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Periodic orbits to Kaplan-Yorke like differential delay equations with two lags of ratio $(2k-1)/2$

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Abstract

In this paper, we study the periodic solutions to a type of differential delay equations with two lags of ratio $(2k-1)/2$ in the form

$$x'(t) = -f\big(x(t-2)\big) - f\big(x\big(t-(2k-1)\big)\big), \quad k \geq 2.$$

The $4k$-periodic solutions are obtained by using the variational method and the method of Kaplan-Yorke coupling system. This is a new type of differential-delay equations compared with all previous researches since the ratio of two lags is not an integer. Three functionals are constructed for a discussion on critical points. An example is given to demonstrate our main results.

MSC: 34B10; 34B15

Keywords: differential delay equation; periodic solutions; critical point theory; variational method

1 Introduction

The differential delay equations have useful applications in various fields such as age-structured population growth, control theory, and any models involving responses with nonzero delays [1–5].

Given $f \in C^0(R^+, R^-)$ with $f(-x) = -f(x), xf(x) > 0, x \neq 0$. Kaplan and Yorke [6] studied the existence of 4-periodic and 6-periodic solutions to the differential delay equations

$$x'(t) = -f\big(x(t-1)\big) \tag{1.1}$$

and

$$x'(t) = -f\big(x(t-1)\big) - f\big(x(t-2)\big), \tag{1.2}$$

respectively. The method they applied is transforming the two equations into adequate ordinary differential equations by regarding the retarded functions $x(t-1)$ and $x(t-2)$ as independent variables. They guessed that the existence of $2(n+1)$-periodic solution to the equation

$$x'(t) = -\sum_{i=1}^{n} f\bigl(x(t-i)\bigr) \tag{1.3}$$

could be studied under the restriction

$$x\bigl(t-(n+1)\bigr) = -x(t),$$

which was proved by Nussbaum [7] in 1978 by use of a fixed point theorem on cones.

After then, a lot of papers [8–21] discussed the existence and multiplicity of $2(n+1)$-periodic solutions to equation (1.3) and its extension

$$x'(t) = -\sum_{i=1}^{n} \operatorname{grad} F\bigl(x(t-i)\bigr), \tag{1.4}$$

where $F \in C^1(R^N, R), F(-x) = F(x), F(0) = 0$.

Recently, Zhang and Ge [22] studied the multiplicity of $2n$-periodic solutions to a type of differential delay equations of the form

$$x'(t) = -f\bigl(x(t-1)\bigr) - f\bigl(x(t-n)\bigr), \quad n \geq 2, \tag{1.5}$$

and obtained new results.

In this paper, we study the periodic orbits to a type of differential delay equations with two lags of ratio $(2k-1)/2$ in the form

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x\bigl(t-(2k-1)\bigr)\bigr), \quad k \geq 2, \tag{1.6}$$

which is different from (1.3) and can be regarded as a new extension of (1.2). The method applied in this paper is the variational approach in the critical point theory [23, 24].

Since the equation

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x(t-2k)\bigr), \quad k \geq 2,$$

can be changed into the form of equation (1.5) by the transformation

$$t = 2s, \qquad x(t) = x(2s) = y(s), \qquad \widehat{f}(y) = 2f(x),$$

this paper completes the research of the equations in the form

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x(t-n)\bigr), \quad n \geq 3. \tag{1.7}$$

In fact, it follows from

$$y'(s) = 2x'(t) = -2f\bigl(x(t-2)\bigr) - 2f\bigl(x(t-2k)\bigr) = -2f\bigl(x\bigl(2(s-1)\bigr)\bigr) - 2f\bigl(x\bigl(2(s-k)\bigr)\bigr)$$

that

$$y'(s) = -\widehat{f}\big(y(s-1)\big) - \widehat{f}\big(y(s-k)\big)$$

for $\widehat{f} = 2f$, which is much the same as equation (1.5).

We suppose that

$$f \in C^0(R,R), \quad f(-x) = -f(x), \tag{1.8}$$

and there are $\alpha, \beta \in R$ such that

$$\lim_{x\to 0} \frac{f(x)}{x} = \alpha, \qquad \lim_{x\to\infty} \frac{f(x)}{x} = \beta. \tag{1.9}$$

Let $F(x) = \int_0^x f(s)\,ds$. Then $F(-x) = F(x)$ and $F(0) = 0$. For convenience, we make the following assumptions.

(S_1) f satisfies (1.8) and (1.9),

(S_2) there exist $M > 0$ and a function $r \in C^0(R^+, R^+)$ satisfying $r(s) \to \infty$, $r(s) \to 0$ as $s \to \infty$ such that

$$\left| F(x) - \frac{1}{2}\beta x^2 \right| > r\big(|x|\big) - M,$$

($S_3^{\pm}$) $\pm[F(x) - \frac{1}{2}\beta x^2] > 0, |x| \to \infty,$

($S_4^{\pm}$) $\pm[F(x) - \frac{1}{2}\alpha x^2] > 0, 0 < |x| \ll 1.$

In this paper, we need the following lemma.

Let X be a Hilbert space, $L : X \to X$ be a linear operator, and $\Phi : X \to R$ be a differentiable functional.

Lemma 1.1 ([24], Theorem 2.4; [8], Lemma 2.4) *Assume that there are two closed s^1-invariant linear subspaces X^+ and X^- and $r > 0$ such that*

(a) $X^+ \cup X^-$ *is closed and of finite codimensions in* X,

(b) $\widehat{L}(X^-) \subset X^-, \widehat{L} = L + P^{-1}A_0$ *or* $\widehat{L} = L + P^{-1}A_\infty$,

(c) *there exists* $c_0 \in R$ *such that*

$$\inf_{x\in X^+} \Phi(x) \ge c_0,$$

(d) *there is* $c_\infty \in R$ *such that* $\Phi(x) \le c_\infty < \Phi(0) = 0, \forall x \in X^- \cap S_r = \{x \in X^- : \|x\| = r\}$,

(e) Φ *satisfies* $(P.S)_c$*-condition,* $c_0 < c < c_\infty$. *Then* Φ *has at least* $\frac{1}{2}[\dim(X^+ \cap X^-) - \operatorname{co}\dim_X(X^+ \cup X^-)]$ *generally different critical orbits in* $\Phi^{-1}([c_0, c_\infty])$ *if*

$$\big[\dim\big(X^+ \cap X^-\big) - \operatorname{co}\dim_X\big(X^+ \cup X^-\big)\big] > 0.$$

Remark 1.1 We may use $(P.S)$-condition to replace condition (e) in Lemma 1.1 since $(P.S)$-condition implies that $(P.S)_c$-condition holds for each $c \in R$.

In order to construct adequate functional whose critical points are the solutions of equation (1.6), we need to distinguish our problem into three cases:

$$k = 3l + 2, \qquad k = 3l + 3,$$

and

$$k = 3l + 4.$$

Then we construct the corresponding three functionals.

2 Space X, functional Φ, and its differential Φ'

2.1 $4k$-Periodic orbits to equation (1.6) when $k = 3l + 2$

We are concerned at the $4k$-periodic solutions to (1.6) and suppose that

$$x\big(t - (6l + 4)\big) = -x(t), \quad l \geq 0. \tag{2.1}$$

We transform (1.6) into

$$x'(t) = -f\big(x(t - 2)\big) - f\big(x\big(t - (6l + 3)\big)\big). \tag{2.2}$$

Let

$$\begin{aligned}
\widehat{X} &= \big\{x \in C_T : x\big(t - (6l + 4)\big) = -x(t)\big\} \\
&= \left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right) : a_i, b_i \in R\right\}, \\
X &= cl\left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right) : a_i, b_i \in R,\right. \\
&\quad \left.\sum_{i=0}^{\infty}(2i+1)\big(a_i^2 + b_i^2\big) < \infty\right\},
\end{aligned}$$

and define $P : X \to L^2$ by

$$\begin{aligned}
Px(t) &= P\left(\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right)\right) \\
&= \sum_{i=0}^{\infty}(2i+1)\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right).
\end{aligned}$$

Then the inverse P^{-1} of P exists. For $x \in X$, define

$$\begin{aligned}
\langle x, y\rangle &= \int_0^{12l+8}\big(Px(t), y(t)\big)\,dt, \quad \|x\| = \sqrt{\langle x, x\rangle}, \\
\langle x, y\rangle_2 &= \int_0^{12l+8}\big(x(t), y(t)\big)\,dt, \quad \|x\|_2 = \sqrt{\langle x, x\rangle_2}.
\end{aligned}$$

Therefore, $(X, \|\cdot\|)$ is an $H^{\frac{1}{2}}$ space.

Define the functional $\Phi : X \to R$ by

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\langle Lx, x\rangle + \int_0^{4k} F\big(x(t)\big)\,dt \\ &= \frac{1}{2}\langle Lx, x\rangle + \int_0^{12l+8} F\big(x(t)\big)\,dt,\end{aligned}\tag{2.3}$$

where

$$Lx = \frac{1}{2}P^{-1}\left[\sum_{i=0}^{2l+1} x'(t-3i) - \sum_{i=0}^{2l} x'(t-3i-1) - \sum_{i=0}^{2l} x'(t-3i-2)\right].\tag{2.4}$$

Let $k_1 = [\frac{2k-2}{3}] = [\frac{6l+2}{3}] = 2l, k_2 = k-1 = 3l+1$, and

$$X(i) = \left\{x(t) = a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4} : a_i, b_i \in R\right\}.$$

We have

$$\begin{aligned}X = \sum_{m=0}^{\infty}\Bigg[&\sum_{i=0}^{k_1}\big(X(2mk+i) + X(2mk+2k-i-1)\big) \\ &+ \sum_{i=k_1+1}^{k_2}\big(X(2mk+i) + X(2mk+2k-i-1)\big)\Bigg].\end{aligned}\tag{2.5}$$

For each

$$x(t) = \sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right) \in X,$$

define $\Omega : X \to X$ by

$$(\Omega x)(t) = \sum_{i=0}^{\infty}\left(b_i \cos\frac{(2i+1)\pi t}{6l+4} - a_i \sin\frac{(2i+1)\pi t}{6l+4}\right)\frac{\sin\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}.$$

If $x_i(t) = a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4} \in X(i), i \in N$, then we have

$$Lx = -\frac{\pi}{12l+8}\left(\Omega x + \sum_{i=0}^{\infty} x_i \frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}\right).\tag{2.6}$$

Obviously, $L|_{X(i)} : X(i) \to X(i)$ is invertible.

By the theorem of Mawhin and Willem [25], Theorem 1.4, the functional Φ is differentiable, and its differential is

$$\Phi'(x) = Lx + K(x),\tag{2.7}$$

where $K(x) = P^{-1}f(x)$. It is easy to prove that $K : (X, \|x\|^2) \to (X, \|x\|_2^2)$ is compact.

It is easy to see that $\langle \Omega x, x\rangle = 0$. Therefore, from (2.6) we have that if

$$x(t) = \sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+4} + b_i \sin\frac{(2i+1)\pi t}{6l+4}\right),$$

then

$$\begin{aligned}
&\langle Lx, x\rangle \\
&\quad = -\sum_{i=0}^{\infty}\frac{(2i+1)\pi}{2}\left(a_i^2 + b_i^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}} \\
&\quad = \sum_{m=0}^{\infty}\Bigg[-\sum_{i=0}^{2l}\frac{(4m(3l+2)+2i+1)\pi}{2}\left(a_{2m(3l+2)+i}^2 + b_{2m(3l+2)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}} \\
&\qquad + \sum_{i=0}^{2l}\frac{(4(m+1)(3l+2)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+2)-i-1}^2 + b_{2(m+1)(3l+2)-i-1}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}} \\
&\qquad - \sum_{i=2l+1}^{3l+1}\frac{(4m(3l+2)+2i+1)\pi}{2}\left(a_{2m(3l+2)+i}^2 + b_{2m(3l+2)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}} \\
&\qquad + \sum_{i=2l+1}^{3l+1}\frac{(4(m+1)(3l+2)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+2)-i-1}^2 + b_{2(m+1)(3l+2)-i-1}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}\Bigg].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\langle P^{-1}\beta x, x\right\rangle &= \sum_{i=0}^{\infty}(6l+4)\beta\left(a_i^2 + b_i^2\right) \\
&= \sum_{m=0}^{\infty}\Bigg[\sum_{i=0}^{2l}(6l+4)\beta\left(a_{2m(3l+2)+i}^2 + b_{2m(3l+2)+i}^2\right) \\
&\quad + \sum_{i=0}^{2l}(6l+4)\beta\left(a_{2m(3l+2)+2(3l+2)-i-1}^2 + b_{2m(3l+2)+2(3l+2)-i-1}^2\right) \\
&\quad + \sum_{i=2l+1}^{3l+1}(6l+4)\beta\left(a_{2m(3l+2)+i}^2 + b_{2m(3l+2)+i}^2\right) \\
&\quad + \sum_{i=2l+1}^{3l+1}(6l+4)\beta\left(a_{2m(3l+2)+2(3l+2)-i-1}^2 + b_{2m(3l+2)+2(3l+2)-i-1}^2\right)\Bigg].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left\langle \left(L + P^{-1}\beta\right)x, x\right\rangle \\
&\quad = (6l+4)\sum_{m=0}^{\infty}\Bigg[\sum_{i=0}^{2l}\left(-\frac{(4m(3l+2)+2i+1)\pi}{12l+8}\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}} + \beta\right) \\
&\qquad \times \left(a_{2m(3l+2)+i}^2 + b_{2m(3l+2)+i}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{2l}\left(\frac{(4(m+1)(3l+2)-2i-1)\pi}{12l+8}\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}+\beta\right)\\
&\times\left(a^2_{2(m+1)(3l+2)-i-1}+b^2_{2(m+1)(3l+2)-i-1}\right)\\
&+\sum_{i=2l+1}^{3l+1}\left(-\frac{(4m(3l+2)+2i+1)\pi}{12l+8}\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}+\beta\right)\left(a^2_{2m(3l+2)+i}+b^2_{2m(3l+2)+i}\right)\\
&+\sum_{i=2l+1}^{3l+1}\left(\frac{(4(m+1)(3l+2)-2i-1)\pi}{12l+8}\frac{\cos\frac{(2i+1)\pi}{12l+8}}{\sin\frac{(2i+1)3\pi}{12l+8}}+\beta\right)\\
&\times\left(a^2_{2(m+1)(3l+2)-i-1}+b^2_{2(m+1)(3l+2)-i-1}\right)\Bigg].
\end{aligned}
\tag{2.8}
$$

Lemma 2.1 *Each critical point of the functional* Φ *is a* $(12l+8)$*-periodic solution of equation* (1.6) *satisfying* (2.1).

Proof Let x be a critical point of the functional Φ. Then $x(t)$ satisfies

$$\frac{1}{2}\left[\sum_{i=0}^{2l+1}x'(t-3i)-\sum_{i=0}^{2l}x'(t-3i-1)-\sum_{i=0}^{2l}x'(t-3i-2)\right]+f\big(x(t)\big)=0. \tag{2.9}$$

Consequently,

$$\frac{1}{2}\left[\sum_{i=0}^{2l+1}x'(t-3i-2)-\sum_{i=1}^{2l}x'(t-3i)-\sum_{i=1}^{2l}x'(t-3i-1)\right]+f\big(x(t-2)\big)=0, \tag{2.10}$$

$$\frac{1}{2}\left[\sum_{i=0}^{2l+1}x'(t-3i+1)-\sum_{i=0}^{2l}x'(t-3i)-\sum_{i=0}^{2l}x'(t-3i-1)\right]+f\big(x(t+1)\big)=0. \tag{2.11}$$

Subtracting (2.10) from (2.11), we have

$$-x'(t)+f\big(x(t+1)\big)-f\big(x(t-2)\big)=0,$$

that is,

$$x'(t)=-f\big(x(t-2)\big)-f\big(x\big(t-(6l+3)\big)\big),$$

which implies that x is a solution to (1.6). □

2.2 4k-Periodic orbits to equation (1.6) when $k = 3l + 3$

We are concerned at the $4k$-periodic solutions to (1.6) and suppose that

$$x\big(t-(6l+6)\big)=-x(t),\quad l\geq 0. \tag{2.12}$$

We transform (1.6) into

$$x'(t)=-f\big(x(t-2)\big)-f\big(x\big(t-(6l+5)\big)\big). \tag{2.13}$$

Let

$$\begin{aligned}\widehat{X} &= \{x \in C_T : x(t-(6l+6)) = -x(t)\}\\ &= \left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6}\right) : a_i, b_i \in R\right\},\\ X &= cl\left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6}\right) : a_i, b_i \in R,\right.\\ &\quad \left.\sum_{i=0}^{\infty}(2i+1)(a_i^2+b_i^2) < \infty\right\},\end{aligned}$$

and define $P : X \to L^2$ by

$$\begin{aligned}Px(t) &= P\left(\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6}\right)\right)\\ &= \sum_{i=0}^{\infty}(2i+1)\left(a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6}\right).\end{aligned}$$

Then the inverse P^{-1} of P exists. For $x \in X$, define

$$\begin{aligned}\langle x, y\rangle &= \int_0^{12l+12}\bigl(Px(t), y(t)\bigr)\,dt, \qquad \|x\| = \sqrt{\langle x, x\rangle},\\ \langle x, y\rangle_2 &= \int_0^{12l+12}\bigl(x(t), y(t)\bigr)\,dt, \qquad \|x\|_2 = \sqrt{\langle x, x\rangle_2}.\end{aligned}$$

Therefore, $(X, \|\cdot\|)$ is an $H^{\frac{1}{2}}$ space.

Define the functional $\Phi : X \to R$ by

$$\Phi(x) = \frac{1}{2}\langle Lx, x\rangle + \int_0^{12l+12} F\bigl(x(t)\bigr)\,dt, \tag{2.14}$$

where

$$Lx = -\frac{1}{2}P^{-1}\sum_{i=0}^{2l+1} x'(t-3i-1). \tag{2.15}$$

Let $k_1 = [\frac{2k-2}{3}] = [\frac{6l+4}{3}] = 2l+1, k_2 = k-1 = 3l+2$, and

$$X(i) = \left\{x(t) = a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6} : a_i, b_i \in R\right\}.$$

We have

$$\begin{aligned}X = \sum_{m=0}^{\infty}\Biggl[&\sum_{i=0}^{k_1}\bigl(X(2mk+i) + X(2mk+2k-i-1)\bigr)\\ &+ \sum_{i=k_1+1}^{k_2}\bigl(X(2mk+i) + X(2mk+2k-i-1)\bigr)\Biggr].\end{aligned} \tag{2.16}$$

Define $\Omega : X \to X$ by

$$(\Omega x)(t) = \sum_{i=0}^{\infty}\left(b_i \cos\frac{(2i+1)\pi t}{6l+6} - a_i \sin\frac{(2i+1)\pi t}{6l+6}\right)\frac{\sin\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}.$$

Then if

$$x_i(t) = a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6} \in X(i), \quad i \in N,$$

then we have

$$Lx = -\frac{\pi}{12l+12}\left(\Omega x + \sum_{i=0}^{\infty} x_i \frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}\right). \tag{2.17}$$

The functional Φ is differentiable, and its differential is

$$\Phi'(x) = Lx + K(x), \tag{2.18}$$

where $K(x) = P^{-1}f(x)$. The mapping $K : (X, \|x\|^2) \to (X, \|x\|_2^2)$ is compact.

Therefore, from (2.17) it follows that, for each

$$x(t) = \sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+6} + b_i \sin\frac{(2i+1)\pi t}{6l+6}\right),$$

we have

$$\begin{aligned}
\langle Lx, x\rangle &= -\sum_{i=0}^{\infty}\frac{(2i+1)\pi}{2}\left(a_i^2 + b_i^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}} \\
&= \sum_{m=0}^{\infty}\Bigg[-\sum_{i=0}^{2l+1}\frac{(4m(3l+3)+2i+1)\pi}{2}\left(a_{2m(3l+3)+i}^2 + b_{2m(3l+3)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}} \\
&\quad + \sum_{i=0}^{2l+1}\frac{(4(m+1)(3l+3)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+3)-i-1}^2 + b_{2(m+1)(3l+3)-i-1}^2\right) \\
&\quad \times \frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}} \\
&\quad - \sum_{i=2l+2}^{3l+2}\frac{(4m(3l+3)+2i+1)\pi}{2}\left(a_{2m(3l+3)+i}^2 + b_{2m(3l+3)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}} \\
&\quad + \sum_{i=2l+2}^{3l+2}\frac{(4(m+1)(3l+3)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+3)-i-1}^2 + b_{2(m+1)(3l+3)-i-1}^2\right) \\
&\quad \times \frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}\Bigg].
\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle P^{-1}\beta x, x\rangle &= \sum_{m=0}^{\infty}(6l+6)\beta\left(a_i^2+b_i^2\right)\\ &= \sum_{m=0}^{\infty}\Bigg[\sum_{i=0}^{2l+1}(6l+6)\beta\left(a_{2m(3l+3)+i}^2+b_{2m(3l+3)+i}^2\right)\\ &\quad+\sum_{i=0}^{2l+1}(6l+6)\beta\left(a_{2(m+1)(3l+3)-i-1}^2+b_{2(m+1)(3l+3)-i-1}^2\right)\\ &\quad+\sum_{i=2l+2}^{3l+2}(6l+6)\beta\left(a_{2m(3l+3)+i}^2+b_{2m(3l+3)+i}^2\right)\\ &\quad+\sum_{i=2l+2}^{3l+2}(6l+6)\beta\left(a_{2(m+1)(3l+3)-i-1}^2+b_{2(m+1)(3l+3)-i-1}^2\right)\Bigg].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left\langle\left(L+P^{-1}\beta\right)x, x\right\rangle\\ &\quad=(6l+6)\sum_{m=0}^{\infty}\Bigg[\sum_{i=0}^{2l+1}\left(-\frac{(4m(3l+3)+2i+1)\pi}{12l+12}\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}+\beta\right)\\ &\qquad\times\left(a_{2m(3l+3)+i}^2+b_{2m(3l+3)+i}^2\right)\\ &\qquad+\sum_{i=0}^{2l+1}\left(\frac{(4(m+1)(3l+3)-i-1)\pi}{12l+12}\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}+\beta\right)\\ &\qquad\times\left(a_{2(m+1)(3l+3)-i-1}^2+b_{2(m+1)(3l+3)-i-1}^2\right)\\ &\qquad+\sum_{i=2l+2}^{3l+2}\left(-\frac{(4m(3l+3)+2i+1)\pi}{12l+12}\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}+\beta\right)\left(a_{2m(3l+3)+i}^2+b_{2m(3l+3)+i}^2\right)\\ &\qquad+\sum_{i=2l+2}^{3l+2}\left(\frac{(4(m+1)(3l+3)-2i-1)\pi}{12l+12}\frac{\cos\frac{(2i+1)\pi}{12l+12}}{\sin\frac{(2i+1)3\pi}{12l+12}}+\beta\right)\\ &\qquad\times\left(a_{2(m+1)(3l+3)-i-1}^2+b_{2(m+1)(3l+3)-i-1}^2\right)\Bigg].\end{aligned}\tag{2.19}$$

Lemma 2.2 *Each critical point of the functional* Φ *is a* $(12l+12)$*-periodic solution of equation* (1.6) *satisfying* (2.12).

Proof Let x be a critical point of the functional Φ. Then $x(t)$ satisfies

$$-\frac{1}{2}\sum_{i=0}^{2l+1}x'(t-3i-1)+f\left(x(t)\right)=0.\tag{2.20}$$

Consequently, we have

$$-\frac{1}{2}\sum_{i=0}^{2l+1} x'(t-3i-3) + f\bigl(x(t-2)\bigr) = 0, \tag{2.21}$$

$$-\frac{1}{2}\sum_{i=0}^{2l+1} x'(t-3i) + f\bigl(x(t+1)\bigr) = 0. \tag{2.22}$$

Subtracting (2.21) from (2.22), we have

$$-x'(t) + f\bigl(x(t+1)\bigr) - f\bigl(x(t-2)\bigr) = 0,$$

that is,

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x\bigl(t-(6l+5)\bigr)\bigr),$$

which implies that x is a solution to (1.6). □

2.3 4*k*-Periodic orbits to equation (1.6) when *k* = 3*l* + 4

We are concerned at the $4k$-periodic solutions to (1.6) and suppose that

$$x\bigl(t-(6l+8)\bigr) = -x(t), \quad l \geq 0. \tag{2.23}$$

We transform (1.6) into

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x\bigl(t-(6l+7)\bigr)\bigr). \tag{2.24}$$

Let

$$\begin{aligned}
\widehat{X} &= \bigl\{x \in C_T : x\bigl(t-(6l+8)\bigr) = -x(t)\bigr\} \\
&= \left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8}\right) : a_i, b_i \in R\right\}, \\
X &= cl\left\{\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8}\right) : a_i, b_i \in R,\right. \\
&\quad \left.\sum_{i=0}^{\infty}(2i+1)\bigl(a_i^2+b_i^2\bigr) < \infty\right\},
\end{aligned}$$

and define $P : X \to L^2$ by

$$\begin{aligned}
Px(t) &= P\left(\sum_{i=0}^{\infty}\left(a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8}\right)\right) \\
&= \sum_{i=0}^{\infty}(2i+1)\left(a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8}\right).
\end{aligned}$$

Then the inverse P^{-1} of P exists. For $x \in X$, define

$$\langle x,y\rangle = \int_0^{12l+16} \bigl(Px(t),y(t)\bigr)\,dt, \qquad \|x\| = \sqrt{\langle x,x\rangle},$$
$$\langle x,y\rangle_2 = \int_0^{12l+16} \bigl(x(t),y(t)\bigr)\,dt, \qquad \|x\|_2 = \sqrt{\langle x,x\rangle_2}.$$

Therefore, $(X, \|\cdot\|)$ is an $H^{\frac{1}{2}}$ space.

Define the functional $\Phi : X \to R$ by

$$\Phi(x) = \frac{1}{2}\langle Lx,x\rangle + \int_0^{12l+16} F\bigl(x(t)\bigr)\,dt, \tag{2.25}$$

where

$$Lx = \frac{1}{2}P^{-1}\left[-\sum_{i=0}^{2l+2} x'(t-3i) - \sum_{i=0}^{2l+2} x'(t-3i-1) + \sum_{i=0}^{2l+1} x'(t-3i-2)\right]. \tag{2.26}$$

Let $k_1 = [\frac{2k-2}{3}] = [\frac{6l+6}{3}] = 2l+2, k_2 = k-1 = 3l+3$, and

$$X(i) = \left\{x(t) = a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8} : a_i, b_i \in R\right\}.$$

We have

$$\begin{aligned} X = \sum_{m=0}^{\infty}\Biggl[&\sum_{i=0}^{k_1}\bigl(X(2mk+i) + X(2mk+2k-i-1)\bigr) \\ &+ \sum_{i=k_1+1}^{k_2}\bigl(X(2mk+i) + X(2mk+2k-i-1)\bigr)\Biggr]. \end{aligned} \tag{2.27}$$

Define $\Omega : X \to X$ by

$$(\Omega x)(t) = \sum_{i=0}^{\infty}\left(b_i \cos\frac{(2i+1)\pi t}{6l+8} - a_i \sin\frac{(2i+1)\pi t}{6l+8}\right)\frac{\sin\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}.$$

Then if

$$x_i(t) = a_i \cos\frac{(2i+1)\pi t}{6l+8} + b_i \sin\frac{(2i+1)\pi t}{6l+8} \in X(i), \quad i \in N,$$

then we have

$$Lx = -\frac{\pi}{12l+16}\left(\Omega x + \sum_{i=0}^{\infty} x_i \frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\right). \tag{2.28}$$

The functional Φ is differentiable, and its differential is

$$\Phi'(x) = Lx + K(x), \tag{2.29}$$

where $K(x) = P^{-1}f(x)$. The mapping $K : (X, \|x\|^2) \to (X, \|x\|_2^2)$ is compact.

For

$$x(t)=\sum_{i=0}^{\infty}\left(a_i\cos\frac{(2i+1)\pi t}{6l+8}+b_i\sin\frac{(2i+1)\pi t}{6l+8}\right),$$

we have

$$\begin{aligned}
&\langle Lx,x\rangle\\
&=-\sum_{i=0}^{\infty}\frac{(2i+1)\pi}{2}\left(a_i^2+b_i^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\\
&=\sum_{m=0}^{\infty}\left[-\sum_{i=0}^{2l+2}\frac{(4m(3l+4)+2i+1)\pi}{2}\left(a_{2m(3l+4)+i}^2+b_{2m(3l+4)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\right.\\
&\quad+\sum_{i=0}^{2l+2}\frac{(4(m+1)(3l+4)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+4)-i-1}^2+b_{2(m+1)(3l+4)-i-1}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\\
&\quad-\sum_{i=2l+3}^{3l+3}\frac{(4m(3l+4)+2i+1)\pi}{2}\left(a_{2m(3l+4)+i}^2+b_{2m(3l+4)+i}^2\right)\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\\
&\quad+\sum_{i=2l+3}^{3l+3}\frac{(4(m+1)(3l+4)-2i-1)\pi}{2}\left(a_{2(m+1)(3l+4)-i-1}^2+b_{2(m+1)(3l+4)-i-1}^2\right)\\
&\quad\left.\times\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\langle P^{-1}\beta x,x\right\rangle&=\sum_{m=0}^{\infty}(6l+8)\beta\left(a_i^2+b_i^2\right)\\
&=\sum_{m=0}^{\infty}\left[\sum_{i=0}^{2l+2}(6l+8)\beta\left(a_{2m(3l+4)+i}^2+b_{2m(3l+4)+i}^2\right)\right.\\
&\quad+\sum_{i=0}^{2l+2}(6l+8)\beta\left(a_{2(m+1)(3l+4)-i-1}^2+b_{2(m+1)(3l+4)-i-1}^2\right)\\
&\quad+\sum_{i=2l+3}^{3l+3}(6l+8)\beta\left(a_{2m(3l+4)+i}^2+b_{2m(3l+4)+i}^2\right)\\
&\quad\left.+\sum_{i=2l+3}^{3l+3}(6l+8)\beta\left(a_{2(m+1)(3l+4)-i-1}^2+b_{2(m+1)(3l+4)-i-1}^2\right)\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left\langle\left(L+P^{-1}\beta\right)x,x\right\rangle\\
&=(6l+8)\sum_{m=0}^{\infty}\left[\sum_{i=0}^{2l+2}\left(-\frac{(4m(3l+4)+2i+1)\pi}{12l+16}\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}}+\beta\right)\right.
\end{aligned}$$

$$\times \left(a^2_{2m(3l+4)+i} + b^2_{2m(3l+4)+i}\right)$$
$$+ \sum_{i=0}^{2l+2}\left(\frac{(4(m+1)(3l+4)-i-1)\pi}{12l+16}\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}} + \beta\right)$$
$$\times \left(a^2_{2(m+1)(3l+4)-i-1} + b^2_{2(m+1)(3l+4)-i-1}\right)$$
$$+ \sum_{i=2l+3}^{3l+3}\left(-\frac{(4m(3l+4)+2i+1)\pi}{12l+16}\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}} + \beta\right)\left(a^2_{2m(3l+4)+i} + b^2_{2m(3l+4)+i}\right)$$
$$+ \sum_{i=2l+3}^{3l+3}\left(\frac{(4(m+1)(3l+4)-2i-1)\pi}{12l+16}\frac{\cos\frac{(2i+1)\pi}{12l+16}}{\sin\frac{(2i+1)3\pi}{12l+16}} + \beta\right)$$
$$\times \left(a^2_{2(m+1)(3l+4)-i-1} + b^2_{2(m+1)(3l+4)-i-1}\right)\Bigg]. \tag{2.30}$$

Lemma 2.3 *Each critical point of the functional* Φ *is a* $(12l+16)$*-periodic solution of equation* (1.6) *satisfying* (2.23).

Proof Let x be a critical point of the functional Φ. Then $x(t)$ satisfies

$$\frac{1}{2}\left[-\sum_{i=0}^{2l+2}x'(t-3i) - \sum_{i=0}^{2l+2}x'(t-3i-1) + \sum_{i=0}^{2l+1}x'(t-3i-2)\right] + f\bigl(x(t)\bigr) = 0. \tag{2.31}$$

Consequently,

$$\frac{1}{2}\left[-\sum_{i=0}^{2l+2}x'(t-3i-2) - \sum_{i=0}^{2l+2}x'(t-3i-3) + \sum_{i=0}^{2l+1}x'(t-3i-4)\right] + f\bigl(x(t-2)\bigr) = 0, \tag{2.32}$$
$$\frac{1}{2}\left[-\sum_{i=0}^{2l+2}x'(t-3i+1) - \sum_{i=0}^{2l+2}x'(t-3i) + \sum_{i=0}^{2l+1}x'(t-3i-1)\right] + f\bigl(x(t+1)\bigr) = 0. \tag{2.33}$$

Subtracting (2.32) from (2.33), we have

$$-x'(t) + f\bigl(x(t+1)\bigr) - f\bigl(x(t-2)\bigr) = 0,$$

that is,

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x\bigl(t-(6l+7)\bigr)\bigr),$$

which implies that x is a solution to (1.6). □

3 Partition of space X and symbols

In fact, in Sections 2.1, 2.2, and 2.3, we could let

$$k_1 = \left[\frac{2k-2}{3}\right], \qquad k_2 = k-1,$$
$$X(i) = \left\{x(t) = a_i\cos\frac{(2i+1)\pi t}{2k} + b_i\sin\frac{(2i+1)\pi t}{2k} : a_i, b_i \in R\right\},$$

and

$$X=\sum_{m=0}^{\infty}\left[\sum_{i=0}^{k_1}\bigl(X(2mk+i)+X(2mk+2k-i-1)\bigr)+\sum_{i=k_1+1}^{k_2}\bigl(X(2mk+i)+X(2mk+2k-i-1)\bigr)\right]$$

and define $\Omega : X\to X$ by

$$(\Omega x)(t)=\sum_{i=0}^{\infty}\left(b_i\cos\frac{(2i+1)\pi t}{2k}-a_i\sin\frac{(2i+1)\pi t}{2k}\right)\frac{\sin\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}.$$

Then if $x_i(t)=a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}\in X(i), i\in N$, then we have

$$Lx=-\frac{\pi}{4k}\left(\Omega x+\sum_{i=0}^{\infty}x_i\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right),$$

and, for each

$$x(t)=\sum_{i=0}^{\infty}\left(a_i\cos\frac{(2i+1)\pi t}{2k}+b_i\sin\frac{(2i+1)\pi t}{2k}\right)\in X,$$

we have

$$\begin{aligned}\langle Lx,x\rangle &=-\sum_{i=0}^{\infty}\frac{(2i+1)\pi}{2}\bigl(a_i^2+b_i^2\bigr)\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\\ &=\sum_{m=0}^{\infty}\left[-\sum_{i=0}^{k_1}\frac{(4mk+2i+1)\pi}{2}\bigl(a_{2mk+i}^2+b_{2mk+i}^2\bigr)\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right.\\ &\quad+\sum_{i=0}^{k_1}\frac{(4(m+1)k-2i-1)\pi}{2}\bigl(a_{2(m+1)k-i-1}^2+b_{2(m+1)k-i-1}^2\bigr)\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\\ &\quad-\sum_{i=k_1+1}^{k_2}\frac{(4mk+2i+1)\pi}{2}\bigl(a_{2mk+i}^2+b_{2mk+i}^2\bigr)\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\\ &\quad\left.+\sum_{i=k_1+1}^{k_2}\frac{(4(m+1)k-2i-1)\pi}{2}\bigl(a_{2(m+1)k-i-1}^2+b_{2(m+1)k-i-1}^2\bigr)\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle P^{-1}\beta x,x\rangle &=\sum_{m=0}^{\infty}2k\beta\bigl(a_i^2+b_i^2\bigr)\\ &=\sum_{m=0}^{\infty}\left[\sum_{i=0}^{k_1}2k\beta\bigl(a_{2mk+i}^2+b_{2mk+i}^2\bigr)\right.\\ &\quad+\sum_{i=0}^{k_1}2k\beta\bigl(a_{2(m+1)k-i-1}^2+b_{2(m+1)k-i-1}^2\bigr)\end{aligned}$$

$$+ \sum_{i=k_1+1}^{k_2} 2k\beta\left(a_{2mk+i}^2 + b_{2mk+i}^2\right)$$
$$+ \sum_{i=k_1+1}^{k_2} 2k\beta\left(a_{2(m+1)k-i-1}^2 + b_{2(m+1)k-i-1}^2\right)\Bigg].$$

Therefore, we have

$$\begin{aligned}
&\left\langle\left(L + P^{-1}\beta\right)x, x\right\rangle \\
&= 2k\sum_{m=0}^{\infty}\Bigg[\sum_{i=0}^{k_1}\left(-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta\right)\left(a_{2mk+i}^2 + b_{2mk+i}^2\right) \\
&+ \sum_{i=0}^{k_1}\left(\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta\right)\left(a_{2(m+1)k-i-1}^2 + b_{2(m+1)k-i-1}^2\right) \\
&+ \sum_{i=k_1+1}^{k_2}\left(-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta\right)\left(a_{2mk+i}^2 + b_{2mk+i}^2\right) \\
&+ \sum_{i=k_1+1}^{k_2}\left(\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta\right)\left(a_{2(m+1)k-i-1}^2 + b_{2(m+1)k-i-1}^2\right)\Bigg].
\end{aligned}$$

Let

$$\begin{aligned}
X_\infty^+ = &\left\{X(2mk+i) : m \geq 0, 0 \leq i \leq k_1, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta > 0\right\} \\
&\cup \Bigg\{X\big(2(m+1)k-i-1\big) : m \geq 0, 0 \leq i \leq k_1, \\
&\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta > 0\Bigg\} \\
&\cup \left\{X(2mk+i) : m \geq 0, k_1+1 \leq i \leq k_2, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta > 0\right\} \\
&\cup \Bigg\{X\big(2(m+1)k-i-1\big) : m \geq 0, k_1+1 \leq i \leq k_2, \\
&\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta > 0\Bigg\},
\end{aligned}$$

$$\begin{aligned}
X_\infty^- = &\left\{X(2mk+i) : m \geq 0, 0 \leq i \leq k_1, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta < 0\right\} \\
&\cup \Bigg\{X\big(2(m+1)k-i-1\big) : m \geq 0, 0 \leq i \leq k_1, \\
&\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta < 0\Bigg\} \\
&\cup \left\{X(2mk+i) : m \geq 0, k_1+1 \leq i \leq k_2, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \beta < 0\right\}
\end{aligned}$$

$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, k_1+1\leq i\leq k_2,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta<0\right\},$$

$$X_0^+=\left\{X(2mk+i): m\geq 0, 0\leq i\leq k_1, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha>0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, 0\leq i\leq k_1,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha>0\right\}$$
$$\cup\left\{X(2mk+i): m\geq 0, k_1+1\leq i\leq k_2, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha>0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, k_1+1\leq i\leq k_2,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha>0\right\},$$

$$X_0^-=\left\{X(2mk+i): m\geq 0, 0\leq i\leq k_1, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha<0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, 0\leq i\leq k_1,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha<0\right\}$$
$$\cup\left\{X(2mk+i): m\geq 0, k_1+1\leq i\leq k_2, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha<0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, k_1+1\leq i\leq k_2,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha<0\right\}.$$

On the other hand,

$$X_\infty^0=\left\{X(2mk+i): m\geq 0, 0\leq i\leq k_1, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta=0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, 0\leq i\leq k_1,\right.$$
$$\left.\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta=0\right\}$$
$$\cup\left\{X(2mk+i): m\geq 0, k_1+1\leq i\leq k_2, -\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta=0\right\}$$
$$\cup\left\{X\big(2(m+1)k-i-1\big): m\geq 0, k_1+1\leq i\leq k_2,\right.$$

$$\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta=0\Bigg\},$$

$$X_0^0=\Bigg\{X(2mk+i):m\geq 0,0\leq i\leq k_1,-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha=0\Bigg\}$$
$$\cup\Bigg\{X\big(2(m+1)k-i-1\big):m\geq 0,0\leq i\leq k_1,$$
$$\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha=0\Bigg\}$$
$$\cup\Bigg\{X(2mk+i):m\geq 0,k_1+1\leq i\leq k_2,-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha=0\Bigg\}$$
$$\cup\Bigg\{X\big(2(m+1)k-i-1\big):m\geq 0,k_1+1\leq i\leq k_2,$$
$$\frac{(4(m+1)k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\alpha=0\Bigg\}.$$

Obviously, $\dim X_\infty^0<\infty$ and $\dim X_0^0<\infty$.

Lemma 3.1 *Under assumptions* (S_1) *and* (S_2), *there is* $\sigma>0$ *such that*

$$\big\langle\big(L+P^{-1}\beta\big)x,x\big\rangle>\sigma\|x\|^2,\quad x\in X_\infty^+,$$

and

$$\big\langle\big(L+P^{-1}\beta\big)x,x\big\rangle<-\sigma\|x\|^2,\quad x\in X_\infty^-.\tag{3.1}$$

Proof First, we have that, for $\beta\geq 0$ and $i\in\{0,1,\dots,k_1\}$,

$$-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta>-\frac{(4m_0^+(i)k+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta>0,$$

where $m_0^+(i)=\max\{m\in N:-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta>0\}$, and

$$-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta<-\frac{(4m_0^-(i)k+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta<0,$$

where $m_0^-(i)=\min\{m\in N:-\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta<0\}$.

In this case, we may choose

$$\sigma_i=\min\Bigg\{-\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\frac{\beta}{4m_0^+(i)k+2i+1},$$
$$\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}-\frac{\beta}{4m_0^-(i)k+2i+1}\Bigg\}>0.$$

and let $\sigma^0 = \min\{\sigma_0, \sigma_1, \ldots, \sigma_{k_1}\} > 0$. Further, for $\beta \geq 0$ and $i \in \{k_1+1, \ldots, k_2\}$, we have

$$\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta > \frac{(4m_1^+(i)k+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta > 0,$$

where $m_1^+(i) = \max\{m \in N : \frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta > 0\}$, and

$$\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta < \frac{(4m_1^-(i)k+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta < 0,$$

where $m_1^-(i) = \min\{m \in N : \frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\beta < 0\}$.

In this case, we may choose

$$\sigma_i = \min\left\{\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}+\frac{\beta}{4m_1^+(i)k+4k-2i-1}, -\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}-\frac{\beta}{4m_1^-(i)k+4k-2i-1}\right\} > 0,$$

and let $\sigma^1 = \min\{\sigma_{k_1+1}, \sigma_{k_1+2}, \ldots, \sigma_{k_2}\} > 0$.

Let $\sigma = \min\{\sigma^0, \sigma^1\} = \min\{\sigma_1, \sigma_2, \ldots, \sigma_{k_2}\}$. The proof for the case $\beta < 0$ is similar. We omit it. The inequalities in (3.1) are proved. □

Lemma 3.2 *Under conditions* (S_1) *and* (S_2), *the functional* Φ *defined by* (2.3) *satisfies* $(P.S)$-*condition if*

$$\left|\Phi(x)-\frac{1}{2}\beta x^2\right| > r(|x|)-M_0, \quad x \in R,$$

for some $M_0 > 0$ *and some function* $r \in C^0(R^+, R^+)$ *that satisfies*

$$r(s) \to \infty, \qquad r(s)/s \to 0 \quad \text{as } s \to \infty. \tag{3.2}$$

Proof Let Π, N, Z be the orthogonal projections from X onto $X_\infty^+, X_\infty^-, X_\infty^0$, respectively. From the second condition in (1.9) it follows that

$$\left|\langle P^{-1}(f(x)-\beta x), x\rangle\right| < \frac{\sigma}{2}\|x\|^2 + M, \quad x \in X, \tag{3.3}$$

for some $M > 0$.

Assume that $\{x_n\} \subset X$ is a subsequence such that $\Phi'(x_n) \to 0$ and $\Phi(x_n)$ is bounded. Let $w_n = \Pi x_n, y_n = Nx_n, z_n = Zx_n$. Then we have

$$\Pi(L+P^{-1}\beta) = (L+P^{-1}\beta)\Pi, \qquad N(L+P^{-1}\beta) = (L+P^{-1}\beta)N. \tag{3.4}$$

From

$$\langle\Phi'(x_n), x_n\rangle = \langle Lx_n + P^{-1}f(x_n), x_n\rangle = \langle(L+P^{-1}\beta)x_n, x_n\rangle + \langle P^{-1}(f(x_n)-\beta x_n), x_n\rangle$$

and (3.4) we have

$$\begin{aligned}\langle \Pi\Phi'(x_n), x_n\rangle &= \langle \Pi(L+P^{-1}\beta)x_n, x_n\rangle + \langle \Pi P^{-1}(f(x_n)-\beta x_n), x_n\rangle \\ &= \langle (L+P^{-1}\beta)w_n, w_n\rangle + \langle \Pi P^{-1}(f(x_n)-\beta x_n), w_n\rangle\end{aligned}$$

and then, by (3.1),

$$\langle (L+P^{-1}\beta)w_n, w_n\rangle + \langle \Pi P^{-1}(f(x_n)-\beta x_n), w_n\rangle > \frac{\sigma}{2}\|w_n\|^2 - M\|w_n\|,$$

which, together with $\Pi\Phi'(x_n) \to 0$, implies the boundedness of w_n. Similarly, we have the boundedness of y_n. At the same time, (S_2) yields

$$\begin{aligned}\Phi(x_n) &= \frac{1}{2}\langle (L+P^{-1}\beta)x_n, x_n\rangle + \int_0^{4k} F(x_n)\,dt - \frac{\beta}{2}\|x_n\|_2^2 \\ &= \frac{1}{2}\langle (L+P^{-1}\beta)w_n, w_n\rangle + \frac{1}{2}\langle (L+P^{-1}\beta)y_n, y_n\rangle \\ &\quad + \int_0^{4k} F(x_n)\,dt - \frac{\beta}{2}\left(\|w_n\|_2^2 + \|y_n\|_2^2 + \|z_n\|_2^2\right).\end{aligned}$$

Then the boundedness of $\Phi(x)$ implies that $\|z_n\|_2$ is bounded. Consequently, $\|z_n\|$ is bounded since X_∞^0 is finite-dimensional. Therefore, $\|x_n\|$ is bounded.

It follows from (2.7) that

$$\begin{aligned}(\Pi+N)\Phi'(x_n) &= (\Pi+N)Lx_n + (\Pi+N)Kx_n \\ &= L(w_n+y_n) + (\Pi+N)Kx_n.\end{aligned}$$

From the compactness of operator K and the boundedness of x_n we have that $K(x_n) \to u$. Then

$$L|_{x_\infty^+ + x_\infty^-}(w_n+y_n) \to -(\Pi+N)u. \tag{3.5}$$

The finite-dimensionality of X_∞^0 and the boundedness of $z_n = Zx_n$ imply $z_n \to \varphi \in X_\infty^0$. Therefore,

$$x_n = z_n + w_n + y_n \to \varphi - (L|_{x_\infty^+ + x_\infty^-})^{-1}(\Pi+N)u,$$

which implies $(P.S)$-condition. □

Lemma 3.3 *Under conditions* (S_1) *and* (S_2), *the functional* Φ *defined by* (2.14) *and* (2.25) *satisfies* $(P.S)$*-condition if*

$$\left|\Phi(x) - \frac{1}{2}\beta x^2\right| > r(|x|) - M_0, \quad x \in R,$$

for some $M_0 > 0$ *and some function* $r \in C^0(R^+, R^+)$, *which satisfies*

$$r(s) \to \infty, \qquad r(s)/s \to 0 \quad \text{as } s \to \infty.$$

The proof of Lemma 3.3 is the same as that of Lemma 3.2, and we omit it.

4 Notation and main results of this paper

We first give some notation.

Denote

$$N(\alpha)=\begin{cases}-\sum_{i=0}^{k_1}\operatorname{card}\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}<-\alpha\}\\ \quad-\sum_{i=k_1+1}^{k_2}\operatorname{card}\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}|<-\alpha\}, & \alpha<0,\\ \sum_{i=0}^{k_1}\operatorname{card}\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}<\alpha\}\\ \quad+\sum_{i=k_1+1}^{k_2}\operatorname{card}\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}|<\alpha\}, & \alpha\geq 0,\end{cases}$$

$$N(\beta)=\begin{cases}-\sum_{i=0}^{k_1}\operatorname{card}\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}<-\beta\}\\ \quad-\sum_{i=k_1+1}^{k_2}\operatorname{card}\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}|<-\beta\}, & \beta<0,\\ \sum_{i=0}^{k_1}\operatorname{card}\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}<\beta\}\\ \quad+\sum_{i=k_1+1}^{k_2}\operatorname{card}\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}|<\beta\}, & \beta\geq 0,\end{cases}$$

and

$$\begin{aligned}N^0(\alpha_-)={}&\sum_{i=0}^{k_1}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}=-\alpha\right\}\\&+\sum_{i=k_1+1}^{k_2}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\left|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right|=-\alpha\right\},\quad\alpha<0,\\ N^0(\alpha_+)={}&\sum_{i=0}^{k_1}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}=\alpha\right\}\\&+\sum_{i=k_1+1}^{k_2}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\left|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right|=\alpha\right\},\quad\alpha\geq 0,\\ N^0(\beta_-)={}&\sum_{i=0}^{k_1}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}=-\beta\right\}\\&+\sum_{i=k_1+1}^{k_2}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\left|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right|=-\beta\right\},\quad\beta<0,\\ N^0(\beta_+)={}&\sum_{i=0}^{k_1}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+2i+1)\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}=\beta\right\}\\&+\sum_{i=k_1+1}^{k_2}\operatorname{card}\left\{m\geq 0:0<\frac{(4mk+4k-2i-1)\pi}{4k}\left|\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}}\right|=\beta\right\},\quad\beta\geq 0.\end{aligned}$$

Now we give the main results of this paper.

Theorem 4.1 *Suppose that* (S_1) *and* (S_2) *hold. Then equation* (1.6) *possesses at least*

$$n = \max\{N(\beta) - N(\alpha) - N^0(\beta_-) - N^0(\alpha_+), N(\alpha) - N(\beta) - N^0(\alpha_-) - N^0(\beta_+)\}$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$, *provided that* $n > 0$.

Theorem 4.2 *Suppose that* (S_1), (S_2), (S_3^+), *and* (S_4^-) *hold. Then equation* (1.6) *possesses at least*

$$n = N(\beta) - N(\alpha) + N^0(\beta_+) + N^0(\alpha_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$, *provided that* $n > 0$.

Theorem 4.3 *Suppose that* (S_1), (S_2), (S_3^-), *and* (S_4^+) *hold. Then equation* (1.6) *possesses at least*

$$n = N(\alpha) - N(\beta) + N^0(\alpha_+) + N^0(\beta_-)$$

$4k$*-periodic solutions satisfying* $x(t-2k) = -x(t)$, *provided that* $n > 0$.

5 Proof of main results of this paper

Proof of Theorem 4.1 Suppose without loss of generality that

$$n = N(\beta) - N(\alpha) - N^0(\beta_-) - N^0(\alpha_+).$$

Let $X^+ = X_\infty^+$ and $X^- = X_0^-$. Then

$$X \setminus (X^+ \cup X^-) = X \setminus (X_\infty^+ \cup X_0^-) \subseteq X_\infty^0 \cup X_0^0 \cup (X_\infty^+ \cap X_0^-).$$

Obviously,

$$\operatorname{co}\dim_X(X^+ + X^-) \le \dim X_\infty^0 + \dim X_0^0 + \dim(X_\infty^+ \cap X_0^-) < \infty,$$

which implies that condition (a) in Lemma 1.1 holds. Let $A_\infty = \beta$. Then condition (b) in Lemma 1.1 holds since for each $j \in N$, we have that $x \in X(j)$ yields $(L + P^{-1}\beta)x \in X(j)$.

At the same time, Lemma 3.2 gives the $(P.S)$-condition.

Now it suffices to show that conditions (c) and (d) in Lemma 1.1 hold under assumptions (S_1) and (S_2).

In fact, condition (S_1) implies that on X^- we have $\Phi(x) > 0$ if $0 < \|x\| \ll 1$, that is, there are $r > 0$ and $c_\infty < 0$ such that

$$\Phi(x) \le c_\infty < 0 = \Phi(0), \quad \forall x \in X^- \cap S_r = \{x \in X : \|x\| = r\}.$$

On the other hand, we have shown in Lemma 3.1 that there is $\sigma > 0$ such that $\langle (L + P^{-1})x, x\rangle > \sigma \|x\|^2, x \in X_\infty^+$. On the other hand, $|F(x) - \frac{1}{2}\beta x^2| < \frac{1}{4}\sigma |x|^2 + M_1, x \in R$, for some $M_1 > 0$.

Then

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\langle (L+P^{-1}\beta)x,x\rangle + \int_0^{4k}\left[F(x(t)) - \frac{1}{2}\beta |x(t)|^2\right]dt \\ &\geq \frac{1}{2}\sigma \|x\|^2 - \frac{1}{4}\sigma\|x\|^2 - 4kM_1 \\ &\geq \frac{1}{4}\sigma\|x\|^2 - 4kM_1\end{aligned}$$

if $x \in X^+$. Clearly, there is $c_0 < c_\infty$ such that $\Phi(x) \geq c_0, x \in X^+$.

Our last task is to compute the value of

$$\begin{aligned}n &= \frac{1}{2}\left[\dim(X^+ \cap X^-) - \operatorname{co}\dim_X(X^+ + X^-)\right] \\ &= \frac{1}{2}\left[\dim(X^+_\infty \cap X^-_0) - \operatorname{co}\dim_X(X^+_\infty + X^-_0)\right] \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\left[\dim(X^+_\infty(j) \cap X^-_0(j)) - \operatorname{co}\dim_{X(j)}(X^+_\infty(j) + X^-_0(j))\right].\end{aligned}$$

By computation we get that, for each $i \in \{0,1,\ldots,k_2\}$,

$$\begin{aligned}&\langle (L+P^{-1}\beta)x,x\rangle = \left(-\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \frac{\beta}{4mk+2i+1}\right)\|x\|^2, \quad x \in X(2mk+i), \\ &\langle (L+P^{-1}\beta)x,x\rangle = \left(\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \frac{\beta}{4mk+4k-2i-1}\right)\|x\|^2, \\ &\quad x \in X(2mk+2k-i-1),\end{aligned}$$

and

$$\begin{aligned}&\langle (L+P^{-1}\alpha)x,x\rangle = \left(-\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \frac{\alpha}{4mk+2i+1}\right)\|x\|^2, \quad x \in X(2mk+i), \\ &\langle (L+P^{-1}\alpha)x,x\rangle = \left(\frac{\pi}{4k}\frac{\cos\frac{(2i+1)\pi}{4k}}{\sin\frac{(2i+1)3\pi}{4k}} + \frac{\alpha}{4mk+4k-2i-1}\right)\|x\|^2, \\ &\quad x \in X(2mk+2k-i-1).\end{aligned}$$

Therefore,

$$\begin{aligned}&X^+_\infty(2mk+i) = X^+_\infty \cap X(2mk+i) = \emptyset, \\ &X^+_\infty(2mk+2k-i-1) = X^+_\infty \cap X(2mk+2k-i-1) = X(2mk+2k-i-1), \\ &X^-_0(2mk+i) = X^-_0 \cap X(2mk+i) = X(2mk+i), \\ &X^-_0(2mk+2k-i-1) = X^-_0 \cap X(2mk+2k-i-1) = \emptyset\end{aligned}$$

if $i \in \{0,1,\ldots,k_1\}$ and $m \geq 0$ is large enough,

$$X^+_\infty(2mk+i) = X^+_\infty \cap X(2mk+i) = X(2mk+i),$$

$$X_\infty^+(2mk+2k-i-1)=X_\infty^+\cap X(2mk+2k-i-1)=\emptyset,$$
$$X_0^-(2mk+i)=X_0^-\cap X(2mk+i)=\emptyset,$$
$$X_0^-(2mk+2k-i-1)=X_0^-\cap X(2mk+2k-i-1)=X(2mk+2k-i-1)$$

if $i\in\{k_1+1,\dots,k_2\}$ and $m\geq 0$ is large enough, which means that there is $M>0$ such that $\dim(X_\infty^+(j)\cap X_0^-(j))-\operatorname{co}\dim_X(X_\infty^+(j)+X_0^-(j))=0, j>M$, from which it follows that

$$\begin{aligned}
n&=\frac{1}{2}\sum_{j=0}^{M}\bigl[\dim\bigl(X_\infty^+(j)\cap X_0^-(j)\bigr)-\operatorname{co}\dim_{X(j)}\bigl(X_\infty^+(j)+X_0^-(j)\bigr)\bigr]\\
&=\frac{1}{2}\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)-2\bigr]\\
&=\frac{1}{2}\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)\bigr]-(M+1).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{j=0}^{M}\dim\bigl(X_\infty^+(j)\bigr)\\
&\quad=2\begin{cases}
N(\beta)+\operatorname{card}\{2mk+2k-i-1:0\leq 2mk+2k-i-1\leq M,0\leq i\leq k_1\}\\
\qquad+\operatorname{card}\{2mk+i:0\leq 2mk+i\leq M,k_1+1\leq i\leq k_2\},\quad \beta>0,\\
N(\beta)-N^0(\beta_-)+\operatorname{card}\{2mk+2k-i-1:0\leq 2mk+2k-i-1\leq M,0\leq i\leq k_1\}\\
\qquad+\operatorname{card}\{2mk+i:0\leq 2mk+i\leq M,k_1+1\leq i\leq k_2\},\quad \beta<0,
\end{cases}\\
&\sum_{j=0}^{M}\dim\bigl(X_0^-(j)\bigr)\\
&\quad=2\begin{cases}
-N(\alpha)-N^0(\alpha_+)+\operatorname{card}\{2mk+i:0\leq 2mk+i\leq M,0\leq i\leq k_1\}\\
\qquad+\operatorname{card}\{2mk+2k-i-1:0\leq 2mk+2k-i-1\leq M,k_1+1\leq i\leq k_2\},\\
\quad\alpha>0,\\
-N(\alpha)+\operatorname{card}\{2mk+i:0\leq 2mk+i\leq M,0\leq i\leq k_1\}\\
\qquad+\operatorname{card}\{2mk+2k-i-1:0\leq 2mk+2k-i-1\leq M,k_1+1\leq i\leq k_2\},\\
\quad\alpha<0,
\end{cases}
\end{aligned}$$

and

$$\sum_{j=0}^{M}\bigl[\dim X_\infty^+(j)+\dim X_0^-(j)\bigr]=2\bigl[N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+)\bigr]+2(M+1). \tag{5.1}$$

Therefore,

$$n=N(\beta)-N(\alpha)-N^0(\beta_-)-N^0(\alpha_+).$$

Theorem 4.1 is proved. □

Proof of Theorem 4.2 *and Theorem* 4.3 Since the proof for the two theorems is almost the same, we prove only Theorem 4.2.

Let $X^+ = X^+_\infty + X^0_\infty, X^- = X^0_- + X^0_0$. Then as in the proof of Theorem 4.1, we check conditions (a), (b), (c), (d), and (e). In the present case, we may suppose that (5.1) still holds for some $M > 0$. Let $X^0_\infty(i) = X^0_\infty \cap X(i), X^0_0(i) = X^0_0 \cap X(i)$. Then

$$\begin{aligned} n &= \frac{1}{2}\sum_{i=0}^{M}\bigl[\dim\bigl(X^+_\infty(i)\cap X^-_0(i)\bigr) - \operatorname{co}\dim_{X(i)}\bigl(X^+_\infty(i) + X^-_0(i)\bigr)\bigr] + \bigl(\dim X^0_\infty + \dim X^0_0\bigr) \\ &= \frac{1}{2}\sum_{i=0}^{M}\bigl[\dim X^+_\infty(i) + \dim X^-_0(i) - 2\bigr] + \bigl(\dim X^0_\infty + \dim X^0_0\bigr) \\ &= \frac{1}{2}\sum_{i=0}^{M}\bigl[\dim X^+_\infty(i) + \dim X^-_0(i)\bigr] - (M+1) + \bigl(\dim X^0_\infty + \dim X^0_0\bigr) \\ &= N(\beta) - N(\alpha) - N^0(\beta_-) - N^0(\alpha_+) + \bigl(N^0(\beta_+) + N^0(\beta_-) + N^0(\alpha_+) + N^0(\alpha_-)\bigr) \\ &= N(\beta) - N(\alpha) + N^0(\beta_+) + N^0(\alpha_-). \end{aligned}$$

Our proof is completed. □

6 An example

Example Suppose that $f \in C^0(R,R)$ is such that

$$f(x) = \begin{cases} \frac{5\pi}{2}x + x^{\frac{1}{3}}, & |x| \gg 1, \\ -\frac{7\pi}{2}x - x^3, & |x| \ll 1. \end{cases}$$

We are to discuss the multiplicity of 12-periodic solutions of the equation

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x(t-5)\bigr). \tag{6.1}$$

In this case, $k = 3, k_1 = 1, k_2 = 2, \alpha = -\frac{7\pi}{2}, \beta = \frac{5\pi}{2}$. This yields that

$$\begin{aligned} N(\alpha) = &-\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+12-1)\pi}{12}\frac{\cos\frac{\pi}{12}}{\sin\frac{3\pi}{12}} < \frac{7\pi}{2}\right\} \\ &-\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+12-3)\pi}{12}\frac{\cos\frac{3\pi}{12}}{\sin\frac{9\pi}{12}} < \frac{7\pi}{2}\right\} \\ &-\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+5)\pi}{12}\left|\frac{\cos\frac{5\pi}{12}}{\sin\frac{15\pi}{12}}\right| < \frac{7\pi}{2}\right\} = -15, \\ N(\beta) = &\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+1)\pi}{12}\frac{\cos\frac{\pi}{12}}{\sin\frac{3\pi}{12}} < \frac{5\pi}{2}\right\} \\ &+\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+3)\pi}{12}\frac{\cos\frac{3\pi}{12}}{\sin\frac{9\pi}{12}} < \frac{5\pi}{2}\right\} \\ &+\operatorname{card}\left\{m \ge 0 : 0 < \frac{(12m+12-5)\pi}{12}\left|\frac{\cos\frac{5\pi}{12}}{\sin\frac{15\pi}{12}}\right| < \frac{5\pi}{2}\right\} = 12, \\ N^0(\alpha_+) = &N^0(\beta_-) = N^0(\alpha_-) = N^0(\beta_+) = 0. \end{aligned}$$

Applying Theorem 4.2, we conclude that equation (6.1) possesses at least 27 different 12-periodic orbits satisfying $x(t-6) = -x(t)$ since f satisfies (S_1), (S_2), (S_3^+), and (S_4^-).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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