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On the non-Lipschitz stochastic differential equations driven by fractional Brownian motion

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Abstract

In this paper, we use a successive approximation method to prove the existence and uniqueness theorems of solutions to non-Lipschitz stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm) with the Hurst parameter $H \in (\frac{1}{2}, 1)$. The non-Lipschitz condition which is motivated by a wider range of applications is much weaker than the Lipschitz one. Due to the fact that the stochastic integral with respect to fBm is no longer a martingale, we definitely lost good inequalities such as the Burkholder-Davis-Gundy inequality which is crucial for SDEs driven by Brownian motion. This point motivates us to carry out the present study.

Keywords: fractional Brownian motion; existence and uniqueness; stochastic differential equations; non-Lipschitz condition

1 Introduction

Stochastic differential equations (SDEs) have been greatly developed and are well known to model diverse phenomena, including but not limited to fluctuating stock prices, physical systems subject to thermal fluctuations, forecasting the growth of a population, from various points of view [1–4]. There is no doubt that the mathematical models under a random disturbance of 'Gaussian white noise' have seen rapid development. However, it is not appropriate to model some real situations where stochastic fluctuations with long-range dependence might exist. Due to the long-range dependence of the fBm which was introduced by Hurst [5], Kolmogorov [6], Mandelbrot [7] originally, SDEs driven by fBm have been used as the models of a number of practical problems in various fields, such as queueing theory, telecommunications, and economics [8–10].

On most occasions, the coefficients of SDEs driven by fBm are assumed to satisfy the Lipschitz condition. The existence and uniqueness of solutions of SDEs driven by fBm with Lipschitz condition have been studied by many scholars [11–14]. However, this Lipschitz condition seemed to be considerably strong when one discusses variable applications in real world. For example, the hybrid square root process and the one-dimensional semi-linear SDEs with Markov switching. Such models appear widely in many branches of science, engineering, industry and finance [15–17]. Therefore, it is important to obtain some weaker condition than the Lipschitz one under which the SDEs still have unique solutions. Fortunately, many researchers have investigated the SDEs under non-Lipschitz condition and they presented many meaningful results [18–22]. But, to the best of our knowledge, the existence and uniqueness of solutions of SDEs driven by fBm with a non-Lipschitz condition have not been considered. Since fBm is neither a semi-martingale nor a Markov process, we definitely lost good inequalities such as the Burkholder-Davis-Gundy inequality, which is crucial for SDEs driven by Brownian motion. Then it seems not to be very easy to obtain the existence and uniqueness of solutions to non-Lipschitz SDEs with fBm. This point motivates us to carry out the present study.

We in the present paper discuss the SDEs with fBm under the non-Lipschitz condition. Using the successive approximation method, the existence and uniqueness theorems of solutions to the following non-Lipschitz SDEs driven by fBm are proved:

$$X(t) = X(0) + \int_0^t b\bigl(s, X(s)\bigr)\,ds + \int_0^t \sigma\bigl(s, X(s)\bigr)\,dB^H(s), \quad t \in [0, T], \tag{1.1}$$

where the initial data $X(0) = \xi$ is a random variable, $0 < T < \infty$, the process $B^H(t)$ represents the fBm with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $b(t, X(t)) : [0, T] \times R \to R$ and $\sigma(t, X(t)) : [0, T] \times R \to R$ are all measurable functions; $\int_0^t \cdot\, dB^H(s)$ stands for the stochastic integral with respect to fBm.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. SDEs with respect to fBm have been interpreted via various stochastic integrals, such as the Wick integral, the Wiener integral, the Skorohod integral, and path-wise integrals [13, 23–26]. In this paper, we consider the path-wise integrals [27] with respect to fBm.

Let $\varphi : R_+ \times R_+ \to R_+$ be defined by

$$\varphi(t, s) = H(2H - 1)|t - s|^{2H-2}, \quad t, s \in R_+,$$

where H is a constant with $\frac{1}{2} < H < 1$.

Let $g : R_+ \to R$ be Borel measurable.

Define

$$L^2_\varphi(R_+) = \biggl\{ g : \|g\|^2_\varphi = \int_{R_+}\int_{R_+} g(t)g(s)\varphi(t, s)\,ds\,dt < \infty \biggr\}.$$

If we equip $L^2_\varphi(R_+)$ with the inner product

$$\langle g_1, g_2\rangle_\varphi = \int_{R_+}\int_{R_+} g_1(t)g_2(s)\varphi(t, s)\,ds\,dt, \quad g_1, g_2 \in L^2_\varphi(R_+),$$

then $L^2_\varphi(R_+)$ becomes a separable Hilbert space.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F(\omega) = f\biggl(\int_0^T \psi_1(t)\,dB^H_t, \ldots, \int_0^T \psi_n(t)\,dB^H_t\biggr),$$

where $n \geq 1$, $f \in \mathcal{C}^\infty_b(R^n)$ (*i.e.* f and all its partial derivatives are bounded), and $\psi_i \in \mathcal{H}$, $i = 1, 2, \ldots, n$. $\mathcal{H}$ is the completion of the measurable functions such that $\|\psi\|^2_\varphi < \infty$ and $\{\psi_n\}$ is a sequence in $\mathcal{H}$ such that $\langle\psi_i, \psi_j\rangle_\varphi = \delta_{ij}$.

The Malliavin derivative D_t^H of a smooth and cylindrical random variable $F \in \mathcal{S}$ is defined as the $\mathcal{H}$-valued random variable:

$$D_t^H F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(\int_0^T \psi_1(t)\,dB_t^H, \ldots, \int_0^T \psi_n(t)\,dB_t^H\right)\psi_i(t).$$

Then, for any $p \geq 1$, the derivative operator D_t^H is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$. Next, we introduce the φ-derivative of F:

$$D_t^\varphi F = \int_{R_+} \varphi(t,v) D_v^H F\,dv.$$

The elements of $\mathcal{H}$ may not be functions but distributions of negative order. Thanks to this, it is convenient to introduce the space $|\mathcal{H}|$ of the measurable function h on $[0,T]$ satisfying

$$\|h\|_{|\mathcal{H}|}^2 = \int_0^T \int_0^T |h(t)||h(s)|\varphi(t,s)\,ds\,dt < \infty.$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}^2$.

In addition, we denote by $D_t^{H,k}$ the iteration of the derivative operator for any integer $k \geq 1$. The Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm, for any $p \geq 1$ ($\bigotimes$ denotes the tensor product),

$$\|F\|_{k,p}^p = \mathbb{E}|F|^p + \mathbb{E}\sum_{j=1}^{k} \|D_t^{H,j} F\|_{\mathcal{H}^{\otimes j}}^p.$$

Similarly, for a Hilbert space U, we denote by $\mathbb{D}^{k,p}(U)$ the corresponding Sobolev space of U-valued random variables. For any $p > 0$ we denote by $\mathbb{D}^{1,p}(|\mathcal{H}|)$ the subspace of $\mathbb{D}^{1,p}(\mathcal{H})$ formed by the elements h such that $h \in |\mathcal{H}|$.

Biagini *et al.* [14], Alos, Mazet and Nualart [24], Hu and Øksendal [9] have given more details as regards the fBm.

Lemma 1 *Let $u(t)$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$, satisfying*

$$\int_0^T \int_0^T |D_s^H u(t)||t-s|^{2H-2}\,ds\,dt < \infty,$$

then the symmetric integral coincides with the forward and backward integrals ($P159$,[14]).

Definition 2 The space $\mathcal{L}_\varphi[0,T]$ of integrands is defined as the family of stochastic processes $u(t)$ on $[0,T]$, such that $\mathbb{E}\|u(t)\|_\varphi^2 < \infty$, $u(t)$ is φ-differentiable, the trace of $D_s^\varphi u(t)$ exists, $0 \leq s \leq T$, $0 \leq t \leq T$, and

$$\mathbb{E}\int_0^T \int_0^T [D_t^\varphi u(s)]^2\,ds\,dt < \infty,$$

and for each sequence of partitions $(\pi_n, n \in \mathbb{N})$ such that $|\pi_n| \to 0$ as $n \to \infty$,

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \left|D_s^\varphi u^\pi\left(t_i^{(n)}\right) D_t^\varphi u^\pi\left(t_j^{(n)}\right) - D_s^\varphi u(t) D_t^\varphi u(s)\right| \,ds\,dt\right]$$

and

$$\mathbb{E}\left[\left\|u^\pi - u\right\|_\varphi^2\right]$$

tend to 0 as $n \to \infty$, where $\pi_n = t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = T$.

Lemma 3 *Let $B^H(t)$ be a fBm with $\frac{1}{2} < H < 1$, and $u(t)$ be a stochastic process in $\mathbb{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$, then for every $T < \infty$,*

$$\mathbb{E}\left[\int_0^T u(s)\,d^\circ B^H(s)\right]^2 \le 2HT^{2H-1}\mathbb{E}\left[\int_0^T \left|u(s)\right|^2 ds\right] + 4T\mathbb{E}\int_0^T \left[D_s^\varphi u(s)\right]^2 ds.$$

The detailed proof of Lemma 3 *can be found in the authors' previous work* [28–30].

In this paper, we always assume the following non-Lipschitz condition, which was proposed by Yamada and Watanabe [22], is satisfied.

Hypothesis 4

(1) There exists a function $\kappa(q) > 0$, $q > 0$, $\kappa(0) = 0$ such that $\kappa(q)$ is a continuous non-decreasing, concave function and $\int_{0+} \frac{dq}{\kappa(q)} = +\infty$,

(2) $b(t,0)$, $\sigma(t,0)$ are locally integral with respect to t,

(3) Furthermore, $\forall t \in [0,T]$, $b(t,\cdot), \sigma(t,\cdot) \in \mathcal{L}_\varphi[0,T] \cap \mathbb{D}^{1,2}(|\mathcal{H}|)$, we have

$$\begin{aligned} &\mathbb{E}\left|b(t,X) - b(t,Y)\right|^2 + \mathbb{E}\left|\sigma(t,X) - \sigma(t,Y)\right|^2 \\ &\quad + \mathbb{E}\left|D_t^\varphi\left(\sigma(t,X) - \sigma(t,Y)\right)\right|^2 \le \kappa\left(\mathbb{E}|X-Y|^2\right). \end{aligned} \tag{2.1}$$

The above-mentioned Hypothesis 4 is the so-called non-Lipschitz condition. The non-Lipschitz condition has a variety of forms [31–34]. Here, we consider one kind of them. In particular, we see clearly that if we let $\kappa(q) = K'q$, then the non-Lipschitz condition reduces to the Lipschitz condition. In other words, the non-Lipschitz condition is weaker than the Lipschitz condition.

Now, we give some concrete examples of the function κ. Let $K' > 0$ and let $\mu \in]0,1[$ be sufficiently small. Define

$$\kappa_1(x) = K'x, \quad x \ge 0,$$

$$\kappa_2(x) = \begin{cases} x\log(x^{-1}), & 0 \le x \le \mu, \\ \mu\log(\mu^{-1}) + \kappa_2'(\mu-)(x-\mu), & x > \mu, \end{cases}$$

$$\kappa_3(x) = \begin{cases} x\log(x^{-1})\log\log(x^{-1}), & 0 \le x \le \mu, \\ \mu\log(\mu^{-1})\log\log(\mu^{-1}) + \kappa_3'(\mu-)(x-\mu), & x > \mu, \end{cases}$$

where κ' denotes the derivative of the function κ. They are all concave and non-decreasing functions satisfying $\int_{0+} \frac{1}{\kappa_i(x)}\,dx = \infty$ $(i = 1,2,3)$.

3 The main theorems

In this section, using an iteration of the Picard type, we will discuss the solutions for non-Lipschitz SDEs with fBm. Let $X_0(t) \equiv \xi$ be a random variable with $\mathbb{E}|\xi|^2 < +\infty$, and construct an approximate sequence of stochastic process $\{X_k(t)\}_{k\geq 1}$ as follows:

$$X_k(t) = \xi + \int_0^t b\big(s, X_{k-1}(s)\big)\,ds + \int_0^t \sigma\big(s, X_{k-1}(s)\big)\,d^\circ B^H(s), \quad k = 1,2,\ldots. \tag{3.1}$$

Hereafter, we assume that $1 \leq T < +\infty$ without losing generality.

First, we given the following four key lemmas. The proofs for Lemma 5 and Lemma 6 will be presented in the Appendix.

Lemma 5 *There exists a positive number K, $\forall b(t,\cdot), \sigma(t,\cdot) \in \mathcal{L}_\varphi[0,T] \cap \mathbb{D}^{1,2}(|\mathcal{H}|)$, $t \in [0,T]$, and we have*

$$\mathbb{E}\big|b(t,X)\big|^2 + \mathbb{E}\big|\sigma(t,X)\big|^2 + \mathbb{E}\big|D_t^\varphi \sigma(t,X)\big|^2 \leq K\big(1 + \mathbb{E}|X|^2\big).$$

Lemma 6 *Under the conclusion of Lemma* 5, *one can get*

$$\mathbb{E}\big|X_k(t)\big|^2 \leq C_1, \quad k = 1,2,\ldots, t \in [0,T], \tag{3.2}$$

where $C_1 = 3(1 + \mathbb{E}|\xi|^2)\exp(12KT^2)$.

Lemma 7 *If $b(t,X)$ and $\sigma(t,X)$ satisfy the Hypothesis* 4, *then for $t \in [0,T]$, $n \geq 1$, $k \geq 1$, we have*

$$\mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2 \leq C_2 \int_0^t \kappa\big(\mathbb{E}\big|X_{n+k-1}(s) - X_{n-1}(s)\big|^2\big)\,ds \tag{3.3}$$

and

$$\sup_{0\leq s\leq t} \mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2 \leq C_3 t,$$

where $C_2 = 8T$ and C_3 is a constant.

Proof For $0 \leq s \leq t$, we show that

$$\begin{aligned}
&\mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2 \\
&\quad\leq 2\mathbb{E}\bigg|\int_0^s \big(b\big(s_1, X_{n+k-1}(s_1)\big) - b\big(s_1, X_{n-1}(s_1)\big)\big)\,ds_1\bigg|^2 \\
&\qquad + 2\mathbb{E}\bigg|\int_0^s \big(\sigma\big(s_1, X_{n+k-1}(s_1)\big) - \sigma\big(s_1, X_{n-1}(s_1)\big)\big)\,d^\circ B^H(s_1)\bigg|^2 \\
&\quad\leq 8T\mathbb{E}\int_0^t \big[\big|b\big(s_1, X_{n+k-1}(s_1)\big) - b(s_1, X_{n-1}(s_1)\big|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\sigma\left(s_1, X_{n+k-1}(s_1)\right) - \sigma\left(s_1, X_{n-1}(s_1)\right|^2 \\
&+ \left|D^{\varphi}_{s_1}\left(\sigma\left(s_1, X_{n+k-1}(s_1)\right) - \sigma\left(s_1, X_{n-1}(s_1)\right)\right)\right|^2\Big] \, ds_1 \\
\le{}& C_2 \int_0^t \kappa\left(\mathbb{E}\left|X_{n+k-1}(s) - X_{n-1}(s)\right|^2\right) ds.
\end{aligned}
$$

Then it is easy to verify

$$
\begin{aligned}
\sup_{0\le s\le t} \mathbb{E}\left|X_{n+k}(s) - X_n(s)\right|^2 &\le C_2 \int_0^t \kappa\left(\mathbb{E}\left|X_{n+k-1}(s) - X_{n-1}(s)\right|^2\right) ds \\
&\le C_2 \int_0^t \kappa(4C_1)\, ds \le C_3 t.
\end{aligned}
$$

This completes the proof of Lemma 7. □

Now, choose $0 < T_1 \le T$, such that $t \in [0, T_1]$, for $\kappa_1(C_3 t) \le C_3$, $\kappa_1(q) = C_2\kappa(q)$ holds. We should note that in the following part, we first of all prove the following main theorem, Theorem 9, in the time interval $[0, T_1]$, then we extend the result in the whole interval $[0, T]$. Fix $k \ge 1$ arbitrarily and define two sequences of functions $\{\phi_n(t)\}_{n=1,2,\ldots}$ and $\{\tilde{\phi}_{n,k}(t)\}_{n=1,2,\ldots}$, where

$$
\begin{aligned}
&\phi_1(t) = C_3 t, \\
&\phi_{n+1}(t) = \int_0^t \kappa_1\left(\phi_n(s)\right) ds, \\
&\tilde{\phi}_{n,k}(t) = \sup_{0\le s\le t} \mathbb{E}\left|X_{n+k}(s) - X_n(s)\right|^2, \quad n = 1, 2, \ldots.
\end{aligned}
$$

Lemma 8 *Under the Hypothesis* 4,

$$
0 \le \tilde{\phi}_{n,k}(t) \le \phi_n(t) \le \phi_{n-1}(t) \le \cdots \le \phi_1(t), \quad t \in [0, T_1], \tag{3.4}
$$

for all positive integer n.

Proof By Lemma 7, we have

$$
\tilde{\phi}_{1,k}(t) = \sup_{0\le s\le t} \mathbb{E}\left|X_{1+k}(s) - X_1(s)\right|^2 \le C_3 t = \phi_1(t), \quad t \in [0, T_1].
$$

Then, since $\kappa_1(q) = C_2\kappa(q)$, $\kappa(q)$ is a concave function and

$$
\mathbb{E}\left|X_{k+1}(s) - X_1(s)\right|^2 \le \sup_{0\le s\le t} \mathbb{E}\left|X_{k+1}(s) - X_1(s)\right|^2 = \tilde{\phi}_{1,k}(t), \quad 0 \le s \le t,
$$

it is easy to verify

$$
\begin{aligned}
\tilde{\phi}_{2,k}(t) &= \sup_{0\le s\le t} \mathbb{E}\left|X_{2+k}(s) - X_2(s)\right|^2 \\
&\le C_2 \int_0^t \kappa\left(\mathbb{E}\left|X_{k+1}(s) - X_1(s)\right|^2\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^t \kappa_1\big(\tilde{\phi}_{1,k}(s)\big)\,ds \leq \int_0^t \kappa_1\big(\phi_1(s)\big)\,ds \\
&= \phi_2(t) = \int_0^t \kappa_1(C_3 s)\,ds \\
&\leq C_3 t = \phi_1(t), \quad t \in [0, T_1].
\end{aligned}
$$

That is to say, for $n = 2$, we have

$$\tilde{\phi}_{2,k}(t) \leq \phi_2(t) \leq \phi_1(t), \quad t \in [0, T_1].$$

Next, assume (3.4) for $n \geq 2$ and by the assumption for n

$$\mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2 \leq \sup_{0\leq s\leq t} \mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2 = \tilde{\phi}_{n,k}(t) \leq \phi_n(t),$$

it is easy to verify for $n + 1$

$$
\begin{aligned}
\tilde{\phi}_{n+1,k}(t) &= \sup_{0\leq s\leq t} \mathbb{E}\big|X_{n+k+1}(s) - X_{n+1}(s)\big|^2 \\
&\leq \int_0^t \kappa_1\big(\mathbb{E}\big|X_{n+k}(s) - X_n(s)\big|^2\big)\,ds \\
&\leq \int_0^t \kappa_1\big(\tilde{\phi}_{n,k}(s)\big)\,ds \\
&\leq \int_0^t \kappa_1\big(\phi_n(s)\big)\,ds = \phi_{n+1}(t) \\
&\leq \int_0^t \kappa_1\big(\phi_{n-1}(s)\big)\,ds = \phi_n(t), \quad t \in [0, T_1].
\end{aligned}
$$

This completes the proof of Lemma 8. □

Theorem 9 *Under the Hypothesis* 4, *then*

$$\lim_{n,i\to\infty} \sup_{0\leq t\leq T} \mathbb{E}\big|X_n(t) - X_i(t)\big|^2 = 0.$$

By Theorem 9, we say that $\{X_k(\cdot)\}_{k\geq 1}$ is a Cauchy sequence and define its limit as $X(\cdot)$. Then letting $k \to \infty$ in (3.1), we finally see that the solutions to (1.1) exist.

Proof *Step* 1: In this step we shall show

$$\lim_{n,i\to\infty} \sup_{0\leq t\leq T_1} \mathbb{E}\big|X_n(t) - X_i(t)\big|^2 = 0.$$

By Lemma 8, we know $\phi_n(t)$ decreases monotonically when $n \to \infty$ and $\phi_n(t)$ is non-negative function on $t \in [0, T_1]$. Therefore, we can define the limit function $\phi(t)$ by $\phi_n(t) \downarrow \phi(t)$. It is easy to verify that $\phi(0) = 0$ and $\phi(t)$ is a continuous function on $t \in [0, T_1]$ [35]. According to the definition of $\phi_n(t)$ and $\phi(t)$, we obtain

$$\phi(t) = \lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \int_0^t \kappa_1\big(\phi_n(s)\big)\,ds = \int_0^t \kappa_1\big(\phi(s)\big)\,ds, \quad t \in [0, T_1]. \tag{3.5}$$

Since $\phi(0) = 0$ and

$$\int_{0+} \frac{dq}{\kappa_1(q)} = \frac{1}{C_2} \int_{0+} \frac{dq}{\kappa(q)} = +\infty,$$

(3.5) implies $\phi(t) \equiv 0$, $t \in [0, T_1]$.

Therefore we obtain

$$0 \leq \lim_{k,n\to\infty} \sup_{0\leq t\leq T_1} \mathbb{E}\left|X_{n+k}(t) - X_n(t)\right|^2 = \lim_{k,n\to\infty} \tilde{\phi}_{n,k}(T_1) \leq \lim_{n\to\infty} \phi_n(T_1) = 0,$$

namely,

$$\lim_{n,i\to\infty} \sup_{0\leq t\leq T_1} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 = 0.$$

Step 2: Define

$$T_2 = \sup\left\{\tilde{T} : \tilde{T} \in [0, T] \text{ and } \lim_{n,i\to\infty} \sup_{0\leq t\leq \tilde{T}} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 = 0\right\}.$$

Immediately, we can observe $0 < T_1 \leq T_2 \leq T$. Now, we shall show

$$\lim_{n,i\to\infty} \sup_{0\leq t\leq T_2} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 = 0.$$

Let $\varepsilon > 0$ be an arbitrary positive number. Choose S_0 so that $0 < S_0 < \min(T_2, 1)$. And

$$C_4 S_0 < \frac{\varepsilon}{10}, \tag{3.6}$$

where $C_4 = 8K(1 + K_1(1 + \mathbb{E}|\xi|^2))S_0$.

From the definition of T_2, we have

$$\lim_{n,i\to\infty} \sup_{0\leq t\leq T_2-S_0} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 = 0.$$

Then, for large enough N, we observe

$$\sup_{0\leq t\leq T_2-S_0} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 < \frac{\varepsilon}{10}, \quad n, i \geq N. \tag{3.7}$$

On the other hand, one can get

$$\begin{aligned} \sup_{T_2-S_0\leq t\leq T_2} \mathbb{E}\left|X_n(t) - X_i(t)\right|^2 &\leq 3 \sup_{T_2-S_0\leq t\leq T_2} \mathbb{E}\left|X_n(t) - X_n(T_2 - S_0)\right|^2 \\ &\quad + 3\mathbb{E}\left|X_n(T_2 - S_0) - X_i(T_2 - S_0)\right|^2 \\ &\quad + 3 \sup_{T_2-S_0\leq t\leq T_2} \mathbb{E}\left|X_i(T_2 - S_0) - X_i(t)\right|^2 \\ &= 3I_1 + 3I_2 + 3I_3. \end{aligned} \tag{3.8}$$

Now, using Lemma 3, we obtain

$$\begin{aligned}
I_1 &= \sup_{T_2-S_0\le t\le T_2} \mathbb{E}\big|X_n(t)-X_n(T_2-S_0)\big|^2 \\
&\le 2S_0\mathbb{E}\int_{T_2-S_0}^{T_2} \big|b\big(s_1,X_{n-1}(s_1)\big)\big|^2\,ds_1 \\
&\quad + 4H{S_0}^{2H-1}\mathbb{E}\int_{T_2-S_0}^{T_2} \big|\sigma\big(s_1,X_{n-1}(s_1)\big)\big|^2\,ds_1 \\
&\quad + 8S_0\mathbb{E}\int_{T_2-S_0}^{T_2} \big|D_{s_1}^{\varphi}\sigma\big(s_1,X_{n-1}(s_1)\big)\big|^2\,ds_1 \\
&\le 8S_0\int_{T_2-S_0}^{T_2} K\big(1+K_1\big(1+\mathbb{E}|\xi|^2\big)\big)\,ds_1 \\
&\le 8S_0^2K\big(1+K_1\big(1+\mathbb{E}|\xi|^2\big)\big).
\end{aligned}$$

Therefore by (3.6) we have

$$I_1 \le \frac{\varepsilon}{10} \tag{3.9}$$

and

$$I_3 \le \frac{\varepsilon}{10}. \tag{3.10}$$

Meanwhile, (3.7) implies

$$I_2 = \mathbb{E}\big|X_n(T_2-S_0)-X_i(T_2-S_0)\big|^2 < \frac{\varepsilon}{10}, \quad n,i\ge N. \tag{3.11}$$

Now putting (3.7)-(3.11) together, we have

$$\begin{aligned}
\sup_{0\le t\le T_2} \mathbb{E}\big|X_n(t)-X_i(t)\big|^2 &\le \sup_{0\le t\le T_2-S_0} \mathbb{E}\big|X_n(t)-X_i(t)\big|^2 \\
&\quad + \sup_{T_2-S_0\le t\le T_2} \mathbb{E}\big|X_n(t)-X_i(t)\big|^2 \\
&\le \frac{\varepsilon}{10} + 3I_1 + 3I_2 + 3I_3 < \varepsilon.
\end{aligned}$$

That is to say,

$$\lim_{n,i\to\infty} \sup_{0\le t\le T_2} \mathbb{E}\big|X_n(t)-X_i(t)\big|^2 = 0.$$

Step 3: Using the method of reduction to absurdity, we shall show $T_2 = T$. Assume $T_2 < T$, we can choose a sequence of numbers $\{a_i\}_{i=1,2,\ldots}$ so that $a_i \downarrow 0$ $(i\to+\infty)$ and for $n > i \ge 1$,

$$\sup_{0\le t\le T_2} \mathbb{E}\big|X_n(t)-X_i(t)\big|^2 \le a_i. \tag{3.12}$$

We shall divide the step into several sub-steps.

First, for $n > i \geq 1$, we shall show

$$\sup_{T_2 \leq s \leq T_2+t} \mathbb{E}\left|X_n(s) - X_i(s)\right|^2 \leq 3a_i + C_5 t, \quad T_2 + t \leq T, \tag{3.13}$$

where $C_5 = 12TK(1 + K_1(1 + \mathbb{E}|\xi|^2))$.

To show this, set

$$J_1^{(i)} = \mathbb{E}\left|X_n(T_2) - X_i(T_2)\right|^2,$$

$$J_2^{(i)}(t) = \sup_{T_2 \leq s \leq T_2+t} \mathbb{E}\left|\int_{T_2}^{s} \big(b\big(s_1, X_{n-1}(s_1)\big) - b\big(s_1, X_{i-1}(s_1)\big)\big)\, ds_1\right|^2,$$

$$J_3^{(i)}(t) = \sup_{T_2 \leq s \leq T_2+t} \mathbb{E}\left|\int_{T_2}^{s} \big(\sigma\big(s_1, X_{n-1}(s_1)\big) - \sigma\big(s_1, X_{i-1}(s_1)\big)\big)\, d^\circ B^H(s_1)\right|^2.$$

Then (3.12) implies $J_1^{(i)} \leq a_i$ and

$$\begin{aligned} J_2^i(t) + J_3^i(t) &\leq 4T\mathbb{E}\int_{T_2}^{T_2+t} \Big[\big|b\big(s_1, X_{n-1}(s_1)\big) - b\big(s_1, X_{i-1}(s_1)\big)\big|^2 \\ &\quad + \big|\sigma\big(s_1, X_{n-1}(s_1)\big) - \sigma\big(s_1, X_{i-1}(s_1)\big)\big|^2 \\ &\quad + \big|D_{s_1}^{\varphi}\big(\sigma\big(s_1, X_{n-1}(s_1)\big) - \sigma\big(s_1, X_{i-1}(s_1)\big)\big)\big|^2\Big]\, ds_1 \\ &\leq 4TK\big(1 + K_1\big(1 + \mathbb{E}|\xi|^2\big)\big)t. \end{aligned}$$

Therefore

$$\begin{aligned} \sup_{T_2 \leq s \leq T_2+t} \mathbb{E}\left|X_n(s) - X_i(s)\right|^2 &\leq 3J_1^{(i)} + 3J_2^{(i)}(t) + 3J_3^{(i)}(t) \\ &\leq 3a_i + C_5 t, \quad T_2 + t \leq T. \end{aligned}$$

Next, we shall show an assertion which is analogous to Lemma 8. To state the assertion, we need to introduce several notations.

Choose a positive number $0 < \eta \leq T - T_2$ and a positive integer $j \geq 1$, so that

$$C_6\kappa(3a_j + C_5 t) \leq C_5, \quad t \in [0, \eta], \kappa_2(q) = C_6\kappa(q), \tag{3.14}$$

where $C_6 = 12T$.

Introduce the sequence of functions $\{\psi_k(t)\}_{k=1,2,\ldots}$, $t \in [0, \eta]$, defined by

$$\psi_1(t) = 3a_j + C_5 t,$$

$$\psi_{k+1}(t) = 3a_{j+k} + \int_0^t \kappa_2\big(\psi_k(s)\big)\, ds,$$

$$\tilde{\psi}_{k,n}(t) = \sup_{T_2 \leq s \leq T_2+t} \mathbb{E}\left|X_{n+k}(s) - X_{j+k}(s)\right|^2.$$

Now, the assertion to be proved is the following:

$$\tilde{\psi}_{k,n}(t) \leq \psi_k(t) \leq \psi_{k-1}(t) \leq \cdots \leq \psi_1(t), \quad t \in [0, \eta], \tag{3.15}$$

for all positive integer k.

Noticing that $\kappa_2(q)$ is a non-decreasing, concave function, and (3.13) holds, from this for $k = 1$, we work out

$$\begin{aligned}
\tilde{\psi}_{1,n}(t) &= \sup_{T_2 \le s \le T_2+t} \mathbb{E}\bigl|X_{n+1}(s) - X_{j+1}(s)\bigr|^2 \\
&\le 3a_{j+1} + C_6 \mathbb{E} \int_{T_2}^{T_2+t} \Bigl[\bigl|b\bigl(s_1, X_n(s_1)\bigr) - b\bigl(s_1, X_j(s_1)\bigr)\bigr|^2 \\
&\quad + \bigl|\sigma\bigl(s_1, X_n(s_1)\bigr) - \sigma\bigl(s_1, X_j(s_1)\bigr)\bigr|^2 \\
&\quad + \bigl|D_{s_1}^{\varphi}\bigl(\sigma\bigl(s_1, X_n(s_1)\bigr) - \sigma\bigl(s_1, X_j(s_1)\bigr)\bigr)\bigr|^2\Bigr]\, ds_1 \\
&\le 3a_{j+1} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\mathbb{E}\bigl|X_n(s_1) - X_j(s_1)\bigr|^2\bigr)\, ds_1 \\
&\le 3a_j + \int_{T_2}^{T_2+t} \kappa_2(3a_j + C_5 s_1)\, ds_1 \le \psi_1(t), \quad t \in [0, \eta].
\end{aligned}$$

On the other hand, using (3.14) we arrive at

$$\begin{aligned}
\tilde{\psi}_{2,n}(t) &\le \sup_{T_2 \le s \le T_2+t} \mathbb{E}\bigl|X_{n+2}(s) - X_{j+2}(s)\bigr|^2 \\
&\le 3a_{j+2} + C_6 \int_{T_2}^{T_2+t} \kappa\bigl(\mathbb{E}\bigl|X_{n+1}(s_1) - X_{j+1}(s_1)\bigr|^2\bigr)\, ds_1 \\
&\le 3a_{j+2} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\tilde{\psi}_{1,n}(t)\bigr)\, ds_1 \\
&\le 3a_{j+1} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\psi_1(t)\bigr)\, ds_1 = \psi_2(t) \\
&\le 3a_j + C_5 t = \psi_1(t), \quad t \in [0, \eta].
\end{aligned}$$

Then we have proved

$$\tilde{\psi}_{2,n}(t) \le \psi_2(t) \le \psi_1(t).$$

Now assume that the assertion holds for $k \ge 2$. Then, by an analogous argument, one can obtain

$$\begin{aligned}
\tilde{\psi}_{k+1,n}(t) &\le 3a_{j+k+1} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\mathbb{E}\bigl|X_{n+k}(s_1) - X_{j+k}(s_1)\bigr|^2\bigr)\, ds_1 \\
&\le 3a_{j+k+1} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\tilde{\psi}_{k,n}(s_1)\bigr)\, ds_1 \\
&\le 3a_{j+k} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\psi_k(s_1)\bigr)\, ds_1 = \psi_{k+1}(t) \\
&\le 3a_{j+k-1} + \int_{T_2}^{T_2+t} \kappa_2\bigl(\psi_{k-1}(s_1)\bigr)\, ds_1 \\
&= \psi_k(t), \quad t \in [0, \eta].
\end{aligned}$$

Therefore, we obtain (3.15) for all k. In terms of (3.15), we can define the function $\psi(t)$ by $\psi_k(t) \downarrow \psi(t)$ ($k \to \infty$). We observe that

$$\begin{aligned}\psi(0) &= \lim_{k\to\infty} \psi_{k+1}(0) \\ &= \lim_{k\to\infty} a_{j+k} = 0.\end{aligned}$$

It is easy to verify that $\psi(t)$ is a continuous function on $[0,\eta]$. Now by the definition of $\psi_{k+1}(t)$ and $\psi(t)$, we have

$$\begin{aligned}\psi(t) &= \lim_{k\to\infty} \psi_{k+1}(t) \\ &= \lim_{k\to\infty}\left[3a_{j+k} + \int_0^t \kappa_2\bigl(\psi_k(s)\bigr)\,ds\right] \\ &= \int_0^t \kappa_2\bigl(\psi(s)\bigr)\,ds.\end{aligned} \tag{3.16}$$

Since $\psi(0) = 0$ and

$$\int_{0+} \frac{dq}{\kappa_2(q)} = \frac{1}{C_6}\int_{0+} \frac{dq}{\kappa(q)} = +\infty,$$

(3.16) implies $\psi(t) = 0$, $t \in [0,\eta]$.

Therefore, we obtain

$$\begin{aligned}\lim_{k\to\infty} \tilde{\psi}_{k,n}(t) &= \lim_{k\to\infty} \sup_{0\le s\le T_2+t} \mathbb{E}\bigl|X_{n+k}(s) - X_{j+k}(s)\bigr|^2 \\ &\le \lim_{k\to\infty} \sup_{0\le s\le T_2} \mathbb{E}\bigl|X_{n+k}(s) - X_{j+k}(s)\bigr|^2 \\ &\quad + \lim_{k\to\infty} \sup_{T_2\le s\le T_2+\eta} \mathbb{E}\bigl|X_{n+k}(s) - X_{j+k}(s)\bigr|^2 \\ &\le \lim_{k\to\infty} \psi_k(\eta) = \psi(\eta) = 0,\end{aligned}$$

namely

$$\lim_{n,i\to\infty} \sup_{0\le t\le T_2+\eta} \mathbb{E}\bigl|X_n(t) - X_i(t)\bigr|^2 = 0.$$

But this conclusion is contradictory to the definition of T_2. In other words, we have already shown that

$$\lim_{n,i\to\infty} \sup_{0\le t\le T} \mathbb{E}\bigl|X_n(t) - X_i(t)\bigr|^2 = 0.$$

The proof of the existence of solutions of SDEs (1.1) is complete. □

Theorem 10 *Under the Hypothesis* 4, *the path-wise uniqueness holds for* (1.1), $t \in [0,T]$.

Proof Let $X(t)$ and $\tilde{X}(t)$ be two solutions of (1.1) on the same probability space and $X(0) = \tilde{X}(0)$. We observe

$$\begin{aligned}
&\mathbb{E}\big|X(t)-\tilde{X}(t)\big|^2 \\
&\quad = \mathbb{E}\bigg|\int_0^t \big(b\big(s,X(s)\big)-b\big(s,\tilde{X}(s)\big)\big)\,ds + \int_0^t \big(\sigma\big(s_1,X(s)\big)-\sigma\big(s,\tilde{X}(s)\big)\big)\,d^\circ B^H(s)\bigg|^2 \\
&\quad \le 2\mathbb{E}\bigg|\int_0^t \big(b\big(s,X(s)\big)-b\big(s,\tilde{X}(s)\big)\big)\,ds\bigg|^2 + 2\mathbb{E}\bigg|\int_0^t \big(\sigma\big(s,X(s)\big)-\sigma\big(s,\tilde{X}(s)\big)\big)\,d^\circ B^H(s)\bigg|^2 \\
&\quad \le 8T\mathbb{E}\int_0^t \Big(\big|b\big(s,X(s)\big)-b\big(s,\tilde{X}(s)\big)\big|^2 + \big|\sigma\big(s,X(s)\big)-\sigma\big(s,\tilde{X}(s)\big)\big|^2 \\
&\qquad + \big|D_s^\varphi\big(\sigma\big(s,X(s)\big)-\sigma\big(s,\tilde{X}(s)\big)\big)\big|^2\Big)\,ds.
\end{aligned}$$

Combining the above inequalities and the Hypothesis 4, one has

$$\mathbb{E}\big|X(t)-\tilde{X}(t)\big|^2 \le 8T\int_0^t \kappa\big(\mathbb{E}\big|X(s)-\tilde{X}(s)\big|^2\big)\,ds. \tag{3.17}$$

Then, noticing that $\int_{0+}\frac{dq}{\kappa(q)} = +\infty$, the above inequality (3.17) implies

$$\mathbb{E}\big|X(t)-\tilde{X}(t)\big|^2 = 0, \quad t\in[0,T].$$

Since T is an arbitrary positive number, we obtain from this $X(t)\equiv\tilde{X}(t)$, for all $0\le t\le T$. Thus the path-wise uniqueness holds for (1.1). □

Appendix

Proof of Lemma 5 Since $\kappa(q)$ is a concave and non-negative function, we can choose two positive constants $a>0$ and $b>0$, so that

$$\kappa(q)\le a+bq, \quad q\ge 0,$$

then, by (2.1), we get

$$\begin{aligned}
&\mathbb{E}\big|\sigma(t,X)\big|^2 + \mathbb{E}\big|b(t,X)\big|^2 + \mathbb{E}\big|D_t^\varphi\sigma(t,X)\big|^2 \\
&\quad \le 2\mathbb{E}\big(\big|\sigma(t,0)\big|^2 + \big|b(t,0)\big|^2 + \big|D_t^\varphi\sigma(t,0)\big|^2\big) + 2\mathbb{E}\big|\sigma(t,X)-\sigma(t,0)\big|^2 \\
&\qquad + 2\mathbb{E}\big|b(t,X)-b(t,0)\big|^2 + 2\mathbb{E}\big|D_t^\varphi(\sigma(t,X)-\sigma(t,0))\big|^2 \\
&\quad \le 2\sup_{0\le t\le T}\mathbb{E}\big(\big|\sigma(t,0)\big|^2 + \big|b(t,0)\big|^2 + \big|D_t^\varphi\sigma(t,0)\big|^2\big) + 2\kappa\big(\mathbb{E}|X|^2\big) \\
&\quad \le K\big(1+\mathbb{E}|X|^2\big),
\end{aligned}$$

where $K = \max[2\sup_{0\le t\le T}\mathbb{E}(|\sigma(t,0)|^2 + |b(t,0)|^2 + |D_t^\varphi\sigma(t,0)|^2) + 2a, 2b] < +\infty$. □

Proof of Lemma 6 Using mathematical induction, we first assume that

$$\mathbb{E}\left|X_k(t)\right|^2 \le 3\mathbb{E}|\xi|^2 \sum_{l=0}^{k} \frac{(12KT)^l}{l!} t^l + \sum_{l=1}^{k} \frac{(12KT)^l}{l!} t^l \tag{A.1}$$

holds, $t \in [0, T]$, $k = 1, 2, \ldots$.

Clearly, by Lemma 3 and Lemma 5, we arrive at

$$\begin{aligned} \mathbb{E}\left|X_1(t)\right|^2 &\le 3\mathbb{E}|\xi|^2 + 3\mathbb{E}\left|\int_0^t b\big(s, X_0(s)\big)\,ds\right|^2 + 3\mathbb{E}\left|\int_0^t \sigma\big(s, X_0(s)\big)\,d^\circ B^H(s)\right|^2 \\ &\le 3\mathbb{E}|\xi|^2 + 12T\mathbb{E}\int_0^t \left(\left|b\big(s, X_0(s)\big)\right|^2 + \left|\sigma\big(s, X_0(s)\big)\right|^2 + \left|D_s^\varphi \sigma\big(s, X_0(s)\big)\right|^2\right) ds \\ &\le 3\mathbb{E}|\xi|^2 + 12KTt\big(1 + \mathbb{E}|\xi|^2\big). \end{aligned} \tag{A.2}$$

Now, assume that (A.1) holds for k, then we have, for $k + 1$,

$$\begin{aligned} \mathbb{E}\left|X_{k+1}(t)\right|^2 &\le 3\mathbb{E}|\xi|^2 + 3\mathbb{E}\left|\int_0^t b\big(s, X_k(s)\big)\,ds\right|^2 + 3\mathbb{E}\left|\int_0^t \sigma\big(s, X_k(s)\big)\,d^\circ B^H(s)\right|^2 \\ &\le 3\mathbb{E}|\xi|^2 + 12T\mathbb{E}\int_0^t \left(\left|b\big(s, X_k(s)\big)\right|^2 + \left|\sigma\big(s, X_k(s)\big)\right|^2 + \left|D_s^\varphi \sigma\big(s, X_k(s)\big)\right|^2\right) ds \\ &\le 3\mathbb{E}|\xi|^2 + 12KT\int_0^t \left(1 + \mathbb{E}\left|X_k(s)\right|^2\right) ds \\ &\le 3\mathbb{E}|\xi|^2 + 12KT\int_0^t \left(1 + 3\mathbb{E}|\xi|^2 \sum_{l=0}^{k} \frac{(12KT)^l}{l!} s^l + \sum_{l=1}^{k} \frac{(12KT)^l}{l!} s^l\right) ds \\ &= 3\mathbb{E}|\xi|^2 + 12KTt + 3\mathbb{E}|\xi|^2 \sum_{l=1}^{k+1} \frac{(12KT)^l}{l!} t^l + \sum_{l=2}^{k+1} \frac{(12KT)^l}{l!} t^l \\ &= 3\mathbb{E}|\xi|^2 \sum_{l=0}^{k+1} \frac{(12KT)^l}{l!} t^l + \sum_{l=1}^{k+1} \frac{(12KT)^l}{l!} t^l. \end{aligned}$$

Therefore, by induction, (A.1) holds for all k.

Now, we obtain the form $C_1 = 3(1 + \mathbb{E}|\xi|^2)\exp(12KT^2)$, then (A.2) implies (3.2). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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