q_{0+}$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times X \to X$ is continuous, and $\alpha \in \mathbb{R}$ is such that $\alpha \neq \frac{2}{\eta^2}$. Here, $(X, \|\cdot\|)$ is a Banach space and $C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

To the best of our knowledge, it seems that no one considered BVP (1.1)-(1.3). Therefore, we will investigate the existence and uniqueness of solutions of the nonlinear BVP (1.1)-(1.3) under some further conditions. We consider the effect of time delays, but the authors do not consider it in the literature [2, 6, 14]. Taking $f(t, u_t, u'(t), {}^cD^\beta_{0+}u(t)) = f(t, u(t))$, $\gamma_1 = \gamma_2 = \delta_1 = 0$, $\delta_2 = \lambda$, BVP (1.1)-(1.3) is changed into the boundary value problem of literature [6]. Let $f(t, u_t, u'(t), {}^cD^\beta_{0+}u(t)) = f(t, u, {}^cD^\sigma_{0+}u(t))$, $\delta_1 = \delta_2 = 0$, $\gamma_1 = \alpha$, $\gamma_2 = \beta$, BVP (1.1)-(1.3) is changed into the boundary value problem of literature [17]. Taking $f(t, u_t, u'(t), {}^cD^\beta_{0+}u(t)) = f(t, u(t))$, $\gamma_1 = \gamma_2 = \delta_1 = 0$, $\delta_2 = \alpha$, BVP (1.1)-(1.3) is changed into the boundary value problem of literature [2]. Therefore, our study improves and extends the previous results in the relevant literature [2, 6, 14].

The rest of this paper is organized as follows. In Section 2, we recall some useful definitions and properties, and present the properties of the Green's function. In Section 3, we give some sufficient conditions for the existence and uniqueness of solutions for boundary value problem (1.1)-(1.3). Some examples are also provided to illustrate our main results in Section 4.

2 Preliminaries

For convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent literature.

Definition 2.1 (see [28, 29]) The Riemann-Liouville fractional integral of order $q > 0$ of a function $f \in L^1[0,\infty)$ is given by

$$I^q_{0+}f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [28, 29]) The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in N$, is defined as

$$D^q_{0+}f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\,ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 (see [28, 29]) The Caputo derivative of order q for a function $f : [0,\infty)\to\mathbb{R}$ can be written as

$$ {}^{c}D_{0+}^{q}f(t) = D_{0+}^{q}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t > 0, n-1 < q < n. $$

Remark 2.1 If $f(t) \in C^n[0,\infty)$, then

$$ {}^{c}D_{0+}^{q}f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\,ds = I_{0+}^{n-q}f^{(n)}(t), \quad t > 0, n-1 < q < n. $$

Lemma 2.1 (see [28]) *Assume that $u \in C[0,\infty)$ with a Caputo fractional derivative of order $q > 0$ that belongs to $u \in C^n[0,\infty)$, then*

$$ I_{0+}^{q}D_{0+}^{q}u(t) = u(t) + c_1 + c_2t + \cdots + c_nt^{n-1}, $$

for some $c_i \in \mathbb{R}$, $i = 1,2,\dots,n$, where n is the smallest integer greater than or equal to q.

Here we introduce the following useful fixed-point theorems.

Lemma 2.2 (see [30]) *Let X be a Banach space with $K \subseteq X$ closed and convex. Assume Ω is a relatively open subset of K with $0 \in \Omega$ and $T : \overline{\Omega}\to K$ is a completely continuous operator. Then either*

(a) *T has a fixed point in $\overline{\Omega}$; or*
(b) *there exist $u \in \partial\Omega$ and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Lemma 2.3 (Schauder fixed point theorem (see [31])) *Let X be a Banach space and Ω be a closed convex subset of X. If the operator $T : \overline{\Omega}\to\overline{\Omega}$ is completely continuous, then the operator T has at least one fixed point $u^* \in \overline{\Omega}$.*

Throughout this paper, we denote $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{R}_0^+ = (0,+\infty)$. For simplicity, we introduce some notations as follows:

$$ \begin{aligned} &P_1 = 1-\gamma_1-\delta_1\eta, \qquad P_2 = 1-\gamma_2-\delta_2\eta, \qquad Q_1 = \gamma_1\eta + \frac{\delta_1\eta^2}{2}, \qquad Q_2 = 1-\gamma_2\eta - \frac{\delta_2\eta^2}{2}, \\ &D_1 = \frac{P_1}{P_1Q_2+P_2Q_1}, \qquad D_2 = \frac{P_2}{P_1Q_2+P_2Q_1}, \\ &K_1 = \frac{Q_1}{P_1Q_2+P_2Q_1}, \qquad K_2 = \frac{Q_2}{P_1Q_2+P_2Q_1}, \\ &m_1 = \left(1+\frac{\delta_2\eta^{\alpha+1}}{\alpha+1}+\gamma_2\eta^\alpha\right)\bigl(|D_1|+|K_1|\bigr) + \left(\frac{\delta_1\eta^{\alpha+1}}{\alpha+1}+\gamma_1\eta^\alpha\right)\bigl(|D_2|+|K_2|\bigr)+1, \\ &m_2 = \left(1+\frac{\delta_2\eta^{\alpha+1}}{\alpha+1}+\gamma_2\eta^\alpha\right)|D_1| + \left(\frac{\delta_1\eta^{\alpha+1}}{\alpha+1}+\gamma_1\eta^\alpha\right)|D_2|+\alpha, \\ &Q = \frac{1}{\Gamma(\alpha+1)}\left(m_1+m_2+\frac{m_2}{\Gamma(2-\beta)}\right). \end{aligned} $$

Now we present the Green's function for the system associated with BVP (1.1)-(1.3).

Lemma 2.4 *If $h \in C[0,1]$ and $P_1Q_2 + P_2Q_1 \neq 0$, then the unique solution of* (2.1)

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) + h(t) = 0, \quad 2 < \alpha < 3, t \in J = (0,1], \\ u(0) - \gamma_1 u(\eta) = \delta_1 \int_0^{\eta} u(s)\,ds, \\ u(1) - \gamma_2 u(\eta) = \delta_2 \int_0^{\eta} u(s)\,ds, \qquad u''(0) = 0 \end{cases} \tag{2.1}$$

is formulated by

$$u(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \begin{cases} g_1(t,s), & 0 \le s \le t \le \eta \le 1 \text{ or } 0 \le s \le \eta \le t \le 1, \\ g_2(t,s), & 0 \le \eta \le s \le t \le 1, \\ g_3(t,s), & 0 \le t \le s \le \eta \le 1, \\ g_4(t,s) & 0 \le t \le \eta \le s \le 1 \text{ or } 0 \le \eta \le t \le s \le 1, \end{cases}$$

$$\begin{aligned} g_1(t,s) &= \frac{\rho_1(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1} - \rho_2(t)\frac{(\eta-s)^{\alpha}}{\alpha} - \rho_3(t)(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ g_2(t,s) &= \frac{\rho_1(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ g_3(t,s) &= \frac{\rho_1(t)(1-s)^{\alpha-1} - \rho_2(t)\frac{(\eta-s)^{\alpha}}{\alpha} - \rho_3(t)(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ g_4(t,s) &= \frac{\rho_1(t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \end{aligned} \tag{2.2}$$

$\rho_1(t) = D_1 t + K_1$, $\rho_2(t) = (\delta_2 D_1 - \delta_1 D_2)t + \delta_2 K_1 + \delta_1 K_2$, $\rho_3(t) = (\gamma_2 D_1 - \gamma_1 D_2)t + \gamma_2 K_1 + \gamma_1 K_2$.

Proof Applying Lemma 2.1, equation (2.1) is changed into an equivalent integral equation,

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha}h(t) + C_1 + C_2 t + C_3 t^2 \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + C_1 + C_2 t + C_3 t^2. \end{aligned} \tag{2.3}$$

From $u''(0) = 0$, we derive $C_3 = 0$. According to $u(0) - \gamma_1 u(\eta) = \delta_1 \int_0^{\eta} u(s)\,ds$, we obtain

$$C_1 - \gamma_1\bigl(-I_{0+}^{\alpha}h(\eta) + C_1 + C_2\eta\bigr) = \delta_1 \int_0^{\eta} u(s)\,ds. \tag{2.4}$$

By $u(1) - \gamma_2 u(\eta) = \delta_2 \int_0^{\eta} u(s)\,ds$, we have

$$-I_{0+}^{\alpha}h(1) + C_1 + C_2 - \gamma_2\bigl(-I_{0+}^{\alpha}h(\eta) + C_1 + C_2\eta\bigr) = \delta_2 \int_0^{\eta} u(s)\,ds. \tag{2.5}$$

By (2.3), we get

$$\begin{aligned}\int_0^\eta u(s)\,ds &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\left[\int_0^s (s-\tau)^{\alpha-1}h(\tau)\,d\tau\right]ds + C_1\int_0^\eta ds + C_2\int_0^\eta s\,ds\\ &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\left[\int_\tau^\eta (s-\tau)^{\alpha-1}\,ds\right]h(\tau)\,d\tau + C_1\eta + \frac{C_2\eta^2}{2}\\ &= -\frac{1}{\alpha\Gamma(\alpha)}\int_0^\eta (\eta-\tau)^\alpha h(\tau)\,d\tau + C_1\eta + \frac{C_2\eta^2}{2},\end{aligned}$$

which yields

$$\int_0^\eta u(s)\,ds = -I_{0+}^{\alpha+1}h(\eta) + C_1\eta + \frac{C_2\eta^2}{2}. \tag{2.6}$$

Substituting (2.6) into (2.4) and (2.5), we obtain

$$P_1C_1 - Q_1C_2 = -\delta_1 I_{0+}^{\alpha+1}h(\eta) - \gamma_1 I_{0+}^\alpha h(\eta) \triangleq A, \tag{2.7}$$

$$P_2C_1 + Q_2C_2 = I_{0+}^\alpha h(1) - \delta_2 I_{0+}^{\alpha+1}h(\eta) - \gamma_2 I_{0+}^\alpha h(\eta) \triangleq B. \tag{2.8}$$

By (2.7) and (2.8), we get $C_1 = \frac{AQ_2 + BQ_1}{P_1Q_2 + P_2Q_1}$, $C_2 = \frac{P_1C_1 - A}{Q_1}$. Therefore, the solution of BVP (2.1) is

$$\begin{aligned}u(t) &= -I_{0+}^\alpha h(t) + C_1 + \frac{P_1C_1 - A}{Q_1}t = -I_{0+}^\alpha h(t) + \left(1 + \frac{P_1 t}{Q_1}\right)C_1 - \frac{A}{Q_1}t\\ &= -I_{0+}^\alpha h(t) + \frac{Q_1 + P_1 t}{Q_1}\frac{AQ_2 + BQ_1}{P_1Q_2 + P_2Q_1} - \frac{A}{Q_1}t\\ &= -I_{0+}^\alpha h(t) + \frac{Q_1 + P_1 t}{Q_1}\frac{BQ_1}{P_1Q_2 + P_2Q_1} + \frac{Q_1 + P_1 t}{Q_1}\frac{AQ_2}{P_1Q_2 + P_2Q_1} - \frac{A}{Q_1}t\\ &= -I_{0+}^\alpha h(t) + \frac{Q_1 + P_1 t}{P_1Q_2 + P_2Q_1}B + \left[\frac{(Q_1 + P_1 t)Q_2}{P_1Q_2 + P_2Q_1} - t\right]\frac{A}{Q_1}\\ &= -I_{0+}^\alpha h(t) + (K_1 + D_1 t)B + (K_2 - D_2 t)A\\ &= -I_{0+}^\alpha h(t) + (K_1 + D_1 t)\left[I_{0+}^\alpha h(1) - \delta_2 I_{0+}^{\alpha+1}h(\eta) - \gamma_2 I_{0+}^\alpha h(\eta)\right]\\ &\quad + (K_2 - D_2 t)\left[-\delta_1 I_{0+}^{\alpha+1}h(\eta) - \gamma_1 I_{0+}^\alpha h(\eta)\right]\\ &= -I_{0+}^\alpha h(t) + (K_1 + D_1 t)I_{0+}^\alpha h(1) - \left[(K_1 + D_1 t)\delta_2 + (K_2 - D_2 t)\delta_1\right]I_{0+}^{\alpha+1}h(\eta)\\ &\quad - \left[(K_1 + D_1 t)\gamma_2 + (K_2 - D_2 t)\gamma_1\right]I_{0+}^\alpha h(\eta)\\ &= -I_{0+}^\alpha h(t) + \rho_1(t)I_{0+}^\alpha h(1) - \rho_2(t)I_{0+}^{\alpha+1}h(\eta) - \rho_3(t)I_{0+}^\alpha h(\eta)\\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{\rho_1(t)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\\ &\quad - \frac{\rho_2(t)}{\Gamma(\alpha+1)}\int_0^\eta (\eta-s)^\alpha h(s)\,ds - \frac{\rho_3(t)}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}h(s)\,ds.\end{aligned}$$

When $t \leq \eta$, we have

$$\begin{aligned}
u(t) &= \frac{\rho_1(t)}{\Gamma(\alpha)}\left(\int_0^t + \int_t^\eta + \int_\eta^1\right)(1-s)^{\alpha-1}h(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds \\
&\quad - \left(\int_0^t + \int_t^\eta\right)\left(\frac{\rho_2(t)(\eta-s)^\alpha}{\Gamma(\alpha+1)} + \frac{\rho_3(t)(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\right)h(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \left(\rho_1(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1} - \rho_2(t)\frac{(\eta-s)^\alpha}{\alpha} - \rho_3(t)(\eta-s)^{\alpha-1}\right)h(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_t^\eta \left(\rho_1(t)(1-s)^{\alpha-1} - \rho_2(t)\frac{(\eta-s)^\alpha}{\alpha} - \rho_3(t)(\eta-s)^{\alpha-1}\right)h(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\eta^1 \rho_1(t)(1-s)^{\alpha-1}h(s)\,ds \\
&= \int_0^1 G(t,s)h(s)\,ds.
\end{aligned}$$

When $t \geq \eta$, we have

$$\begin{aligned}
u(t) &= \frac{\rho_1(t)}{\Gamma(\alpha)}\left(\int_0^\eta + \int_\eta^t + \int_t^1\right)(1-s)^{\alpha-1}h(s)\,ds - \frac{1}{\Gamma(\alpha)}\left(\int_0^\eta + \int_\eta^t\right)(t-s)^{\alpha-1}h(s)\,ds \\
&\quad - \int_0^\eta \left(\frac{\rho_2(t)(\eta-s)^\alpha}{\Gamma(\alpha+1)} + \frac{\rho_3(t)(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\right)h(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^\eta \left(\rho_1(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1} - \rho_2(t)\frac{(\eta-s)^\alpha}{\alpha} - \rho_3(t)(\eta-s)^{\alpha-1}\right)h(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_\eta^t \left(\rho_1(t)(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right)h(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_t^1 \rho_1(t)(1-s)^{\alpha-1}h(s)\,ds \\
&= \int_0^1 G(t,s)h(s)\,ds,
\end{aligned}$$

where $G(t,s)$ is defined by (2.2).

Next, we will prove the uniqueness of solution for BVP (2.1). In fact, let $u_1(t)$, $u_2(t)$ be any two solutions of (2.1). Denote $w(t) = u_1(t) - u_2(t)$, then (2.1) is changed into the following system:

$$\begin{cases}
{}^cD_{0+}^\alpha w(t) = 0, \quad 2 < \alpha < 3, t \in J = (0,1], \\
w(0) - \gamma_1 w(\eta) = \delta_1 \int_0^\eta w(s)\,ds, \\
w(1) - \gamma_2 w(\eta) = \delta_2 \int_0^\eta w(s)\,ds, \quad w''(0) = 0.
\end{cases}$$

Similar to the above discussion, we get $w(t) = 0$, namely, $u_1(t) = u_2(t)$, which indicates that the solution for BVP (2.1) is unique. The proof is complete. □

Lemma 2.5 *If $P_1Q_2 + P_2Q_1 \neq 0$, then the Green's function $G(t,s)$ defined by* (2.2) *possesses the following properties*:

(i) $\int_0^1 |G(t,s)|\,ds \leq \frac{m_1}{\Gamma(\alpha+1)}$, *for all $t \in [0,1]$*.

(ii) $\int_0^1 |\frac{\partial}{\partial t}G(t,s)|\,ds \leq \frac{m_2}{\Gamma(\alpha+1)}$, *for all $t \in [0,1]$*.

(iii) $\int_0^1 |\frac{\partial^2}{\partial t^2}G(t,s)|\,ds \leq \frac{\alpha-1}{\Gamma(\alpha)}$, *for all $t \in [0,1]$*.

Proof In fact, according to the expression of $G(t,s)$, we have, for $t,s \in [0,1]$,

$$\begin{aligned}
\int_0^1 \bigl|G(t,s)\bigr|\,ds &\le \frac{|\rho_1(t)|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \\
&\quad + \frac{|\rho_2(t)|}{\Gamma(\alpha+1)}\int_0^\eta (\eta-s)^{\alpha}\,ds + \frac{|\rho_3(t)|}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\,ds \\
&\le \frac{1}{\Gamma(\alpha+1)}\biggl(\bigl|\rho_1(t)\bigr| + t^{\alpha} + \frac{|\rho_2(t)|\eta^{\alpha+1}}{\alpha+1} + \bigl|\rho_3(t)\bigr|\eta^{\alpha}\biggr) \\
&\le \frac{1}{\Gamma(\alpha+1)}\biggl(|D_1| + |K_1| + 1 + \bigl[\bigl(|D_1| + |K_1|\bigr)\delta_2 + \bigl(|D_2| + |K_2|\bigr)\delta_1\bigr]\frac{\eta^{\alpha+1}}{\alpha+1} \\
&\quad + \bigl[\bigl(|D_1| + |K_1|\bigr)\gamma_2 + \bigl(|D_2| + |K_2|\bigr)\gamma_1\bigr]\eta^{\alpha}\biggr) = \frac{m_1}{\Gamma(\alpha+1)}, \\
\int_0^1 \biggl|\frac{\partial}{\partial t}G(t,s)\biggr|\,ds &\le \biggl|\frac{\partial}{\partial t}\rho_1(t)\biggr|\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}\,ds \\
&\quad + \biggl|\frac{\partial}{\partial t}\rho_2(t)\biggr|\frac{1}{\Gamma(\alpha+1)}\int_0^\eta (\eta-s)^{\alpha}\,ds + \biggl|\frac{\partial}{\partial t}\rho_3(t)\biggr|\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\,ds \\
&\le \frac{1}{\Gamma(\alpha+1)}\biggl(\biggl|\frac{\partial}{\partial t}\rho_1(t)\biggr| + \alpha t^{\alpha-1} + \biggl|\frac{\partial}{\partial t}\rho_2(t)\biggr|\frac{\eta^{\alpha+1}}{\alpha+1} + \biggl|\frac{\partial}{\partial t}\rho_3(t)\biggr|\eta^{\alpha}\biggr) \\
&\le \frac{1}{\Gamma(\alpha+1)}\biggl(|D_1| + \alpha + \bigl(|D_1|\delta_2 + |D_2|\delta_1\bigr)\frac{\eta^{\alpha+1}}{\alpha+1} + \bigl(|D_1|\gamma_2 + |D_2|\gamma_1\bigr)\eta^{\alpha}\biggr) \\
&= \frac{m_2}{\Gamma(\alpha+1)}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \biggl|\frac{\partial^2}{\partial t^2}G(t,s)\biggr|\,ds &\le \biggl|\frac{\partial^2}{\partial t^2}\rho_1(t)\biggr|\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds + \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-3}\,ds \\
&\quad + \biggl|\frac{\partial^2}{\partial t^2}\rho_2(t)\biggr|\frac{1}{\Gamma(\alpha+1)}\int_0^\eta (\eta-s)^{\alpha}\,ds \\
&\quad + \biggl|\frac{\partial^2}{\partial t^2}\rho_3(t)\biggr|\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\,ds \\
&\le \frac{(\alpha-1)t^{\alpha-2}}{\Gamma(\alpha)} \le \frac{\alpha-1}{\Gamma(\alpha)}.
\end{aligned}$$

The proof is complete. □

3 Main results

In this section, we will discuss the existence and uniqueness of solutions for BVP (1.1)-(1.3) by the Schauder fixed point theorem, the Leray-Schauder nonlinear alternative, and the Banach contraction principle.

For $r > 0$, C_r represents the Banach space of all continuous functions φ, φ', ${}^cD_{0+}^{\beta}\varphi$: $[-r,0]\to\mathbb{R}$ endowed with the sup-norm $\|\varphi\|_{[-r,0]} = \sup_{-r\le s\le 0}\{|\varphi(s)| + |\varphi'(s)| + |{}^cD_{0+}^{\beta}\varphi(s)|\}$.

Let $X = \{u | u \in C[-r,1], u' \in C[-r,1], {}^cD^{\beta}_{0+}u \in C[-r,1], \beta \in (0,1)\}$ denote a real Banach space with the norm $\|\cdot\|$ defined by

$$\|u\| = \max_{t\in I}|u(t)| + \max_{t\in I}|u'(t)| + \max_{t\in I}|{}^cD^{\beta}_{0+}u(t)|, \tag{3.1}$$

where $u \in C(I)$, $I = [-r,1]$. $C(I)$ and $C^1(I)$ represent the sets of continuous and continuously differentiable functions on I.

From Lemma 2.4, we can obtain the following lemma.

Lemma 3.1 *Suppose that f is continuous, then $u \in X$ is a solution of BVP* (1.1)-(1.3) *if and only if $u \in X$ is a solution of the integral equation*

$$u(t) = \begin{cases} \int_0^1 G(t,s)f(s,u_s,u'(s),{}^cD^{\beta}_{0+}u(s))\,ds, & t \in J, \\ \varphi(t), & t \in [-r,0]. \end{cases}$$

Define $T : X \to X$ as the operator

$$(Tu)(t) = u(t) = \begin{cases} \int_0^1 G(t,s)f(s,u_s,u'(s),{}^cD^{\beta}_{0+}u(s))\,ds, & t \in J, \\ \varphi(t), & t \in [-r,0]. \end{cases} \tag{3.2}$$

By Lemma 3.1, the fixed point of operator T coincides with the solution of BVP (1.1)-(1.3).

Theorem 3.1 *Assume that $P_1Q_2 + P_2Q_1 \neq 0$ and $f : J \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. Suppose further that the conditions* (H_1)-(H_2) *are satisfied*:

(H_1) *There exist the nonnegative functions $a_1, a_2 \in L^1(J)$ and a nonnegative nondecreasing function $\phi(x,y,z)$ with respect to each variable x, y, z, such that*

$$|f(t,x,y,z)| \le a_1(t) + a_2(t)\phi(|x|,|y|,|z|).$$

(H_2) *There exists a constant $R_0 > k_1$ such that $\phi(R_0,R_0,R_0) \le \frac{R_0-k_1}{k_2}$, where $k_i = Q\int_0^1 |a_i(t)|\,dt = Q\|a_i\|$, $i = 1,2$.*

Then BVP (1.1)-(1.3) *has at least one solution.*

Proof Define a closed ball of Banach space X as follows:

$$U = \{u \in X : \|u\| \le R_0\}. \tag{3.3}$$

Now we will show that $T(U) \subset U$. In fact, any $u \in U$, $t \in J$, from Lemma 2.5 and (H_1), we have

$$\begin{aligned} |(Tu)(t)| &\le \int_0^1 |G(t,s)||f(s,u_s,u'(s),{}^cD^{\beta}_{0+}u(s))|\,ds \\ &\le \int_0^1 |G(t,s)||a_1(s) + a_2(s)\phi(R_0,R_0,R_0)|\,ds \\ &\le \frac{m_1(\|a_1\| + \|a_2\|\phi(R_0,R_0,R_0))}{\Gamma(\alpha+1)} \end{aligned}$$

and

$$\begin{aligned}\left|(Tu)'(t)\right| &\le \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| \left|f\left(s,u_s,u'(s),{}^cD_{0+}^{\beta}u(s)\right)\right| ds \\ &\le \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| \left|a_1(s)+a_2(s)\phi(R_0,R_0,R_0)\right| ds \\ &\le \frac{m_2(\|a_1\|+\|a_2\|\phi(R_0,R_0,R_0))}{\Gamma(\alpha+1)}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|{}^cD_{0+}^{\beta}(Tu)(t)\right| &= \left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\,ds\right| \\ &\le \left(\frac{m_2(\|a_1\|+\|a_2\|\phi(R_0,R_0,R_0))}{\Gamma(\alpha+1)}\right)\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\,ds \\ &\le \frac{m_2(\|a_1\|+\|a_2\|\phi(R_0,R_0,R_0))}{\Gamma(2-\beta)\Gamma(\alpha+1)}.\end{aligned}$$

In view of (3.1), (3.3), and (H_2), we have

$$\begin{aligned}\left\|(Tu)(t)\right\| &\le \frac{\|a_1\|+\|a_2\|\phi(R_0,R_0,R_0)}{\Gamma(\alpha+1)}\left(m_1+m_2+\frac{m_2}{\Gamma(2-\beta)}\right) \\ &\le \left(\|a_1\|+\|a_2\|\phi(R_0,R_0,R_0)\right)Q \\ &\le k_1+k_2\phi(R_0,R_0,R_0)\le R_0,\end{aligned}$$

which implies that $T(U)\subset U$. The continuity of the operator T follows from the continuity of f and G.

Next, we shall show that T is a completely continuous operator through the following three cases. Indeed, let $L \triangleq \max_{t\in J}|f(t,u_t,u'(t),{}^cD_{0+}^{\beta}u(t))|+1$, $u\in U$, and t_1, $t_2\in[-r,1]$ with $t_1<t_2$.

Case 1. When $0<t_1<t_2\le 1$, from Lemma 2.5, we have

$$\begin{aligned}\left|(Tu)(t_2)-(Tu)(t_1)\right| &= \left|\int_{t_1}^{t_2}(Tu)'(s)\,ds\right| \le \int_{t_1}^{t_2}\left|(Tu)'(s)\right| ds \\ &\le \int_{t_1}^{t_2}\left(\int_0^1\left|\frac{\partial}{\partial s}G(s,\tau)\right|\left|f\left(\tau,u_\tau,u'(\tau),{}^cD_{0+}^{\beta}u(\tau)\right)\right|d\tau\right)ds \\ &\le \frac{m_2L}{\Gamma(\alpha+1)}(t_2-t_1), \\ \left|(Tu)'(t_2)-(Tu)'(t_1)\right| &= \left|\int_{t_1}^{t_2}(Tu)''(s)\,ds\right| \le \int_{t_1}^{t_2}\left|(Tu)''(s)\right| ds \le \frac{(\alpha-1)L}{\Gamma(\alpha)}(t_2-t_1)\end{aligned}$$

and

$$\begin{aligned}&\left|{}^cD_{0+}^{\beta}(Tu)(t_2)-{}^cD_{0+}^{\beta}(Tu)(t_1)\right| \\ &\quad= \frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_2}(t_2-s)^{-\beta}(Tu)'(s)\,ds-\int_0^{t_1}(t_1-s)^{-\beta}(Tu)'(s)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_2}(t_2-s)^{-\beta}(Tu)'(s)\,ds-\int_0^{t_1}(t_2-s)^{-\beta}(Tu)'(s)\,ds\right| \\
&\quad+\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}(t_2-s)^{-\beta}(Tu)'(s)\,ds-\int_0^{t_1}(t_1-s)^{-\beta}(Tu)'(s)\,ds\right| \\
&\leq \frac{1}{\Gamma(1-\beta)}\left|\int_{t_1}^{t_2}(t_2-s)^{-\beta}\big|(Tu)'(s)\big|\,ds+\int_0^{t_1}\big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\big)\big|(Tu)'(s)\big|\,ds\right| \\
&\leq \frac{1}{\Gamma(1-\beta)}\left|\int_{t_1}^{t_2}(t_2-s)^{-\beta}\left(\int_0^1\left|\frac{\partial}{\partial s}G(s,\tau)\right|\big|f\big(\tau,u_\tau,u'(\tau),{}^cD_{0+}^{\beta}u(\tau)\big)\big|\,d\tau\right)ds\right| \\
&\quad+\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}\big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\big)\right. \\
&\quad\times\left.\left(\int_0^1\left|\frac{\partial}{\partial s}G(s,\tau)\right|\big|f\big(\tau,u_\tau,u'(\tau),{}^cD_{0+}^{\beta}u(\tau)\big)\big|\,d\tau\right)ds\right| \\
&\leq \frac{m_2L}{\Gamma(1-\beta)\Gamma(\alpha+1)}\left[\int_{t_1}^{t_2}(t_2-s)^{-\beta}\,ds+\int_0^{t_1}\big((t_1-s)^{-\beta}-(t_2-s)^{-\beta}\big)\,ds\right] \\
&\leq \frac{m_2L}{\Gamma(2-\beta)\Gamma(\alpha+1)}\big[2(t_2-t_1)^{1-\beta}+{t_1}^{1-\beta}-{t_2}^{1-\beta}\big].
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\big\|(Tu)(t_2)-(Tu)(t_1)\big\| &\leq \frac{m_2L}{\Gamma(\alpha+1)}(t_2-t_1)+\frac{(\alpha-1)L}{\Gamma(\alpha)}(t_2-t_1)+\frac{m_2L}{\Gamma(2-\beta)\Gamma(\alpha+1)} \\
&\quad\times\big[2(t_2-t_1)^{1-\beta}+{t_1}^{1-\beta}-{t_2}^{1-\beta}\big]\to 0\quad\text{as } t_1\to t_2.
\end{aligned}
$$

Case 2. When $-r\leq t_1<0<t_2\leq 1$, $|t_2-t_1|$ is small enough, namely, $|t_2-t_1|\to 0$ as $t_1\to t_2$ means that $t_1\to 0^-$ and $t_2\to 0^+$. Then we obtain

$$
\begin{aligned}
\big|(Tu)(t_2)-(Tu)(t_1)\big| &\leq \big|(Tu)(t_2)-(Tu)(0)\big|+\big|(Tu)(0)-(Tu)(t_1)\big| \\
&\leq \int_0^{t_2}\big|(Tu)'(s)\big|\,ds+\big|\varphi(0)-\varphi(t_1)\big| \\
&\leq \frac{m_2L}{\Gamma(\alpha+1)}t_2+\big|\varphi(0)-\varphi(t_1)\big|, \\
\big|(Tu)'(t_2)-(Tu)'(t_1)\big| &\leq \big|(Tu)'(t_2)-(Tu)'(0)\big|+\big|(Tu)'(0)-(Tu)'(t_1)\big| \\
&\leq \int_0^{t_2}\big|(Tu)''(s)\big|\,ds+\big|\varphi'(0)-\varphi'(t_1)\big| \\
&\leq \frac{(\alpha-1)L}{\Gamma(\alpha)}t_2+\big|\varphi'(0)-\varphi'(t_1)\big|
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|{}^cD_{0+}^{\beta}(Tu)(t_2)-{}^cD_{0+}^{\beta}(Tu)(t_1)\big| \\
&\quad\leq \big|{}^cD_{0+}^{\beta}(Tu)(t_2)-{}^cD_{0+}^{\beta}(Tu)(0)\big|+\big|{}^cD_{0+}^{\beta}(Tu)(0)-{}^cD_{0+}^{\beta}(Tu)(t_1)\big| \\
&\quad\leq \frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_2}(t_2-s)^{-\beta}(Tu)'(s)\,ds-0\right|+\big|{}^cD_{0+}^{\beta}\varphi(0)-{}^cD_{0+}^{\beta}\varphi(t_1)\big|
\end{aligned}
$$

$$\leq \frac{m_2 L}{\Gamma(1-\beta)\Gamma(\alpha+1)} \int_0^{t_2} (t_2 - s)^{-\beta}\, ds + \left| {}^c D_{0+}^{\beta} \varphi(0) - {}^c D_{0+}^{\beta} \varphi(t_1) \right|$$
$$\leq \frac{m_2 L}{\Gamma(2-\beta)\Gamma(\alpha+1)} {t_2}^{1-\beta} + \left| {}^c D_{0+}^{\beta} \varphi(0) - {}^c D_{0+}^{\beta} \varphi(t_1) \right|,$$

which implies that

$$\begin{aligned} &\left\| (Tu)(t_2) - (Tu)(t_1) \right\| \\ &\quad \leq \frac{m_2 L}{\Gamma(\alpha+1)} t_2 + \left| \varphi(0) - \varphi(t_1) \right| + \frac{(\alpha-1)L}{\Gamma(\alpha)} t_2 + \left| \varphi'(0) - \varphi'(t_1) \right| \\ &\qquad + \frac{m_2 L}{\Gamma(2-\beta)\Gamma(\alpha+1)} {t_2}^{1-\beta} + \left| {}^c D_{0+}^{\beta} \varphi(0) - {}^c D_{0+}^{\beta} \varphi(t_1) \right| \to 0 \quad \text{as } t_1 \to t_2. \end{aligned}$$

Case 3. When $-r \leq t_1 < t_2 \leq 0$, we have

$$\begin{aligned} \left\| (Tu)(t_2) - (Tu)(t_1) \right\| &= \left| \varphi(t_2) - \varphi(t_1) \right| + \left| \varphi'(t_2) - \varphi'(t_1) \right| \\ &\quad + \left| {}^c D_{0+}^{\beta} \varphi(t_2) - {}^c D_{0+}^{\beta} \varphi(t_1) \right| \to 0 \quad \text{as } t_1 \to t_2. \end{aligned}$$

Thus, for any $\epsilon > 0$ (small enough), there exists $\sigma = \sigma(\epsilon) > 0$ with independent of t_1, t_2, and u such that $\|(Tu)(t_2) - (Tu)(t_1)\| \leq \epsilon$, whenever $|t_2 - t_1| \leq \sigma$. Therefore $T : X \to X$ is completely continuous. In view of Lemma 2.3, T has at least one fixed point $u \in \overline{U}$ which is the solution of BVP (1.1)-(1.3). The proof is complete. □

From Theorem 3.1, we easily obtain the following corollaries.

Corollary 3.1 *Assume that* $P_1 Q_2 + P_2 Q_1 \neq 0$ *and* $f : J \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous. Suppose that the conditions* (H_3)-(H_4) *are satisfied*:

(H_3) *There exists a nonnegative function* $a_3 \in L^1(J)$ *and a nonnegative nondecreasing function* $\phi(x,y,z)$ *with respect to each variable* x, y, z, *such that*

$$\left| f(t,x,y,z) \right| \leq a_3(t) \phi\bigl(|x|,|y|,|z|\bigr).$$

(H_4) *There exists a positive constant* R_1 *such that* $\phi(R_1,R_1,R_1) \leq \frac{R_1}{k_3}$, *where* $k_3 = Q \int_0^1 |a_3(t)|\, dt = Q\|a_3\|$.

Then BVP (1.1)-(1.3) *has at least one solution.*

Corollary 3.2 *Assume that* $P_1 Q_2 + P_2 Q_1 \neq 0$ *and* $f : J \times \mathbb{R}^3 \to \mathbb{R}$ *are continuous. Suppose that the condition* (H_5) *is satisfied*:

(H_5) *There exists a nonnegative function* $a_4 \in L^1(J)$ *such that* $|f(t,x,y,z)| \leq a_4(t)$.

Then BVP (1.1)-(1.3) *has at least one solution.*

Theorem 3.2 *Assume that* $P_1 Q_2 + P_2 Q_1 \neq 0$ *and* $f : J \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous. Suppose further that the hypotheses* (H_6)-(H_7) *are satisfied*:

(H_6) *There exist a function* $\sigma \in C(J, \mathbb{R}^+)$ *and a nondecreasing function* $\psi : \mathbb{R} \to \mathbb{R}_0^+$ *such that*

$$\left|f(t,x,y,z)\right| \leq \sigma(t)\psi\left(|x| + |y| + |z|\right) \quad \textit{for } (t,x,y,z) \in J \times \mathbb{R}^3.$$

(H_7) *There exists a constant* $M > 0$ *such that* $\frac{M}{Q\|\sigma\|\psi(M)} > 1$, *where* $\|\sigma\| = \max_{t\in J} |\sigma(t)|$.

Then BVP (1.1)-(1.3) *has at least one solution.*

Proof Let $\Omega = \{u \in X : \|u\| < M\}$ and define the operator $T : X \to X$ as (3.2). Similar to Theorem 3.1, we know that $T : \overline{\Omega} \to \overline{\Omega}$ is completely continuous. If $\exists u \in \partial\Omega$, $\lambda \in (0,1)$ such that

$$u = \lambda Tu, \tag{3.4}$$

then

$$u' = \lambda(Tu)' \tag{3.5}$$

and

$${}^cD_{0+}^{\beta} u = \lambda {}^cD_{0+}^{\beta}(Tu). \tag{3.6}$$

When $t \in [-r, 0]$, in the light of (3.2) and $\varphi(t) \not\equiv 0$, we clearly can conclude that (3.4)-(3.6) do not hold. When $t \in J$, from Lemma 2.5, conditions (H_6)-(H_7), and (3.4)-(3.6), we have

$$\begin{aligned}
\|u\| &= \max_{t\in J}\left|u(t)\right| + \max_{t\in J}\left|u'(t)\right| + \max_{t\in J}\left|{}^cD_{0+}^{\beta}u(t)\right| \\
&= \max_{t\in J}\left|\lambda(Tu)(t)\right| + \max_{t\in J}\left|\lambda(Tu)'(t)\right| + \max_{t\in J}\left|\lambda {}^cD_{0+}^{\beta}(Tu)(t)\right| \\
&\leq \max_{t\in J}\left|(Tu)(t)\right| + \max_{t\in J}\left|(Tu)'(t)\right| + \max_{t\in J}\left|{}^cD_{0+}^{\beta}(Tu)(t)\right| \\
&= \max_{t\in J}\left|\int_0^1 G(t,s) f\left(s, u_s, u'(s), {}^cD_{0+}^{\beta}u(s)\right) ds\right| \\
&\quad + \max_{t\in J}\left|\int_0^1 \frac{\partial}{\partial t} G(t,s) f\left(s, u_s, u'(s), {}^cD_{0+}^{\beta}u(s)\right) ds\right| \\
&\quad + \max_{t\in J}\left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left|(Tu)'(s)\right| ds\right| \\
&\leq \max_{t\in J}\int_0^1 \left|G(t,s)\right|\left|f\left(s, u_s, u'(s), {}^cD_{0+}^{\beta}u(s)\right)\right| ds \\
&\quad + \max_{t\in J}\int_0^1 \left|\frac{\partial}{\partial t} G(t,s)\right|\left|f\left(s, u_s, u'(s), {}^cD_{0+}^{\beta}u(s)\right)\right| ds \\
&\quad + \max_{t\in J}\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left|(Tu)'(s)\right| ds \\
&\leq \max_{t\in J}\int_0^1 \left|G(t,s)\right|\sigma(s)\psi\left(|u_s| + \left|u'(s)\right| + \left|{}^cD_{0+}^{\beta}u(s)\right|\right) ds \\
&\quad + \max_{t\in J}\int_0^1 \left|\frac{\partial}{\partial t} G(t,s)\right|\sigma(s)\psi\left(|u_s| + \left|u'(s)\right| + \left|{}^cD_{0+}^{\beta}u(s)\right|\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\max_{t\in J}\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\big|(Tu)'(s)\big|\,ds \\
&\leq \frac{m_1}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big)+\frac{m_2}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big)+\frac{m_2}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big) \\
&\quad\times\max_{t\in J}\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\,ds \\
&= \frac{m_1}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big)+\frac{m_2}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big) \\
&\quad+\frac{m_2}{\Gamma(\alpha+1)}\|\sigma\|\psi\big(\|u\|\big)\max_{t\in J}\frac{t^{1-\beta}}{\Gamma(2-\beta)} \\
&\leq \left[\frac{m_1}{\Gamma(\alpha+1)}+\frac{m_2}{\Gamma(\alpha+1)}+\frac{m_2}{\Gamma(\alpha+1)\Gamma(2-\beta)}\right]\|\sigma\|\psi\big(\|u\|\big) \\
&= Q\|\sigma\|\psi(M)<M,
\end{aligned}
$$

which is in contradiction with $u\in\partial\Omega$, that is, $\|u\|=M$. According to Lemma 2.2, we can conclude that T has a fixed point $u\in\overline{\Omega}$. Then BVP (1.1)-(1.3) has at least one solution. The proof is complete. □

Theorem 3.3 *Assume that* $P_1Q_2+P_2Q_1\neq 0$ *and* $f:J\times\mathbb{R}^3\to\mathbb{R}$ *be continuous. Suppose that the conditions* (H_8)-(H_9) *are satisfied*:

(H_8) *There exists a nonnegative function* $b\in L^1(J)$ *and a nonnegative nondecreasing function* $\phi(x,y,z)$ *with respect to each variable* x, y, z *such that*

$$
\big|f(t,x,y,z)-f(t,\overline{x},\overline{y},\overline{z})\big|\leq b(t)\phi\big(|x-\overline{x}|,|y-\overline{y}|,|z-\overline{z}|\big).
$$

(H_9) *For any* $R>0$, $\phi(R,R,R)\leq R$ *and* $\int_0^1|b(t)|\,dt=\|b\|<\frac{1}{3Q}$.

Then BVP (1.1)-(1.3) *has a unique solution.*

Proof Now, we will use Banach contraction principle to prove that $T:X\to X$ defined by (3.2) has a fixed point. We first show that T is a contraction. In fact, when $t\in J$, from Lemma 2.5 and (3.1), we obtain

$$
\begin{aligned}
&\big|(Tu)(t)-(Tv)(t)\big| \\
&\quad\leq\int_0^1\big|G(t,s)\big|\big|f\big(s,u_s,u'(s),{}^cD_{0+}^{\beta}u(s)\big)-f\big(s,v_s,v'(s),{}^cD_{0+}^{\beta}v(s)\big)\big|\,ds \\
&\quad\leq\max_{t\in J}\big|b(t)\big|\phi\big(\|u-v\|,\|u-v\|,\|u-v\|\big)\int_0^1\big|G(t,s)\big|\,ds \\
&\quad\leq\frac{m_1\|b\|}{\Gamma(\alpha+1)}\|u-v\|<Q\|b\|\|u-v\|=\varrho\|u-v\|, \qquad (3.7)\\
&\big|(Tu)'(t)-(Tv)'(t)\big| \\
&\quad\leq\int_0^1\left|\frac{\partial}{\partial t}G(t,s)\right|\big|f\big(s,u_s,u'(s),{}^cD_{0+}^{\beta}u(s)\big)-f\big(s,v_s,v'(s),{}^cD_{0+}^{\beta}v(s)\big)\big|\,ds \\
&\quad\leq\max_{t\in J}\big|b(t)\big|\phi\big(\|u-\overline{u}\|,\|u-\overline{u}\|,\|u-\overline{u}\|\big)\int_0^1\left|\frac{\partial}{\partial t}G(t,s)\right|\,ds
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{m_2\|b\|}{\Gamma(\alpha+1)}\|u-v\| < Q\|b\|\|u-v\| \\ &= \varrho\|u-v\|\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\left|{}^cD_{0+}^{\beta}(Tu)(t) - {}^cD_{0+}^{\beta}(Tv)(t)\right| \\ &\quad = \left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\bigl((Tu)'(s) - (Tv)'(s)\bigr)\,ds\right| \\ &\quad \leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left|(Tu)'(s) - (Tv)'(s)\right|\,ds \\ &\quad \leq \frac{m_2\|b\|}{\Gamma(2-\beta)\Gamma(\alpha+1)}\|u-v\| < Q\|b\|\|u-v\| = \varrho\|u-v\|,\end{aligned} \tag{3.9}$$

where $\varrho = Q\|b\| < \frac{1}{3}$. According to (3.7)-(3.9), we get $\|Tu - Tv\| < 3\varrho\|u-v\|$, for all $u, v \in X$, $0 < t \leq 1$. When $t \in [-r, 0]$, it is obvious that $\|Tu - Tv\| = \|\varphi(t) - \varphi(t)\| = 0 < 3\varrho\|u-v\|$. So, for all $u, v \in X$, $t \in [0,1]$, we obtain $\|Tu - Tv\| < 3\varrho\|u-v\|$, namely, T is a contraction. In view of the Banach contraction principle, we conclude that T has the unique fixed point which is the unique solution of BVP (1.1)-(1.3). The proof is complete. □

4 Some examples

Example 4.1 Consider the following BVP of nonlinear fractional differential equations with time delays:

$$\begin{cases} {}^cD_{0+}^{\frac{5}{2}}u(t) + t^8 + \frac{8}{15}(t-\frac{1}{2})^4 e^{\frac{u_t+u'(t)}{2}-1} + \frac{|\sin 2\pi t|}{10\pi}|{}^cD_{0+}^{\frac{1}{2}}u(t)| = 0, \\ u(0) = \frac{6}{5}u(\frac{2}{3}), \qquad u(1) = \frac{5}{4}u(\frac{2}{3}), \qquad u''(0) = 0, \\ u(s) = \varphi(s), \quad s \in [-1, 0], \end{cases} \tag{4.1}$$

where $t \in J = [0,1]$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma_1 = \frac{6}{5}$, $\gamma_2 = \frac{5}{4}$, $\delta_1 = 0$, $\delta_2 = 0$, $\eta = \frac{2}{3}$, $u_t = u(t+\theta)$ $(-1 \leq \theta \leq 0)$, $\varphi \in C([-1,0])$. By simple computation, we have $P_1 = -\frac{1}{5}$, $P_2 = -\frac{1}{4}$, $Q_1 = \frac{4}{5}$, $Q_2 = \frac{1}{6}$, $Q = 5.107258$, $P_1Q_2 + P_2Q_1 = -\frac{7}{30} \neq 0$. Let

$$f(t,x,y,z) = t^8 + \frac{8}{15}\left(t - \frac{1}{2}\right)^4 e^{\frac{x+y}{2}-1} + \frac{|\sin 2\pi t|}{10\pi}z$$

and choose $a_1(t) = t^8$, $a_2(t) = \max_{t\in J}\{\frac{8}{15}(t-\frac{1}{2})^4, \frac{|\sin 2\pi t|}{10\pi}\} = \frac{1}{30}$, $\phi(x,y,z) = e^{\frac{x+y}{2}-1} + z$. Clearly, $\phi(x,y,z)$ is nondecreasing function with respect to each variable x, y, z, and

$$f(t,x,y,z) \leq a_1(t) + a_2(t)\phi(x,y,z),$$

that is, (H_1) holds. Next, we check the condition (H_2). Since

$$\begin{aligned}&k_1 = Q\|a_1(t)\| = Q\int_0^1 \left|t^8\right|dt = \frac{Q}{9} \approx 0.567473, \\ &k_2 = Q\|a_2(t)\| = Q\int_0^1 \frac{1}{30}\,dt = \frac{Q}{30} \approx 0.170242,\end{aligned}$$

choose $R_0 = 1 > k_1$, we have

$$\phi(R_0, R_0, R_0) = e^{\frac{R_0+R_0}{2}-1} + R_0 = e^0 + 1 = 2 < \frac{R_0 - k_1}{k_2} = \frac{1 - 0.567473}{0.170242} \approx 2.540660,$$

which implies that (H_2) is satisfied. Hence BVP (4.1) has at least one solution by Theorem 3.1.

Remark 4.1 In BVP (4.1), the nonlinear function f involves exponential growth, but the results of [17] are only allowed to have power growth, that is, BVP (4.1) cannot be solved by using the results of [17]. So the results obtained in this paper give a significant improvement of the previous work in [17].

Example 4.2 Consider the following time-delay integral BVP of nonlinear fractional differential equations:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}u(t) + \sqrt{t^3+8}\big(2 + \frac{|u_t|+|u'(t)|+|{}^cD_{0^+}^{\frac{1}{2}}u(t)|}{1+|u_t|+|u'(t)|+|{}^cD_{0^+}^{\frac{1}{2}}u(t)|}\big) = 0, \\ u(0) - \frac{1}{2}u(\frac{1}{3}) = 2\int_0^{\frac{1}{3}} u(s)\,ds, \\ u(1) - \frac{2}{3}u(\frac{1}{3}) = 3\int_0^{\frac{1}{3}} u(s)\,ds, \qquad u''(0) = 0, \\ u(s) = \varphi(s), \quad s \in [-1, 0], \end{cases} \tag{4.2}$$

where $t \in J = [0,1]$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{2}{3}$, $\delta_1 = 2$, $\delta_2 = 3$, $\eta = \frac{1}{3}$, $u_t = u(t+\theta)$ $(-1 \leq \theta \leq 0)$, $\varphi \in C([-1,0])$. By simple computation, we get $P_1 = -\frac{1}{6}$, $P_2 = -\frac{2}{3}$, $Q_1 = \frac{5}{18}$, $Q_2 = \frac{11}{18}$, $Q = 2.916156$, $P_1Q_2 + P_2Q_1 = -\frac{31}{108} \neq 0$. Let

$$\begin{aligned} f\big(t, u_t, u'(t), {}^cD_{0^+}^{\beta}u(t)\big) &= \sqrt{t^3+8}\left(2 + \frac{|u_t| + |u'(t)| + |{}^cD_{0^+}^{\frac{1}{2}}u(t)|}{1 + |u_t| + |u'(t)| + |{}^cD_{0^+}^{\frac{1}{2}}u(t)|}\right) \\ &\leq \sigma(t)\psi\big(\|u\|\big), \end{aligned}$$

with $\sigma(t) = \sqrt{t^3+8}$ and $\psi(\|u\|) = 3$. Noting that we have $\|\sigma\| = 3$ and condition (H_7), we have $M > \psi(\|u\|)Q\|\sigma\| \approx 26.245404$. Thus all the conditions of Theorem 3.2 are satisfied. In conclusion, BVP (4.2) has at least one solution.

Example 4.3 Consider the following delayed integral BVP of nonlinear fractional differential equations:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}u(t) + \frac{|u_t|+|u'(t)|+|{}^cD_{0^+}^{\frac{1}{2}}u(t)|}{t^{12}(4+|u_t|+|u'(t)|+|{}^cD_{0^+}^{\frac{1}{2}}u(t)|)} + \sqrt{e^t + \cos\pi t} = 0, \\ u(0) = u''(0) = 0, \qquad u(1) = \frac{16}{9}\int_0^{\frac{3}{4}} u(s)\,ds, \\ u(s) = \varphi(s), \quad s \in [-1, 0], \end{cases} \tag{4.3}$$

where $t \in J = [0,1]$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma_1 = \gamma_2 = \delta_1 = 0$, $\delta_2 = \frac{16}{9}$, $\eta = \frac{3}{4}$, $u_t = u(t+\theta)$ $(-1 \leq \theta \leq 0)$, $\varphi \in C([-1,0])$. By simple computation, we obtain $P_1 = 1$, $P_2 = -\frac{1}{3}$, $Q_1 = 0$, $Q_2 = \frac{1}{2}$, $P_1Q_2 +$

$P_2Q_1 = \frac{1}{2} \neq 0$, $Q = 4.134025$. Set

$$f(t,x,y,z) = \frac{|x| + |y| + |z|}{t^{12}(4 + |x| + |y| + |z|)} + \sqrt{e^t + \cos \pi t},$$

for $x, \overline{x}, y, \overline{y}, z, \overline{z} \in \mathbb{R}$, $t \in J$, we have

$$\begin{aligned}
&\left|f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z})\right| \\
&\quad = \frac{1}{t^{12}}\left|\frac{|x| + |y| + |z|}{4 + |x| + |y| + |z|} - \frac{|\overline{x}| + |\overline{y}| + |\overline{z}|}{4 + |\overline{x}| + |\overline{y}| + |\overline{z}|}\right| \\
&\quad = \frac{1}{t^{12}}\left|\frac{4(|x| - |\overline{x}| + |y| - |\overline{y}| + |z| - |\overline{z}|)}{(4 + |x| + |y| + |z|)(4 + |\overline{x}| + |\overline{y}| + |\overline{z}|)}\right| \\
&\quad \leq \frac{4(|x - \overline{x}| + |y - \overline{y}| + |z - \overline{z}|)}{t^{12}(4 + |x| + |y| + |z|)(4 + |\overline{x}| + |\overline{y}| + |\overline{z}|)} \leq \frac{|x - \overline{x}| + |y - \overline{y}| + |z - \overline{z}|}{4t^{12}}.
\end{aligned}$$

Thus, choose

$$\phi(x,y,z) = \frac{|x| + |y| + |z|}{4}, \qquad b(t) = \frac{1}{t^{12}},$$

then

$$\left|f(t,x,y,z) - f(t,\overline{x},\overline{y},\overline{z})\right| \leq b(t)\phi\bigl(|x - \overline{x}|, |y - \overline{y}|, |z - \overline{z}|\bigr),$$

and $\phi(x,y,z)$ is a nonnegative nondecreasing function with respect to each variable x, y, z. This means that (H_8) holds.

Now, we check the condition (H_9). Since, for any $R > 0$, we have

$$\phi(R,R,R) = \frac{R + R + R}{4} < R$$

and

$$\|b(t)\| = \int_0^1 \frac{1}{t^{12}}\,dt = \frac{1}{13} \approx 0.076923 < \frac{1}{3Q} = 0.080632,$$

(H_9) is satisfied. We conclude that BVP (4.3) has a unique solution by Theorem 3.3.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the anonymous referees for their useful and valuable suggestions. This work is supported by the National Natural Sciences Foundation of Peoples Republic of China under Grant (No. 11161025, No. 11661047), and the Yunnan Province natural scientific research fund project (No. 2011FZ058).

Received: 7 July 2016 Accepted: 24 October 2016 Published online: 04 November 2016

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