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Stability and approximate analytic solutions of the fractional Lotka-Volterra equations for three competitors

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Abstract

This paper studies the fractional Lotka-Volterra equations for three competitors, since the fractional derivatives possess the properties of good memory and have great biological significance. First of all, the equilibrium points and asymptotic stability for the equations are studied by the stability analysis method. As expected, the fractional-order differential equations are, at least, as stable as their integer-order counterpart. Second, some approximate analytic solutions for this systems are obtained by the stability analysis method and the homotopy perturbation method, which are expressed in the form of the Mittag-Leffler function. The results show that it takes less time for the population to get close to the equilibrium point as the time derivatives increase. Comparing with the classical ones, the fractional Lotka-Volterra equations, no matter whether the fractional derivatives have big or small order, if both take more time, even multiplied time, for the system to reach the equilibrium point, then that may explain the memory properties. Furthermore, some numerical analyses are carried out and verify the theoretical analysis.

Keywords: fractional Lotka-Volterra equations; asymptotic stability; approximate analytic solutions; homotopy perturbation method

1 Introduction

The classical Lotka-Volterra problem was introduced as a model for undamped oscillating chemical reactions and the dynamics of ecological systems with predator-prey interactions by Lotka [1] and Volterra [2], respectively. For the model with a linear functional response of predator it is given by

$$\frac{dx}{dt} = x(a_1 + b_1 x + c_1 y), \tag{1}$$

$$\frac{dy}{dt} = y(a_2 + b_2 x + c_2 y), \tag{2}$$

where $x = x(t)$, $y = y(t)$ are the densities of the prey and predator populations at tine t. Generally for the n competing species, the following first-order differential equations describe the dynamics of the competing populations [3]:

$$\frac{dx_i}{dt} = k_i x_i \left(1 - \sum_{j=1}^{n} c_{ij} x_j \right), \tag{3}$$

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where $x_i = x_i(t)$ is the number of individuals in the ith population at time t, k_i is the intrinsic growth rate of the ith population, and the competition coefficients c_{ij} describe the extent to which the jth species affects the growth rate of the ith species.

Since the classical Lotka-Volterra model has been applied to all kinds of problem in population biology, neural networks, and others, this model has been studied by many ecologists and mathematicians. For now, many important and interesting results have been found, such as the existence and uniqueness of solutions [4, 5], global asymptotic stability [6], Hopf bifurcation [7], and extinction [8]. For some other research results, see [9, 10] and the references therein.

However, most biological systems have memory properties, and the evolution of each species is better described by the generation of Brownian motions. Thus one needs some innovative model to describe the evolution behavior. Fortunately, a new tool called fractional calculus has been found and applied in various systems since fractional derivatives possess the properties of good memory and being hereditary [11]. Also, fractional calculus has been successfully applied in mathematical biology, such as the nonlocal epidemics [12], or the fractional logistic model [13]. More importantly, the corresponding fractional Lotka-Volterra equations are established in describing the dynamical behaviors for an arbitrary number of competitors:

$$D_t^{\alpha_i} x_i = k_i x_i \left(1 - \sum_{j=1}^{n} c_{ij} x_j\right), \tag{4}$$

where $D_t^{\alpha_i}$ is for the fractional derivatives and $0 < \alpha_i < 1$. The fractional Lotka-Volterra equations are obtained from the classical equations by replacing the first-order time derivatives by the derivatives of fractional order. Although the system (4) is simplified under the assumption that the competition functions between the species are linear, they still show difficulty in research. Some researchers have studied the system for two species. Ahmed *et al.* studied the equilibrium points and stability for the fractional-order predator-prey and rabies models [14]. Srivastava *et al.* analyzed the synchronization of chaotic [15]. Meanwhile, some approximate solutions are also searched for by various methods, such as the homotopy perturbation method [16, 17], or the variational iteration method [18]. Some other important results, such as stability, bifurcation, and numerical solutions are also being carried out; see [19, 20] and the references therein.

But for the fractional Lotka-Volterra equations between multi-populations, there are few biological and mathematical results to the best of our knowledge. Therefore, in this paper, we are trying to consider some basic questions for the multi-population fractional Lotka-Volterra equations, and we come first to consider the following initial problem for three competitors, labeled X, Y and Z:

$$D_t^{\alpha} x = k_1 x (1 - c_{11} x - c_{12} y - c_{13} z), \tag{5}$$

$$D_t^{\beta} y = k_2 y (1 - c_{21} x - c_{22} y - c_{23} z), \tag{6}$$

$$D_t^{\gamma} z = k_3 z (1 - c_{31} x - c_{32} y - c_{33} z), \tag{7}$$

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0. \tag{8}$$

For the three competitor system, there are 12 parameters. Although some parameters can be absorbed by renormalizing the populations, there are left most disposable parameters

characterizing the system. So in order to simplify the problem, we make the following symmetry assumptions: (1) $k_1 = k_2 = k_3$; (2) with respect to competition, Y affects X as Z affects Y as X affects Z, which means $c_{12} = c_{23} = c_{31} = a$; (3) similarly one arrives at $c_{21} = c_{32} = c_{13} = b$. Furthermore, we rescale the populations so that effectively $c_{11} = c_{22} = c_{33} = 1$, and we rescale t so that in effect $k = 1$, then we have the following equivalent system:

$$D_t^\alpha x = x(1 - x - ay - bz), \tag{9}$$

$$D_t^\beta y = y(1 - bx - y - az), \tag{10}$$

$$D_t^\gamma z = z(1 - ax - by - z), \tag{11}$$

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0. \tag{12}$$

Thus in the following, we mainly study the initial problem for the system (9)-(12). Specifically, we are going to study the asymptotic stability and approximate analytic solutions for the system with the help of a stability analysis and homotopy perturbation method (HPM). The HPM was first proposed by He to find the approximate analytical solution for linear and nonlinear problems [21–23]. This method was successfully applied to all kinds of partial differential equations [16, 17, 24], since it does not require small parameters in the equations, which overcomes the limitations of the other traditional perturbation techniques. One should notice that it is plausible that the qualitative features of the system (5)-(8) will remain true in the more general asymmetric case. We will discuss the situation in our further study.

The rest of paper is organized as follows. In Section 2, we briefly give some preliminaries. In Section 3, we will study the equilibrium points, asymptotic stability, and the approximate analytic solutions for the fractional Lotka-Volterra system (9)-(12) with the help of a stability analysis. In Section 4, other forms of approximate analytic solutions are studied by the homotopy perturbation method. In Section 5, some numerical results are presented to verify the theoretical analysis. In Section 6, we draw some conclusions.

2 Some preliminaries

In this section, we will briefly give some preliminaries of the fractional calculus and the homotopy perturbation method, which will be used in the next sections.

2.1 Fractional calculus

Since the fractional calculus was proposed in the last two centuries, there are mainly three kinds of definitions to describe the fractional-order derivatives: the Grünwald-Letnikov derivative, the Riemann-Liouville derivative, and the Caputo derivative [11]. However, the Caputo derivative, which we will use in this paper, is the one most commonly employed in real applications, since it has the advantage of only requiring initial conditions given in terms of integer-order derivatives, unlike the Riemann-Liouville fractional derivative, requiring initial conditions to be expressed in terms of fractional integrals and their derivatives, while the initial conditions represent well-understood features of a physical situation. The fractional derivative of $f(t)$ in the Caputo sense is defined as [25]

$$D_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} J_t^{m-\alpha}\left(\frac{\partial^m f(t)}{\partial t^m}\right), & \text{for } m - 1 < \alpha < m, \\ \frac{\partial^m f(t)}{\partial t^m}, & \text{for } \alpha = m \in N, \end{cases} \tag{13}$$

where $\alpha > 0$ and $J_t^\beta f(t)$ $(\beta > 0)$ is the Riemann-Liouville fractional integral of $f(t)$ defined by

$$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(\tau)\, d\tau. \tag{14}$$

There $\Gamma(\cdot)$ is Euler's gamma function. D_t^α and J_t^β are called the Caputo fractional derivative operator and Riemann-Liouville fractional integral operator, respectively.

For the properties of the fractional integrals and derivatives, we have the following results [11].

Lemma 1 *If $m - 1 < \alpha \le m$, and $f(t) \in L_1[0, T]$ the usual Lebesque space, then*

$$J_t^\alpha C = \frac{C}{\Gamma(1 + \alpha)} t^\alpha, \quad C = \text{constant}, \tag{15}$$

$$J_t^\alpha t^\mu = \frac{\Gamma(1 + \mu)}{\Gamma(1 + \mu + \alpha)} t^{\mu+\alpha}, \quad \mu > 0, \tag{16}$$

$$J_t^\alpha J_t^\beta f(t) = J_t^\beta J_t^\alpha f(t) = J_t^{\alpha+\beta} f(t), \quad \beta > 0, \tag{17}$$

$$D_t^\alpha C = 0, \quad C = \text{constant}, \tag{18}$$

$$D_t^\alpha t^\mu = \frac{\Gamma(1 + \mu)}{\Gamma(1 + \mu - \alpha)} t^{\mu-\alpha}, \quad \mu > \alpha - 1, \tag{19}$$

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0_+) \frac{t^k}{k!}. \tag{20}$$

Lemma 2 *If $0 < \alpha < 1$, and considering the following initial value problem for a non-homogeneous fractional equation*:

$$\begin{cases} D_t^\alpha u(t) - u(t) = f(t), \quad t > 0, \\ u(0) = 0, \end{cases} \tag{21}$$

then the solution for the problem can be solved as

$$u(t) = \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}\big((t - \tau)^\alpha\big) f(\tau)\, d\tau, \tag{22}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by [11]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0. \tag{23}$$

Proof These results can be proved easily with the help of the Laplace transform and the properties of the Mittag-Leffler function. We omit the details for simplicity. □

Remark The Mittag-Leffler function (23) was in fact introduced by Agarwal [26] and is the two-parameter generalization of the exponential function e^z, which plays a very important role in the theory of population biology. The exponential function can be found

in the classical Malthus model and logistic model, and thus the Mittag-Leffler function is more generalized to describe a real-world growing population.

Lemma 3 *Let $\alpha > 0$, $\beta > 0$, and $\gamma > 0$, then for the Mittag-Leffler function $E_{\alpha,\beta}(z)$, we have the following results* [11]:

$$\int_0^\infty e^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm at^\alpha)\, dt = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \quad (\mathrm{Re}(s) > |a|^{1/\alpha}), \tag{24}$$

where $E_{\alpha,\beta}^{(k)}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y)$. And

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(\lambda\tau^\alpha)\, d\tau = t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha) \quad (\beta > 0); \tag{25}$$

$$\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha)\tau^\beta E_{\beta,\beta+1}(\tau^\beta)\, d\tau$$

$$= t^{\alpha+\beta} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{t^{\alpha m+\beta n}}{\Gamma(\alpha m + \beta n + \alpha + \beta + 1)}. \tag{26}$$

2.2 Homotopy perturbation method

In order to illustrate the basic ideas of the homotopy perturbation method [21–23], we consider the following boundary value problem for the nonlinear differential equation:

$$\begin{cases} A(u) + f(\mathbf{r}) = 0, & \mathbf{r} \in \Omega, \\ B(u, \partial u/\partial n) = 0, & \mathbf{r} \in \boldsymbol{\Gamma}, \end{cases} \tag{27}$$

where A is a general differential operator, which can be divided into linear parts L and nonlinear parts N generally, B is a boundary operator, $f(\mathbf{r})$ is a well-known analytic function, $\boldsymbol{\Gamma}$ is the boundary of the domain Ω.

According to the homotopy technique, one can construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies

$$\mathcal{H}(v,p) = (1-p)\big[L(v) - L(u_0)\big] + p\big[A(v) + f(\mathbf{r})\big] = 0, \quad p \in [0,1], \mathbf{r} \in \Omega, \tag{28}$$

or

$$\mathcal{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p\big[N(v) - f(\mathbf{r})\big] = 0, \tag{29}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions.

Let us suppose that the solution of (28) or (29) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots. \tag{30}$$

Then setting $p = 1$ in the above equation one can obtain the approximate solution for (27)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{31}$$

The convergence of the series of (31) has been proved in [21–23].

3 Equilibrium points and their approximate solutions

In this section, we first consider the equilibrium points and the asymptotic stability for the system (9)-(12), and furthermore we obtain their approximate solutions by the Mittag-Leffler functions. First for evaluating the equilibrium points, one lets $D_t^\alpha x = 0$, $D_t^\beta y = 0$, $D_t^\gamma z = 0$, which imply

$$
\begin{cases}
D_t^\alpha x = f_1(x,y,z) = x(1 - x - ay - bz) = 0, \\
D_t^\beta y = f_2(x,y,z) = y(1 - bx - y - az) = 0, \\
D_t^\gamma z = f_3(x,y,z) = z(1 - ax - by - z) = 0.
\end{cases}
\tag{32}
$$

Then the equilibrium points $(x^{\mathrm{eq}}, y^{\mathrm{eq}}, z^{\mathrm{eq}})$ can be solved from the above systems. On the other hand, to evaluate the asymptotic stability, let $\xi = [\xi_1(t), \xi_2(t), \xi_3(t)]^T$ and

$$
x(t) = x^{\mathrm{eq}} + \xi_1(t), \qquad y(t) = y^{\mathrm{eq}} + \xi_2(t), \qquad z(t) = z^{\mathrm{eq}} + \xi_3(t).
\tag{33}
$$

Substituting (33) into the system (32) and noticing $f_i(x^{\mathrm{eq}}, y^{\mathrm{eq}}, z^{\mathrm{eq}}) = 0$ $(i = 1, 2, 3)$, then we have

$$
\begin{cases}
D_t^\alpha \xi_1(t) \approx \frac{\partial f_1}{\partial x}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + \frac{\partial f_1}{\partial y}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 + \frac{\partial f_1}{\partial z}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3 \\
\quad = (1 - 2x - ay - bz)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + (-ax)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 \\
\qquad + (-bx)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3, \\
D_t^\beta \xi_2(t) \approx \frac{\partial f_2}{\partial x}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + \frac{\partial f_2}{\partial y}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 + \frac{\partial f_2}{\partial z}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3, \\
\quad = (-by)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + (1 - bx - 2y - az)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 \\
\qquad + (-ay)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3, \\
D_t^\gamma \xi_3(t) \approx \frac{\partial f_3}{\partial x}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + \frac{\partial f_3}{\partial y}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 + \frac{\partial f_3}{\partial z}\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3 \\
\quad = (-az)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_1 + (-by)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_2 \\
\qquad + (1 - ax - by - 2z)\big|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}\xi_3, \\
\xi_1(0) = x_0 - x^{\mathrm{eq}}, \qquad \xi_2(0) = y_0 - y^{\mathrm{eq}}, \qquad \xi_3(0) = z_0 - z^{\mathrm{eq}}.
\end{cases}
\tag{34}
$$

Then we can rewrite the system as

$$
D_t^* \xi = A\xi,
\tag{35}
$$

where

$$
D_t^* \xi = \begin{bmatrix} D_t^\alpha \xi_1 \\ D_t^\beta \xi_2 \\ D_t^\gamma \xi_3 \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},
\tag{36}
$$

and

$$
a_{ij} = \frac{\partial f_i}{\partial x_j}\bigg|_{(x^{\mathrm{eq}},y^{\mathrm{eq}},z^{\mathrm{eq}})}, \quad x_j = [x,y,z]^T, i,j = 1,2,3.
\tag{37}
$$

On the other hand, for the coefficient matrix A, we can get the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and the eigenvectors B of A such that

$$
B^{-1}AB = C,
\tag{38}
$$

where C is a diagonal matrix given by

$$C = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{39}$$

From (38), we find $A = BCB^{-1}$ and we substitute into (35); we have

$$D_t^* \xi = BCB^{-1}\xi \tag{40}$$

and

$$D_t^* \left(B^{-1}\xi\right) = C\left(B^{-1}\xi\right). \tag{41}$$

If we set

$$\eta = B^{-1}\xi, \tag{42}$$

where $\eta = [\eta_1(t), \eta_2(t), \eta_3(t)]^T$, then we obtain $D_t^*\eta = C\eta$, and

$$\begin{cases} D_t^\alpha \eta_1 = \lambda_1 \eta_1, \\ D_t^\beta \eta_2 = \lambda_2 \eta_2, \\ D_t^\gamma \eta_3 = \lambda_3 \eta_3. \end{cases} \tag{43}$$

The solutions of (43) are given by the Mittag-Leffler functions [11]:

$$\eta_1(t) = \sum_{k=0}^{\infty} \frac{(\lambda_1)^k t^{k\alpha}}{\Gamma(1+k\alpha)} \eta_1(0) = E_\alpha\left(\lambda_1 t^\alpha\right)\eta_1(0), \tag{44}$$

$$\eta_2(t) = \sum_{k=0}^{\infty} \frac{(\lambda_2)^k t^{k\beta}}{\Gamma(1+k\beta)} \eta_2(0) = E_\beta\left(\lambda_2 t^\beta\right)\eta_2(0), \tag{45}$$

$$\eta_3(t) = \sum_{k=0}^{\infty} \frac{(\lambda_3)^k t^{k\gamma}}{\Gamma(1+k\gamma)} \eta_3(0) = E_\gamma\left(\lambda_3 t^\gamma\right)\eta_3(0), \tag{46}$$

where

$$\begin{bmatrix} \eta_1(0) \\ \eta_2(0) \\ \eta_3(0) \end{bmatrix} = B^{-1} \begin{bmatrix} \xi_1(0) \\ \xi_2(0) \\ \xi_3(0) \end{bmatrix}. \tag{47}$$

Finally, combining (33), (42), and (45)-(47), we can obtain the solutions for the system (9)-(12).

Furthermore, using the result of Matignon [27] then if the eigenvalues of the matrix A are negative ($|\arg(\lambda_1)| > \alpha\pi/2$, $|\arg(\lambda_2)| > \beta\pi/2$, $|\arg(\lambda_3)| > \gamma\pi/2$), the equilibrium point $(x^{\mathrm{eq}}, y^{\mathrm{eq}}, z^{\mathrm{eq}})$ is locally asymptotically stable. This result confirms the statement that fractional-order differential equations are, at least, as stable as their integer-order counterpart [14].

In order to illustrate the points specifically, we use the *Maple* software to compute each of the terms. First, the equilibrium points for the system (32) can be solved easily and they are $(x^{\text{eq}}, y^{\text{eq}}, z^{\text{eq}}) = (0,0,0), (0,0,1), (0,1,0), (1,0,0), (0, \frac{a-1}{ab-1}, \frac{b-1}{ab-1}), (\frac{b-1}{ab-1}, 0, \frac{a-1}{ab-1}), (\frac{a-1}{ab-1}, \frac{b-1}{ab-1}, 0), (\frac{1}{a+b+1}, \frac{1}{a+b+1}, \frac{1}{a+b+1})$. In the following, we take $(x^{\text{eq}}, y^{\text{eq}}, z^{\text{eq}}) = (\frac{1}{a+b+1}, \frac{1}{a+b+1}, \frac{1}{a+b+1})$ for consideration as an example, the other equilibrium points can be analyzed in the same way. Second, the matrices A and B and the eigenvalue λ can be calculated accordingly:

$$\lambda_1 = -1, \qquad \lambda_2 = \frac{a+b-2-i\sqrt{3}(a-b)}{2(a+b+1)}, \qquad \lambda_3 = \frac{a+b-2+i\sqrt{3}(a-b)}{2(a+b+1)}, \tag{48}$$

$$A = \begin{bmatrix} -\frac{1}{a+b+1} & -\frac{a}{a+b+1} & -\frac{b}{a+b+1} \\ -\frac{b}{a+b+1} & -\frac{1}{a+b+1} & -\frac{a}{a+b+1} \\ -\frac{a}{a+b+1} & -\frac{b}{a+b+1} & -\frac{1}{a+b+1} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 1 & 1 & 1 \end{bmatrix}. \tag{49}$$

Then from (34) and (47), we have

$$\begin{bmatrix} \eta_1(0) \\ \eta_2(0) \\ \eta_3(0) \end{bmatrix} = B^{-1} \begin{bmatrix} \xi_1(0) \\ \xi_2(0) \\ \xi_3(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-1+\sqrt{3}i}{6} & \frac{-1-\sqrt{3}i}{6} & \frac{1}{3} \\ \frac{-1-\sqrt{3}i}{6} & \frac{-1+\sqrt{3}i}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_0 - \frac{1}{a+b+1} \\ y_0 - \frac{1}{a+b+1} \\ z_0 - \frac{1}{a+b+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(x_0 + y_0 + z_0 - \frac{3}{a+b+1}) \\ \frac{-1+\sqrt{3}i}{6}x_0 + \frac{-1-\sqrt{3}i}{6}y_0 + \frac{1}{3}z_0 \\ \frac{-1-\sqrt{3}i}{6}x_0 + \frac{-1+\sqrt{3}i}{6}y_0 + \frac{1}{3}z_0 \end{bmatrix}. \tag{50}$$

Thus from (33), (42), and (44)-(46), we can obtain the solutions for the system (9)-(12):

$$x(t) = x^{\text{eq}} + \xi_1(t)$$

$$= \frac{1}{a+b+1} + \eta_1(t) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\eta_2(t) - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\eta_3(t)$$

$$= \frac{1}{a+b+1} + E_\alpha(-t^\alpha)\eta_1(0)$$

$$- \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)E_\beta\big(\big((a+b-2-i\sqrt{3}(a-b))/(2(a+b+1))\big)t^\beta\big)\eta_2(0)$$

$$- \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)E_\gamma\big(\big((a+b-2+i\sqrt{3}(a-b))/(2(a+b+1))\big)t^\beta\big)\eta_3(0), \tag{51}$$

$$y(t) = y^{\text{eq}} + \xi_2(t)$$

$$= \frac{1}{a+b+1} + \eta_1(t) - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\eta_2(t) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\eta_3(t)$$

$$= \frac{1}{a+b+1} + E_\alpha(-t^\alpha)\eta_1(0)$$

$$- \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)E_\beta\big(\big((a+b-2-i\sqrt{3}(a-b))/(2(a+b+1))\big)t^\beta\big)\eta_2(0)$$

$$- \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)E_\gamma\big(\big((a+b-2+i\sqrt{3}(a-b))/(2(a+b+1))\big)t^\beta\big)\eta_3(0), \tag{52}$$

and

$$z(t) = z^{\text{eq}} + \xi_3(t)$$

$$= \frac{1}{a+b+1} + \eta_1(t) + \eta_2(t) + \eta_3(t)$$

$$= \frac{1}{a+b+1} + E_\alpha\left(-t^\alpha\right)\eta_1(0)$$

$$+ E_\beta\left(\left(\left(a+b-2-i\sqrt{3}(a-b)\right)/\left(2(a+b+1)\right)\right)t^\beta\right)\eta_2(0)$$

$$+ E_\gamma\left(\left(\left(a+b-2+i\sqrt{3}(a-b)\right)/\left(2(a+b+1)\right)\right)t^\beta\right)\eta_3(0), \tag{53}$$

where $\eta_1(0)$, $\eta_2(0)$, and $\eta_3(0)$ are obtained in (49).

At last, according to asymptotic stability results, the equilibrium point $(x^{\text{eq}}, y^{\text{eq}}, z^{\text{eq}}) = (\frac{1}{a+b+1}, \frac{1}{a+b+1}, \frac{1}{a+b+1})$ is locally asymptotically stable if $0 < a + b < 2$. That is to say, if the competition affects the coefficients among the three populations a and b satisfying the condition $0 < a + b < 2$, then the densities of the populations X, Y, and Z will eventually trend to be stable and close to the equilibrium point $(x^{\text{eq}}, y^{\text{eq}}, z^{\text{eq}})$. The numerical simulations in the following section will also support this result.

4 Approximate solutions by HPM

In this section, we will employ the HPM to obtain the approximate solutions for the fractional multispecies Lotka-Volterra (9)-(12). First according to the basic idea of the HAM, we can construct the following homotopies (u, v, w) for the system:

$$\left(D_t^\alpha u - u\right) - \left(D_t^\alpha x_0 - x_0\right) + p\left(D_t^\alpha x_0 - x_0\right) + p\left(u^2 + auv + buw\right) = 0, \tag{54}$$

$$\left(D_t^\beta v - v\right) - \left(D_t^\beta y_0 - v_0\right) + p\left(D_t^\beta y_0 - y_0\right) + p\left(buv + v^2 + avw\right) = 0, \tag{55}$$

$$\left(D_t^\gamma w - w\right) - \left(D_t^\gamma z_0 - z_0\right) + p\left(D_t^\gamma z_0 - z_0\right) + p\left(auw + bvw + w^2\right) = 0, \tag{56}$$

where the homotopy parameter $p \in [0, 1]$. Then we suppose that the solutions of the system (54)-(56) can be expressed as a power series in p as follows:

$$u(t) = \sum_{i=0}^{\infty} p^i u_i(t) = u_0(t) + pu_1(t) + p^2 u_2(t) + p^3 u_3(t) + \cdots, \tag{57}$$

$$v(t) = \sum_{i=0}^{\infty} p^i v_i(t) = v_0(t) + pv_1(t) + p^2 v_2(t) + p^3 v_3(t) + \cdots, \tag{58}$$

$$w(t) = \sum_{i=0}^{\infty} p^i w_i(t) = w_0(t) + pw_1(t) + p^2 w_2(t) + p^3 w_3(t) + \cdots, \tag{59}$$

where the $u_i(t)$, $v_i(t)$, $w_i(t)$ ($i = 0, 1, \ldots$) are functions yet to be determined. And we suppose that $u_i(0)$, $v_i(0)$, $w_i(0)$ for $i = 1, 2, 3, \ldots$. Substituting (57)-(59) into (54)-(56) and equating the terms with identical powers of p, we can obtain the following set of linear differential equations:

$$p^0: \quad D_t^\alpha u_0 - u_0 - \left(D_t^\alpha x_0 - x_0\right) = 0,$$

$$D_t^\alpha v_0 - v_0 - \left(D_t^\alpha y_0 - y_0\right) = 0, \tag{60}$$

$$D_t^\alpha w_0 - w_0 - \left(D_t^\alpha z_0 - z_0\right) = 0,$$

$$p^1: \quad D_t^\alpha u_1 - u_1 = -(D_t^\alpha x_0 - x_0) - (u_0^2 + au_0 v_0 + bu_0 w_0),$$

$$D_t^\beta v_1 - v_1 = -(D_t^\beta y_0 - y_0) - (bu_0 v_0 + v_0^2 + av_0 w_0), \tag{61}$$

$$D_t^\gamma w_1 - w_1 = -(D_t^\gamma z_0 - z_0) - (au_0 w_0 + bv_0 w_0 + w_0^2),$$

$$p^2: \quad D_t^\alpha u_2 - u_2 = -[2u_0 u_1 + a(u_0 v_1 + u_1 v_0) + b(u_0 w_1 + u_1 w_0)],$$

$$D_t^\beta v_2 - v_2 = -[b(u_0 v_1 + u_1 v_0) + 2v_0 v_1 + a(v_0 w_1 + v_1 w_0)], \tag{62}$$

$$D_t^\gamma w_2 - w_2 = -[a(u_0 w_1 + u_1 w_0) + b(v_0 w_1 + v_1 w_0) + 2w_0 w_1],$$

$$p^3: \quad D_t^\alpha u_3 - u_3 = -[(u_0 u_2 + u_1^2 + u_2 u_0) + a(u_0 v_2 + u_1 v_1 + u_2 v_0)$$
$$+ b(u_0 w_2 + u_1 w_1 + u_2 w_0)],$$

$$D_t^\beta v_3 - v_3 = -[b(u_0 v_2 + u_1 v_1 + u_2 v_0) + (v_0 v_2 + v_1^2 + v_2 v_0)$$
$$+ a(v_0 w_2 + v_1 w_1 + v_2 w_0)], \tag{63}$$

$$D_t^\gamma w_3 - w_3 = -[a(u_0 w_2 + u_1 w_1 + u_2 w_0) + b(v_0 w_2 + v_1 w_1 + v_2 w_0)$$
$$+ (w_0 w_2 + w_1^2 + w_2 w_0)],$$

and the general equations for p^k:

$$p^k: \quad D_t^\alpha u_k - u_k = -\sum_{i=0}^{k-1}(u_i u_{k-1-i} + au_i v_{k-1-i} + bu_i w_{k-1-i}),$$

$$D_t^\beta v_k - v_k = -\sum_{i=0}^{k-1}(bu_i v_{k-1-i} + v_i v_{k-1-i} + av_i w_{k-1-i}), \tag{64}$$

$$D_t^\gamma w_k - w_k = -\sum_{i=0}^{k-1}(au_i w_{k-1-i} + bv_i w_{k-1-i} + w_i w_{k-1-i}).$$

Then we can obtain $u_i(t)$, $v_i(t)$, $w_i(t)$ $(i = 0, 1, \ldots)$ in a recurrent manner. Indeed, when computing $u_k(t)$, $v_k(t)$, $w_k(t)$ $(k \geq 1)$, one can find the right hand sides of (64) to depend only on $u_i(t)$, $v_i(t)$, $w_i(t)$ $(i = 0, 1, k-1)$, which are already known. Then applying the results of (22) in Lemma 2, we can determine the terms $u_k(t)$, $v_k(t)$, $w_k(t)$. Thus every term of $u_i(t)$, $v_i(t)$, $w_i(t)$ $(i = 0, 1, \ldots)$ can be calculated.

Using *Maple* software and combining with Lemma 3, we can obtain the explicit formulations for the $u_i(t)$, $v_i(t)$, $w_i(t)$ $(i = 0, 1, \ldots)$. The first few terms are derived as follows:

$$\begin{cases} u_0(t) = x_0, \\ v_0(t) = y_0, \\ w_0(t) = z_0, \end{cases} \tag{65}$$

$$\begin{cases} u_1(t) = A_1 * (t^{\alpha-1} \cdot E_{\alpha,\alpha}(t^\alpha)) \\ \qquad = \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\lambda \tau^\alpha) A_1(t - \tau) \, d\tau \\ \qquad = A_1 t^\alpha E_{\alpha,\alpha+1}(t^\alpha), \\ v_1(t) = A_2 * (t^{\beta-1} \cdot E_{\beta,\beta}(t^\beta)) \\ \qquad = A_2 t^\beta E_{\beta,\beta+1}(t^\beta), \\ w_1(t) = A_3 * (t^{\gamma-1} \cdot E_{\gamma,\gamma}(t^\gamma)) \\ \qquad = A_3 t^\gamma E_{\gamma,\gamma+1}(t^\gamma), \end{cases} \tag{66}$$

where $A_1 = x_0 - x_0^2 - ax_0y_0 - bx_0z_0$, $A_2 = y_0 - bx_0y_0 - y_0^2 - ay_0z_0$, $A_3 = z_0 - ax_0z_0 - by_0z_0 - z_0^2$ are constants depending on the initial values. In the same way, we have

$$
\begin{cases}
u_2(t) = -(2x_0 + ay_0 + bz_0)A_1 t^{2\alpha} \sum_{k=0}^{\infty} \frac{(k+1)t^{\alpha k}}{T(\alpha k + 2\alpha + 1)} \\
\qquad - ax_0 A_2 t^{\alpha+\beta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\alpha m+\beta n}}{\Gamma(\alpha m+\beta n+\alpha+\beta+1)} \\
\qquad - bx_0 A_3 t^{\alpha+\gamma} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\alpha+\gamma+1)}, \\
v_2(t) = -by_0 A_1 t^{\alpha+\beta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\alpha m+\beta n}}{\Gamma(\alpha m+\beta n+\alpha+\beta+1)} \\
\qquad - (bx_0 + 2y_0 + az_0)A_2 t^{2\beta} \sum_{k=0}^{\infty} \frac{(k+1)t^{\beta k}}{T(\beta k+2\beta+1)} \\
\qquad - ay_0 A_3 t^{\beta+\gamma} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\beta m+\gamma n}}{\Gamma(\beta m+\gamma n+\beta+\gamma+1)}, \\
w_2(t) = -az_0 A_1 t^{\alpha+\gamma} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\alpha+\gamma+1)} \\
\qquad - bz_0 A_2 t^{\beta+\gamma} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^{\beta m+\gamma n}}{\Gamma(\beta m+\gamma n+\beta+\gamma+1)} \\
\qquad - (ax_0 + by_0 + 2z_0)A_3 t^{2\gamma} \sum_{k=0}^{\infty} \frac{(k+1)t^{\gamma k}}{T(\gamma k+2\gamma+1)}.
\end{cases}
\tag{67}
$$

Finally, we can get the approximate solutions $x(t)$, $y(t)$, $z(t)$ for the system (9)-(12) readily

$$
\begin{cases}
x(t) = \lim_{p\to 1} u(t) = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i u_i(t) = x_0 + A_1 E_{\alpha,\alpha+1}(t^\alpha)t^\alpha + \cdots, \\
y(t) = \lim_{p\to 1} v(t) = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i v_i(t) = y_0 + A_2 E_{\beta,\beta+1}(t^\beta)t^\beta + \cdots, \\
z(t) = \lim_{p\to 1} w(t) = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i w_i(t) = z_0 + A_3 E_{\gamma,\gamma+1}(t^\gamma)t^\gamma + \cdots,
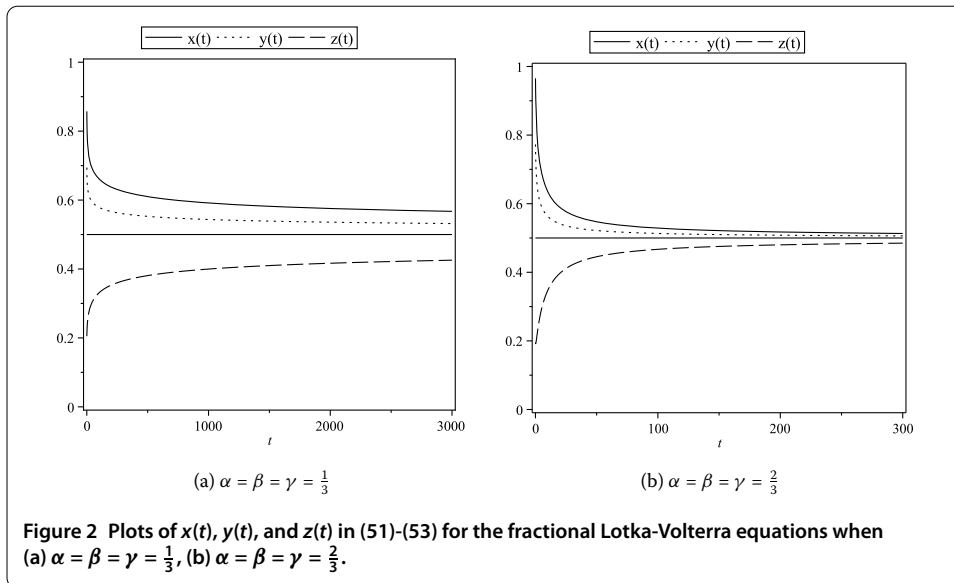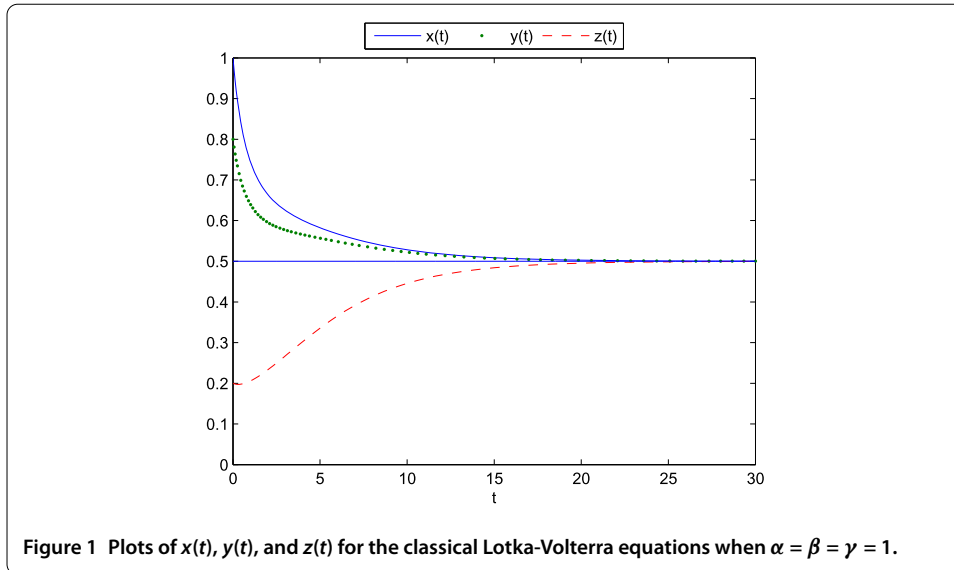\end{cases}
\tag{68}
$$

where the coefficients can be expressed and calculated explicitly.

5 Numerical results and discussion

In the previous sections, we have obtained the approximate analytic solutions as found in (51)-(53) and (68) for the fractional Lotka-Volterra equations (9)-(12) by the stability analysis and HPM. Thus in this section, we present some numerical simulation. For simplicity, we take $a = b = 0.5$, which means the competition affecting coefficients among the three populations are the same, and we choose the initial values $x_0 = 1$, $y_0 = 0.8$, $z_0 = 0.2$. Thus according to the asymptotic stability results, we know the positive equilibrium point is $(x^{eq}, y^{eq}, z^{eq}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and this equilibrium point is locally asymptotically stable.
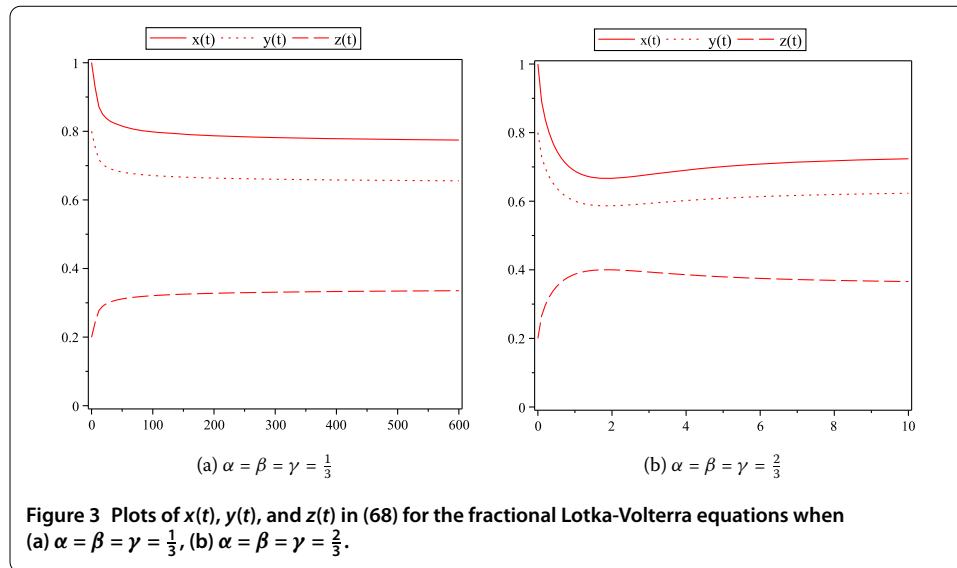
When $\alpha = \beta = \gamma = 1$, the system (9)-(12) reduced the classical Lotka-Volterra equations, which possesses the locally asymptotically stable equilibrium point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The numerical solutions can be presented in Figure 1. As seen in the figure, all the solutions move toward the equilibrium point as the time t increases.

When $0 < \alpha, \beta, \gamma < 1$, we first adopt (51)-(53) to obtain the numerical solutions for various fractional orders. Figure 2 displays the numerical results of the fractional Lotka-Volterra equations when $\alpha = \beta = \gamma = \frac{1}{3}$ and $\alpha = \beta = \gamma = \frac{2}{3}$. As it is expected that fractional-order differential equations are, at least, as stable as their integer-order counterpart, all the numerical solutions in both plots are moving toward the equilibrium point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. What is more, we can find that it takes less time for the every population to get close to the equilibrium point as the time derivatives increase. Compared the classical Lotka-Volterra equations, which possess ordinary derivatives on the left hand side, we can see that the fractional Lotka-Volterra equations, no matter whether or not the order of the fractional derivatives is big or small, both take more time, even multiplied for populations, to reach the equilibrium, which may be the reason that the natural biological systems have memory

Figure 1 **Plots of *x*(*t*), *y*(*t*), and *z*(*t*) for the classical Lotka-Volterra equations when $\alpha = \beta = \gamma = 1$.**

(a) $\alpha = \beta = \gamma = \frac{1}{3}$ (b) $\alpha = \beta = \gamma = \frac{2}{3}$

Figure 2 **Plots of *x*(*t*), *y*(*t*), and *z*(*t*) in (51)-(53) for the fractional Lotka-Volterra equations when (a) $\alpha = \beta = \gamma = \frac{1}{3}$, (b) $\alpha = \beta = \gamma = \frac{2}{3}$.**

properties, and the fractional derivatives provide an excellent instrument in comparison with the classical integer-order counterparts.

On the other hand, we apply the HPM and take only two terms in (68) to obtain the numerical solutions for various fractional orders $0 < \alpha, \beta, \gamma < 1$. Figure 3 represents the numerical simulation results. As we can see, all the numerical solutions in both plots are moving toward some points asymptotically. However, the points are different from the equilibrium points. That is because we just take only two terms in equations (68) to compute the numerical simulation for simplicity, since the following terms are tedious and time-consuming to calculate. But we do believe that the more terms we take in equations (68), the more precise the numerical results will be. At the same time, we also find that it takes less time for the species to get close to the equilibrium point as the time derivatives increases, which expresses the same character as the previous stability analysis method. Thus the analytical method HPM is a useful tool for solving the nonlinear fractional equa-

Figure 3 Plots of $x(t)$, $y(t)$, and $z(t)$ in (68) for the fractional Lotka-Volterra equations when (a) $\alpha = \beta = \gamma = \frac{1}{3}$, (b) $\alpha = \beta = \gamma = \frac{2}{3}$.

tions. However, the convergence and the accuracy for the numerical results need to be considered for further study.

6 Conclusions

Since natural biological systems have memory properties, fractional differential equations provide an excellent instrument in this respect in comparison with the classical integer-order counterparts. Thus in this paper, we studied the fractional Lotka-Volterra equations for three competitors under some symmetry assumptions. First we studied the equilibrium points and asymptotic stability for the equations by the stability analysis method, and we argued that the fractional-order differential equations are, at least, as stable as their integer-order counterpart. Second, we applied the stability analysis method and the homotopy perturbation method to obtain the approximate analytic solutions for the fractional systems, which are expressed in the form of the Mittag-Leffler function. As the numerical results also show, it takes less time for the population to get close to the equilibrium point as the time derivatives increases. However, compared with the classical Lotka-Volterra equations, we can see that the fractional Lotka-Volterra equations, no matter whether the order of the fractional derivatives is big or small, if both take more time, even multiplied for the system to reach the equilibrium, that may explain the memory properties and have great biological significance.

Authors' contributions
CG conceived of the study, performed the numerical experiments, and wrote this paper. SF revised and edited the paper. Both authors contributed equally in preparing this paper and approved this version.

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