Advances in Difference Equations
a SpringerOpen Journal

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Existence and stability of a unique almost periodic solution for a prey-predator system with impulsive effects and multiple delays

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Abstract

In this paper, a nonautonomous almost periodic prey-predator system with impulsive effects and multiple delays is considered. By the mean-value theorem of multiple variables, integral inequalities, differential inequalities, and other mathematical analysis skills, sufficient conditions which guarantee the permanence of the system are obtained. Furthermore, by constructing a series of Lyapunov functionals, we derive that there exists a unique almost periodic solution of the system which is uniformly asymptotically stable. Finally, a numerical example and some simulations are presented to support our theoretical results.

Keywords: impulsive effects; delays; permanence; almost periodic solution; asymptotical stability

1 Introduction

It is widely known that predation is a very common ecological phenomenon in the natural world, and research on the dynamics of the prey-predator system is extremely meaningful and important in many fields, such as protecting varieties of the biological species, maintaining ecological balance, agricultural pests control and management, *etc.* As research of the prey-predator system is concerned, the earliest work dates back to the great contribution of Lotka (1925) and Volterra (1926). However, based on the classic Lotka-Volterra model, many ecologists found that the predation rate was not just simply proportional to the product of the density of the predator and prey. It was always different for different systems. Then the conception of the functional response was proposed. Functional response refers to the predation rate per predator with a response to changes in the prey density, *i.e.*, predation effects of predators on the prey.

Since then, studies depending on all kinds of functional responses sprang up quickly, such as of Holling type [1–3], Michaelis-Menton type [4], Beddington-DeAngelis type [5], Ivlev type [6], Hassell-Varley type [7], and so on. In 1989, Crowley and Martin [8] proposed a new functional response that can accommodate interference between predators. It is similar to the well-known Beddington-DeAngelis functional response but has an additional term in the first right term equation modeling mutual interference between the predator terms.

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Crowley-Martin type functional response is used for data sets that indicate feeding rate that is affected by predator density. It is assumed that predator feeding rate decreases by higher predator density even when prey density is high, and effects of predator interference in feeding rate remain important all the time whenever an individual predator is handling or searching for a prey at a given instant of time. And the formula of the per capita feeding rate Crowley and Martin proposed in [8] is as follows:

$$f(x,y) = \frac{mx}{(1+\alpha x)(1+\beta y)}. \tag{1}$$

Here m, α, β are positive parameters that describe the effects of capture rate handling time and the magnitude of interference among predators, respectively, on the feeding rate.

On the one hand, since models with Crowly-Martin functional response can be generated by a number of natural mechanisms and admits rich but biological dynamics. However, to the best of our knowledge, references reported on this functional response seems to much less until recent several years (see [9–13]) for the form of the functional response is relatively complex, for the form of this functional response is relatively complex, and it is worthwhile to further study the models with it.

Recently, Liu *et al.* in [12] considered a stochastic prey-predator system with Crowley-Martin type functional response as follows:

$$\begin{cases} dx(t) = x(t)[r_1 - b_1 x(t) - \frac{c_1 y(t)}{(1+\alpha x(t))(1+\beta y(t))}] \, dt + x(t)[\delta_1 \, dB_1(t) + \mu_2 \, dB_2(t)], \\ dy(t) = y(t)[r_2 - b_2 y(t) + \frac{c_2 x(t)}{(1+\alpha x(t))(1+\beta y(t))}] \, dt + y(t)[\delta_2 \, dB_1(t) + \mu_1 \, dB_2(t)], \end{cases} \tag{2}$$

in which they obtained the condition of the global existence of a unique positive solution and the stochastic permanence of the positive solution to the model. Furthermore, it is shown that both the prey and the predator species will become extinct with probability one if the noise is sufficiently large.

Yin *et al.* in [13] studied the following modified Leslie-Gower predator-prey model with Crowley-Martin functional response and spatial diffusion under a homogeneous Neumann boundary condition:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \triangle u + u(1 - u - \frac{v}{(1+au)(1+bv)}), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \triangle v + cv(1 - \frac{dv}{u+e}), & x \in \Omega, t > 0, \end{cases} \tag{3}$$

in which they obtained some important qualitative properties, including the existence of the global positive solution, the dissipation and persistence of the two species, the local and global asymptotic stability of the constant equilibria, and Hopf bifurcation around the interior constant equilibrium.

On the other hand, it is well known that there are many natural or man-made factors in the real world, such as earthquake, flooding, drought, crop-dusting, planting, hunting and harvesting. These kinds of factors can bring sudden changes to an ecological system, and the intrinsic discipline of the environment or the species in the system will often undergoes these changes in a relatively short interval, usually we could regard it happens at some fixed times. From the viewpoint of mathematics, such sudden changes could be described by the impulsive effects to the system. In addition, when a prey-predator system is studied, it is more reasonable to consider time delay during the predation, because there is often

a digest time instead of transforming food into intrinsic growth rate of the predator. As far as the impulsive system is concerned, there have appeared much excellent work in the last decades, such as impulsive effects in ecological systems (see [14–21]), in epidemic systems (see [22]) and in the neural network models (see [23–25]). Besides, there are many important monographs in this field (see [26–28]).

Particularly, research on the impulsive system with delays still seems to be a hot issue (see [29–37]), and these kinds of hybrid systems are usually called impulsive functional systems. When an impulsive functional system is concerned, the difficulty both in the impulsive differential equation and in the functional differential equation will occur at the same time, then the dynamical behaviors, such as permanence, periodic solution, almost periodic solution, and its asymptotical stability properties, as well as bifurcations and chaotic behaviors *etc.*, might be richer, more complex, and more interesting.

Enlightened by the above literature, we propose an almost periodic prey-predator system with Crowly-Martin type functional response and impulsive effects in this paper, in which digest delays are also considered in the process of predation, and the final model is as follows:

$$
\begin{cases}
\frac{dx}{dt} = x(t)[r_1(t) - d_1(t)x(t-\tau_1) - \frac{c_1(t)y(t-\tau_4)}{(1+\alpha(t)x(t-\tau_3))(1+\beta(t)y(t-\tau_4))}], \\
\frac{dy}{dt} = y(t)[r_2(t) - d_2(t)y(t-\tau_2) + \frac{c_2(t)x(t-\tau_3)}{(1+\alpha(t)x(t-\tau_3))(1+\beta(t)y(t-\tau_4))}], \\
x(t_k^+) = (1 + h_{1k})x(t_k), \\
y(t_k^+) = (1 + h_{2k})y(t_k),
\end{cases}
\quad
\begin{array}{l}
t \neq t_k, k \in N, \\
\\
t = t_k, k \in N,
\end{array}
\tag{4}
$$

where $x(t)$, $y(t)$ are population densities of the predator and the prey at time t, respectively. τ_i ($i = 1,2,3,4$) are all nonnegative constants, $d_j(t)$ ($j = 1,2$) denote the inner density resistance of the predator and of the prey. $h_{jk} > -1$, $j = 1,2$, $k \in N$, when $h_{jk} > 0$, the impulsive effects represent planting, while $h_{jk} < 0$ represents the impulsive denote harvesting.

Throughout the present paper, we define

$$
f^u = \sup_{t \in R^+} f(t), \qquad f^l = \inf_{t \in R^+} f(t),
$$

for any bounded function $f(t)$ defined on $R^+ = [0, +\infty)$.

Further, we assume that

(C1) $r_i(t)$, $c_i(t)$, $d_i(t)$ ($i = 1,2$), $\alpha(t)$ and $\beta(t)$ are all bounded and positive almost periodic functions;

(C2) $H_i(t) = \prod_{0 < t_k < t}(1 + h_{ik})$, $i = 1,2$, $k \in N$ is almost period functions and there exist positive constants H_i^u and H_i^l such that $H_i^l \leq H_i(t) \leq H_i^u$.

The rest of this article is organized as follows: In Section 2, we will give some definitions and several useful lemmas for the proof of our main results. In Section 3, we will state and prove our main results such as permanence of the system, existence, and the uniqueness of an almost periodic solution which is uniformly asymptotically stable by constructing a series of Lyapunov functionals. In the last section, we give a numerical examples to support our theoretical results, then provide a brief discussion and summary of our main results.

2 Preliminaries

In this section, we will state the following definitions and lemmas, which will be used in the next sections.

We have $K = \{t_k \in R | t_k < t_{k+1}, k \in N, \lim_{k \to \pm\infty} t_k = \pm\infty\}$, and we thus denote the set of all sequences that are unbounded and increasing. Let $D \subset \Omega$, $\Omega \neq \Phi$, $\tau = \max_{1 \leq i \leq 4}\{2\tau_i\}$, $\xi_0 \in R$. Also, we denote $PC(\xi_0)$ is the space of all functions $\phi : [\xi_0 - \tau, \xi_0] \to \Omega$ having points of discontinuity at $\mu_1, \mu_2, \ldots \in [\xi_0 - \tau, \xi_0]$ of the first kind and left continuous at these points.

For $J \subset R$, $PC(J, R)$ is the space of all piecewise continuous functions from J to R with points of discontinuity of the first kind t_k, at which we have left continuity.

Let $\phi_1, \phi_2 \in PC(0)$, denote by $x(t) = x(t; 0, \phi_1)$, $y(t) = y(t; 0, \phi_2)$, $x, y \in \Omega$ the solution of system (4) satisfying the following initial conditions:

$$0 \leq x(s; 0, \phi_1) = \phi_1(s) < \infty, \qquad 0 \leq y(s; 0, \phi_2) = \phi_2(s) < \infty, \quad s \in [-\tau, 0],$$
$$x(0; 0, \phi_1) = \phi_1(0) > 0, \qquad y(0; 0, \phi_2) = \phi_2(0) > 0. \tag{5}$$

Since the solution of system (4) with initial conditions (5) is a piecewise continuous function with points of discontinuity of the first kind, t_k, $k \in Z$, we introduce the following definitions for the almost periodicity.

Definition 2.1 (see [38]) The set of sequence $\{t_k^j\}$, $t_j^k = t_{k+j} - t_k$, $k, j \in N$, $t_k \in K$ is said to be uniformly almost periodic, if for arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periodic common for any sequences.

Definition 2.2 (see [38]) The function $\varphi \in PC(R, R)$ is said to be almost periodic if the following conditions hold:
(1) The set of sequences $\{t_k^j\}$, $t_j^k = t_{k+j} - t_k$, $k, j \in N$, $t_k \in K$ is uniformly almost periodic.
(2) For any $\varepsilon > 0$, there exists a real number $\delta > 0$, such that if the points t_1 and t_2 belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t_1 - t_2| < \delta$, then $|\varphi(t_1) - \varphi(t_2)| < \varepsilon$.
(3) For any $\varepsilon > 0$, there exists a relatively dense set T such that if $\eta \in T$, then $|\varphi(t + \eta) - \varphi(t)| < \varepsilon$ for all $t \in R$ satisfying the condition $|t - t_k| > \varepsilon$, $k \in N$. The elements of T are called ε-almost periods.

For the following system:

$$\begin{cases} \frac{du}{dt} = u(t)[r_1(t) - D_1(t)u(t - \tau_1) - \frac{C_1(t)v(t-\tau_4)}{(1+A(t)u(t-\tau_3))(1+B(t)v(t-\tau_4))}], \\ \frac{dv}{dt} = v(t)[r_2(t) - D_2(t)v(t - \tau_2) + \frac{C_2(t)u(t-\tau_3)}{(1+A(t)u(t-\tau_3))(1+B(t)v(t-\tau_4))}], \end{cases} \tag{6}$$

with initial value

$$u(s) = \phi_1(s), \qquad v(s) = \phi_2(s), \quad \phi_i \in PC(0), i = 1, 2,$$
$$0 < u(s; 0, \phi_1) = \phi_1(s) < +\infty, \qquad 0 < v(s; 0, \phi_2) = \phi_2(s) < +\infty, \quad s \in (-\tau, 0]. \tag{7}$$

Here the expressions of the functions $A(t)$, $B(t)$, $C_i(t)$, $D_i(t)$, $i = 1, 2$, are given as follows:

$$A(t) = H_1(t)\alpha(t) = \prod_{0 < t_k < t}(1 + h_{1k})\alpha(t),$$

$$B(t) = H_2(t)\beta(t) = \prod_{0 < t_k < t}(1 + h_{2k})\beta(t),$$

$$C_1(t) = H_1(t)c_1(t) = \prod_{0 < t_k < t} (1 + h_{1k})c_1(t),$$

$$C_2(t) = H_2(t)c_2(t) = \prod_{0 < t_k < t} (1 + h_{2k})c_2(t),$$

$$D_1(t) = H_1(t)d_1(t) = \prod_{0 < t_k < t} (1 + h_{1k})d_1(t),$$

$$D_2(t) = H_2(t)d_2(t) = \prod_{0 < t_k < t} (1 + h_{2k})d_2(t).$$

In the following, we will give some useful lemmas, which will be used in the next sections.

Lemma 2.1 *Assume that $(u(t), v(t))^T$ is any solution of system* (6) *with the initial conditions* (7), *then $u(t) > 0$, $v(t) > 0$ for all $t \in R^+$.*

Proof Let

$$M(t) = r_1(t) - D_1(t)u(t - \tau_1) - \frac{C_1(t)v(t - \tau_4)}{(1 + A(t)u(t - \tau_3))(1 + B(t)v(t - \tau_4))},$$

$$N(t) = r_2(t) - D_2(t)v(t - \tau_2) + \frac{C_2(t)u(t - \tau_3)}{(1 + A(t)u(t - \tau_3))(1 + B(t)v(t - \tau_4))}.$$

Then from system (6) we have

$$u(t) = u(0) \exp \left\{ \int_0^t M(s)\, ds \right\} = \phi_1(0) \exp \left\{ \int_0^t M(s)\, ds \right\} > 0,$$

$$v(t) = v(0) \exp \left\{ \int_0^t N(s)\, ds \right\} = \phi_2(0) \exp \left\{ \int_0^t N(s)\, ds \right\} > 0.$$

This completes the proof of this lemma. $\square$

Lemma 2.2 *For system* (4) *and system* (6), *we have the following results*:
(1) *If $(u(t), v(t))^T$ is a solution of system* (6), *then*

$$\left(x(t), y(t) \right)^T = \left(\prod_{0 < t_k < t} (1 + h_{1k})u(t), \prod_{0 < t_k < t} (1 + h_{2k})v(t) \right)^T$$

is a solution of system (4).
(2) *If $(x(t), y(t))^T$ is a solution of system* (4), *then*

$$\left(u(t), v(t) \right)^T = \left(\prod_{0 < t_k < t} (1 + h_{1k})^{-1}x(t), \prod_{0 < t_k < t} (1 + h_{2k})(1 + h_{1k})^{-1}y(t) \right)^T$$

is a solution of system (6).

Proof (1) $(u(t), v(t))^T$ is a solution of system (6). That is, for any $t \neq t_k$, $k \in N$,

$$u(t) = \prod_{0 < t_k < t} (1 + h_{1k})^{-1}x(t), \qquad v(t) = \prod_{0 < t_k < t} (1 + h_{2k})^{-1}y(t)$$

is a solution of system (6), which yields

$$\prod_{0<t_k<t} (1 + h_{1k})^{-1} x'(t)$$

$$= \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t) \left\{ r_1(t) - D_1(t) \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t - \tau_1) \right.$$

$$\left. - \frac{C_1(t) \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t - \tau_4)}{(1 + A(t) \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t - \tau_3))(1 + B(t) \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t - \tau_4))} \right\}, \quad (8)$$

$$\prod_{0<t_k<t} (1 + h_{2k})^{-1} y'(t)$$

$$= \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t) \left\{ r_2(t) - D_2(t) \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t - \tau_2) \right.$$

$$\left. + \frac{C_2(t) \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t - \tau_3)}{(1 + A(t) \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t - \tau_3))(1 + B(t) \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t - \tau_4))} \right\}. \quad (9)$$

By the previous definitions of the functions $A(t)$, $B(t)$, $C_i(t)$, and $D_i(t)$, $i = 1, 2$, it follows from the simplification of (8) and (9) that

$$\begin{cases} \frac{dx}{dt} = x(t)[r_1(t) - d_1(t)x(t - \tau_1) - \frac{c_1(t)y(t-\tau_4)}{(1+\alpha(t)x(t-\tau_3))(1+\beta(t)y(t-\tau_4))}], \\ \frac{dy}{dt} = y(t)[r_2(t) - d_2(t)y(t - \tau_2) + \frac{c_2(t)x(t-\tau_3)}{(1+\alpha(t)x(t-\tau_3))(1+\beta(t)y(t-\tau_4))}]. \end{cases} \quad (10)$$

On the other hand, when $t = t_k$, $k \in N$, we have

$$x(t_k^+) = \lim_{t \to t_k^+} \prod_{0<t_k<t} (1 + h_{1k})u(t) = \prod_{0<t_j \le t_k} (1 + h_{1j})u(t_k)$$

$$= (1 + h_{1k}) \prod_{0<t_k<t} (1 + h_{1k})u(t_k) = (1 + h_{1k})x(t_k), \quad (11)$$

$$y(t_k^+) = \lim_{t \to t_k^+} \prod_{0<t_k<t} (1 + h_{2k})v(t) = \prod_{0<t_j \le t_k} (1 + h_{2j})v(t_k)$$

$$= (1 + h_{2k}) \prod_{0<t_k<t} (1 + h_{2k})v(t_k) = (1 + h_{2k})y(t_k). \quad (12)$$

Combined (10) with (11) and (12), we can see that $(x(t), y(t))^T$ is the solution of the impulsive system (4).

(2) Because $u(t)$ and $v(t)$ are continuous on each interval $(t_k, t_{k+1}]$, we only need to check the continuity of $u(t)$ and $v(t)$ at the impulsive point $t = t_k$, $k \in N$.

Since

$$u(t) = \prod_{0<t_k<t} (1 + h_{1k})^{-1} x(t), \qquad v(t) = \prod_{0<t_k<t} (1 + h_{2k})^{-1} y(t),$$

we have

$$u(t_k^+) = \prod_{0<t_j \le t_k} (1 + h_{1j})^{-1} x(t_k^+) = \prod_{0<t_j<t_k} (1 + h_{1j})^{-1} x(t_k) = u(t_k), \quad (13)$$

$$u\left(t_k^-\right) = \prod_{0<t_j<t_k} (1 + h_{1j})^{-1} x\left(t_k^-\right) = \prod_{0<t_j<t_k} (1 + h_{1j})^{-1} x(t_k) = u(t_k). \tag{14}$$

Similarly, we can check that

$$v\left(t_k^+\right) = v\left(t_k^-\right) = v(t_k). \tag{15}$$

Thus, $u(t)$ and $v(t)$ are continuous on $[0, +\infty)$.

On the other hand, similar to the above proof of step (1), we can verify that $(u(t), v(t))^T$ satisfies system (6).

Then $(u(t), v(t))^T$ is a solution of the non-impulsive system (6).

This completes the proof of Lemma 2.2. □

Lemma 2.3 (see [39])

(1) *Assume that for $w(t) > 0$, $t \geq 0$, we have*

$$w'(t) \leq w(t)\big(a - bw(t - \tau)\big)$$

with initial conditions $w(s) = \phi(s) \geq 0$, $s \in [-\tau, 0]$, where a, b are positive constants, then there exists a positive constant w^* such that

$$\limsup_{t \to \infty} w(t) \leq w^* := \frac{ae^{a\tau}}{b}.$$

(2) *Assume that for $w(t) > 0$, $t \geq 0$, we have*

$$w'(t) \geq w(t)\big(c - dw(t - \tau)\big)$$

with initial conditions $w(s) = \phi(s) \geq 0$, $s \in [-\tau, 0]$, where c, d are positive constants, then there exists a positive constant w_* such that

$$\liminf_{t \to \infty} w(t) \geq w_* := \frac{ce^{(c-dw^*)\tau}}{d}.$$

Let R^2 be the plane Euclidean space with element $X = (x, y)^T$ and norm $|X|_0 = |x| + |y|$, $C = C([-\tau, 0], R^2)$, $B \in R^+$, and denote $C_B = \{\varphi = (\varphi_1(s), \varphi_2(s))^T \in C | \|\varphi\| \leq B\}$ with $\|\varphi\| = \sup_{s \in [-\tau, 0]} |\varphi(s)|_0 = \sup_{s \in [-\tau, 0]} (|\varphi_1(s)| + |\varphi_2(s)|)$.

Consider the following almost periodic system with delay:

$$x'(t) = f(t, x_t), \quad t \in R, \tag{16}$$

where $f(t, \varphi)$ is continuous in $(t, \varphi) \in R \times C_B$ and almost periodic in t uniformly for $\varphi \in C_B$, $\forall \rho > 0$, $\exists M(\rho) > 0$ such that $|f(t, \varphi)| \leq M(\rho)$ as $t \in R$, $\varphi \in C_\rho$, while $x_t \in C_B$ is defined as $x_t(s) = x(t + s)$ for $s \in [-\tau, 0]$.

The associated product system of (16) is in the form of

$$x'(t) = f(t, x_t), \qquad y'(t) = f(t, y_t), \quad t \in R. \tag{17}$$

Then by the conclusions of [38, 40], we have Lemma 2.4.

Lemma 2.4 (see [40]) *For $\phi, \psi \in C_B$, suppose that there exists a Lyapunov function $V(t, \phi, \psi)$ defined on $R^+ \times C_B \times C_B$ satisfying the following three conditions*:

(1) $u(\|\phi - \psi\|) \leq V(t, \phi, \psi) \leq v(\|\phi - \psi\|)$, *where $u, v \in \mathcal{P} = \{u : R^+ \to R^+ | u$ is continuous increasing function and $u(s) \to 0$, as $s \to 0\}$*;

(2) $|V(t, \bar{\phi}, \bar{\psi}) - V(t, \hat{\phi}, \hat{\psi})| \leq L(\|\bar{\phi} - \hat{\phi}\| + \|\bar{\psi} - \hat{\psi}\|)$, *where L is a positive constant*;

(3) $D^+ V(t, \phi, \psi)|_{(17)} \leq -\gamma V(t, \phi, \psi)$, *where γ is a positive constant*.

Further, assume that (16) *has a solution $x(t, v, \phi)$ such that $|x(t, v, \phi)| \leq B_1$ for $t \geq v \geq 0$, $B > B_1 > 0$. Then system* (16) *has a unique almost periodic solution which is uniformly asymptotically stable.*

Lemma 2.5 (see [36]) *For any $x, y, u, v \in R$, we have the following important inequality*:

$$\left| |x - y| - |u - v| \right| \leq |x - u| + |y - v|. \tag{18}$$

3 Main results

First in this section we will give some notations as follows:

$$u^* = \frac{r_1^u}{D_1^l} \exp\{r_1^u \tau_1\}, \qquad u_* = \frac{r_1^l B^l - C_1^u}{B^l D_1^u} \exp\left\{ \left(\frac{r_1^l B^l - C_1^u}{B^l} - D_1^u u^* \right) \tau_1 \right\},$$

$$v^* = \frac{A^l r_2^u + C_2^u}{A^l D_2^l} \exp\left\{ \frac{A^l r_2^u + C_2^u}{A^l} \tau_2 \right\}, \qquad v_* = \frac{r_2^l}{D_2^u} \exp\{(r_2^l - D_2^u v^*) \tau_2\}.$$

Theorem 3.1 *Assume that* (C1) *and* (C2) *hold, suppose further that*

(C3) $C_1^u < B^l r_1^l$,

then any positive solution $(u(t), v(t))^T$ of system (6) *satisfies*

$$u_* \leq \liminf_{t \to +\infty} u(t) \leq \limsup_{t \to +\infty} u(t) \leq u^*,$$

$$v_* \leq \liminf_{t \to +\infty} v(t) \leq \limsup_{t \to +\infty} v(t) \leq v^*.$$

Proof Let $(u(t), v(t))^T$ be any solution of system (6), then from the first equation of (6),

$$u'(t) \leq u(t)\left[r_1^u - D_1^l u(t - \tau_1)\right].$$

By the first conclusion of Lemma 2.3, we have

$$\limsup_{t \to +\infty} u(t) \leq \frac{r_1^u}{D_1^l} \exp\{r_1^u \tau_1\} = u^*. \tag{19}$$

Similarly, by the second equation of system (6),

$$v'(t) \leq v(t)\left[r_2^u - D_2^l v(t - \tau_2) + \frac{C_2^u u(t - \tau_3)}{1 + A^l u(t - \tau_3)}\right] \leq v(t)\left[\left(r_2^u + \frac{C_2^u}{A^l}\right) - D_2^l v(t - \tau_2)\right],$$

which leads to

$$\limsup_{t \to +\infty} v(t) \leq \frac{A^l r_2^u + C_2^u}{A^l D_2^l} \exp\left\{ \frac{A^l r_2^u + C_2^u}{A^l} \tau_2 \right\} = v^*. \tag{20}$$

Therefore, there exists a $T_1 > 0$, such that $v(t) \leq v^*$ when $t > T_1$.

Thus, when $t > T_1 + \tau$, from the first equation of system (6) again we have

$$u'(t) \geq u(t)\left[r_1^l - D_1^u u(t - \tau_1) - \frac{C_1^u v^*}{1 + B^l v^*}\right] \geq u(t)\left[\left(r_1^l - \frac{C_1^u}{B^l}\right) - D_1^u u(t - \tau_1)\right].$$

Then by the second conclusion of Lemma 2.3, we have

$$\liminf_{t \to +\infty} u(t) \geq \frac{r_1^l B^l - C_1^u}{B^l D_1^u} \exp\left\{\left(\frac{r_1^l B^l - C_1^u}{B^l} - D_1^u u^*\right)\tau_1\right\} = u_*. \tag{21}$$

On the other hand, from the second equation of system (6) again, we can get

$$v(t) \geq v(t)\left[r_2^l - D_2^u v(t - \tau_2)\right].$$

Then, by the second conclusion of Lemma 2.3 again, we have

$$\liminf_{t \to +\infty} v(t) \geq \frac{r_2^l}{D_2^u} \exp\left\{\left(r_2^l - D_2^u v^*\right)\tau_2\right\} = v_*. \tag{22}$$

Thus, we complete the proof of this theorem by combining (19), (20), (21), and (22). □

Theorem 3.2 *Assume that* (C1)-(C3) *hold, then any positive solution* $(x(t), y(t))^T$ *of system* (4) *satisfies*

$$H_1^l u_* \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq H_1^u u^*,$$

$$H_2^l v_* \leq \liminf_{t \to +\infty} y(t) \leq \limsup_{t \to +\infty} y(t) \leq H_2^u v^*.$$

Proof Since $(x(t), y(t))^T$ is a solution of system (4), then by the second conclusion of Lemma 2.2,

$$\left(u(t), v(t)\right)^T = \left(\frac{x(t)}{H_1(t)}, \frac{y(t)}{H_2(t)}\right)^T \tag{23}$$

is a solution of system (6).

Then it follows from Theorem 3.1 that

$$u_* \leq \liminf_{t \to +\infty} \frac{x(t)}{H_1(t)} \leq \liminf_{t \to +\infty} u(t) \leq \limsup_{t \to +\infty} u(t) = \limsup_{t \to +\infty} \frac{x(t)}{H_1(t)} \leq u^*, \tag{24}$$

$$v_* \leq \liminf_{t \to +\infty} \frac{y(t)}{H_2(t)} \leq \liminf_{t \to +\infty} v(t) \leq \limsup_{t \to +\infty} v(t) = \limsup_{t \to +\infty} \frac{y(t)}{H_2(t)} \leq v^*, \tag{25}$$

which implies that

$$H_1^l u_* \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq H_1^u u^*, \tag{26}$$

$$H_2^l v_* \leq \liminf_{t \to +\infty} y(t) \leq \limsup_{t \to +\infty} y(t) \leq H_2^u v^*. \tag{27}$$

This completes the proof of this theorem. □

Remark 3.1 Suppose that (C1)-(C3) hold, then system (6) is permanent.

Before we discuss the existence and uniformly asymptotically stability of a unique almost periodic solution of system (4), we will introduce the following notations first:

$$\alpha_1 = K_4\tau_2 + K_5\tau_4; \qquad \beta_1 = D_1^l u_* - \sqrt{K_1} - (K_2 + K_3)\tau_1 - (K_1 + K_3)\tau_3;$$

$$\alpha_2 = D_2^l v_* + P - (\tau_2 + \tau_4)L_3 - L_4\tau_2 - L_5\tau_4; \qquad \beta_2 = L_1\tau_1 + L_2\tau_3,$$

where

$$K_1 = \left[\frac{A^u C_1^u u^* v^*}{M^2 N}\right]^2, \qquad K_2 = \left(D_1^u u^*\right)^2;$$

$$K_3 = \frac{A^u C_1^u D_1^u (u^*)^2 v^*}{M^2 N}, \qquad K_4 = \frac{C_2^u D_2^u u^* v^*}{M^2 N}, \qquad K_5 = \frac{B^u (C_2^u u^*)^2 v^*}{M^3 N^3};$$

$$L_1 = \frac{C_1^u D_1^u u^* v^*}{MN^2}, \qquad L_2 = \frac{A^u (C_1^u v^*)^2 u^*}{M^3 N^3}, \qquad L_3 = \frac{B^u C_2^u D_2^u u^* (v^*)^2}{MN^2};$$

$$L_4 = \left(D_2^u v^*\right)^2, \qquad L_5 = \left[\frac{B^u C_2^u u^* v^*}{MN^2}\right]^2, \qquad M_1 = \frac{C_2^u u^*}{M^2 N}, \qquad M_2 = \frac{C_1^u v^*}{MN^2};$$

$$M = 1 + A^l u_*, \qquad N = 1 + B^l v_*, \qquad P = \frac{B^l C_2^l u_* v_*}{(1 + A^u u^*)(1 + B^u v^*)^2}.$$

Theorem 3.3 *Assume that* (C1)-(C3) *hold, further assume that*

(C4) *positive λ_1 and λ_2 exist, such that $\frac{\alpha_1}{\beta_1} < \frac{\lambda_1}{\lambda_2} < \frac{\alpha_2}{\beta_2}$,*

then system (4) *admits a unique almost periodic solution which is uniformly asymptotically stable.*

Proof At first, we prove that system (6) has a unique uniformly asymptotically stable almost periodic solution.

In order to achieve this aim, we take a transformation

$$u(t) = e^{x_1(t)}, \qquad v(t) = e^{y_1(t)}, \quad t \in R^+. \tag{28}$$

Then system (6) is transformed into following system:

$$\begin{cases} x_1'(t) = r_1(t) - D_1(t)e^{x_1(t-\tau_1)} - \frac{C_1(t)e^{y_1(t-\tau_4)}}{(1+A(t)e^{x_1(t-\tau_3)})(1+B(t)e^{y_1(t-\tau_4)})}, \\ y_1'(t) = r_2(t) - D_2(t)e^{y_1(t-\tau_2)} + \frac{C_2(t)e^{x_1(t-\tau_3)}}{(1+A(t)e^{x_1(t-\tau_3)})(1+B(t)e^{y_1(t-\tau_4)})}. \end{cases} \tag{29}$$

Suppose that $U_1(t) = (x_1(t), y_1(t))^T$ and $U_2(t) = (x_2(t), y_2(t))^T$ are any two solutions of system (29), then the product system of (29) reads

$$\begin{cases} x_1'(t) = r_1(t) - D_1(t)e^{x_1(t-\tau_1)} - \frac{C_1(t)e^{y_1(t-\tau_4)}}{(1+A(t)e^{x_1(t-\tau_3)})(1+B(t)e^{y_1(t-\tau_4)})}, \\ y_1'(t) = r_2(t) - D_2(t)e^{y_1(t-\tau_2)} + \frac{C_2(t)e^{x_1(t-\tau_3)}}{(1+A(t)e^{x_1(t-\tau_3)})(1+B(t)e^{y_1(t-\tau_4)})}, \\ x_2'(t) = r_1(t) - D_1(t)e^{x_2(t-\tau_1)} - \frac{C_1(t)e^{y_2(t-\tau_4)}}{(1+A(t)e^{x_2(t-\tau_3)})(1+B(t)e^{y_2(t-\tau_4)})}, \\ y_2'(t) = r_2(t) - D_2(t)e^{y_2(t-\tau_2)} + \frac{C_2(t)e^{x_2(t-\tau_3)}}{(1+A(t)e^{x_2(t-\tau_3)})(1+B(t)e^{y_2(t-\tau_4)})}. \end{cases} \tag{30}$$

Denote

$$S^* = \left\{ \phi = (x_{1t}, y_{1t})^T \in C\big([-\tau, 0], R^2\big) \,|\, \ln u_* \le x_{1t} \le \ln u^*, \ln v_* \le y_{1t} \le \ln v^*, t \in R^+ \right\}.$$

For any $\Phi = (\phi_1, \phi_2)^T = (x_{1t}, y_{1t})^T \in S^*$, we can choose $\Psi = (\psi_1, \psi_2)^T = (x_{2t}, y_{2t})^T \in S^*$ such that

$$\big|\Phi(0) - \Psi(0)\big|_0 = \big|\phi_1(0) - \psi_1(0)\big| + \big|\phi_1(0) - \psi_1(0)\big| > 0. \tag{31}$$

Consider a Lyapunov functional $V(t) = V(t, \Phi, \Psi) = V(t, (x_{1t}, y_{1t})^T, (x_{2t}, y_{2t})^T)$ defined on $R^+ \times S^* \times S^*$ as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{32}$$

where

$$V_1(t) = \lambda_1 \big|x_1(t) - x_2(t)\big| + \lambda_2 \big|y_1(t) - y_2(t)\big|; \tag{33}$$

$$
\begin{aligned}
V_2(t) = {}& \lambda_1 K_1 \int_{-2\tau_3}^{-\tau_3} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds \\
& + \lambda_1 K_2 \int_{-2\tau_1}^{-\tau_1} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds \\
& + \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds \\
& + \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds \\
& + \lambda_2 K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds \\
& + \lambda_2 K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} \int_{t+s}^{t} \big|x_1(r) - x_2(r)\big| \, dr \, ds;
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
V_3(t) = {}& \lambda_1 L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds \\
& + \lambda_1 L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds \\
& + \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds \\
& + \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds \\
& + \lambda_2 L_4 \int_{-2\tau_2}^{-\tau_2} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds \\
& + \lambda_2 L_5 \int_{-2\tau_4}^{-\tau_4} \int_{t+s}^{t} \big|y_1(r) - y_2(r)\big| \, dr \, ds;
\end{aligned}
\tag{35}
$$

$$V_4(t) = \lambda_2 M_1 \int_{t-\tau_3}^{t} \big|x_1(s) - x_2(s)\big| \, ds + \lambda_1 M_2 \int_{t-\tau_4}^{t} \big|y_1(s) - y_2(s)\big| \, ds. \tag{36}$$

According to the definitions of S^* and $V(t) = V(t, \Phi, \Psi)$, there is some positive constant M large enough such that

$$V(t) = V(t, \Phi, \Psi) \leq M. \tag{37}$$

By the structure of $V(t)$, it is easy to see that

$$V(t) \geq V_1(t) \geq \min\{\lambda_1, \lambda_2\}\big(\big|x_1(t) - x_2(t)\big| + \big|y_1(t) - y_2(t)\big|\big) = \underline{\lambda}\big|\Phi(0) - \Psi(0)\big|_0 > 0, \tag{38}$$

where $\underline{\lambda} = \min\{\lambda_1, \lambda_2\} > 0$.

Moreover, by the integrative inequality and the absolute-value inequality properties we have

$$\begin{aligned}
V(t) \leq \frac{1}{2}\Big\{ &2(\lambda_1 + \lambda_2) + \lambda_1 \tau^2 (K_1 + K_2 + 2K_3 + L_1 + L_2) \\
&+ \lambda_2 \tau^2 (K_4 + K_5 + 2L_3 + L_4 + L_5) + \tau(\lambda_2 M_1 + \lambda_1 M_2) \Big\} \\
&\times \sup_{s \in [-\tau, 0]} \big[\big|x_{1t}(s) - x_{2t}(s)\big| + \big|y_{1t}(s) - y_{2t}(s)\big| \big] \\
= &\, \bar{\lambda} \|\Phi - \Psi\|,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\bar{\lambda} := \frac{1}{2}\Big\{ &2(\lambda_1 + \lambda_2) + \lambda_1 \tau^2 (K_1 + K_2 + 2K_3 + L_1 + L_2) \\
&+ \lambda_2 \tau^2 (K_4 + K_5 + 2L_3 + L_4 + L_5) + \tau(\lambda_2 M_1 + \lambda_1 M_2) \Big\}.
\end{aligned}$$

Let $u, v \in C(R^+, R^+)$, choose $u = \underline{\lambda}s$, $v = \bar{\lambda}s$, then the first condition of Lemma 2.4 is satisfied.

For $\forall \bar{\Phi} = (x_{1t}, y_{1t})^T$, $\bar{\Psi} = (x_{2t}, y_{2t})^T$, $\hat{\Phi} = (x_{1t}^*, y_{1t}^*)^T$, $\hat{\Psi} = (x_{2t}^*, y_{2t}^*)^T \in S^*$, then from the previous definitions of $V_i(t)$, $i = 1, 2, 3, 4$, it is easy to calculate that

$$\begin{aligned}
&\big|V(t, \bar{\Phi}, \bar{\Psi}) - V(t, \hat{\Phi}, \hat{\Psi})\big| \\
&\quad = \lambda_1 \big| \big|x_1(t) - x_2(t)\big| - \big|x_1^*(t) - x_2^*(t)\big| \big| + \lambda_2 \big| \big|y_1(t) - y_2(t)\big| - \big|y_1^*(t) - y_2^*(t)\big| \big| \\
&\qquad + \lambda_1 K_1 \int_{-2\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds \\
&\qquad + \lambda_1 K_2 \int_{-2\tau_1}^{-\tau_1} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds \\
&\qquad + \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds \\
&\qquad + \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds \\
&\qquad + \lambda_2 K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds \\
&\qquad + \lambda_2 K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} \int_{t+s}^{t} \big| \big|x_1(r) - x_2(r)\big| - \big|x_1^*(r) - x_2^*(r)\big| \big|\, dr\, ds
\end{aligned}$$

$$+ \lambda_1 L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_4 \int_{-2\tau_2}^{-\tau_2} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_5 \int_{-2\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_2(r)| - |y_1^*(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 M_1 \int_{t-\tau_3}^{t} \big| |x_1(s) - x_2(s)| - |x_1^*(s) - x_2^*(s)| \big| \, ds$$

$$+ \lambda_1 M_2 \int_{t-\tau_4}^{t} \big| |y_1(s) - y_2(s)| - |y_1^*(s) - y_2^*(s)| \big| \, ds.$$

Then it follows from inequality (19) in Lemma 2.5 that

$$\big| V(t, \bar{\Phi}, \bar{\Psi}) - V(t, \hat{\Phi}, \hat{\Psi}) \big|$$

$$\leq \lambda_1 \big| |x_1(t) - x_1^*(t)| + |x_2(t) - x_2^*(t)| \big| + \lambda_2 \big| |y_1(t) - y_1^*(t)| + |y_2(t) - y_2^*(t)| \big|$$

$$+ \lambda_1 K_1 \int_{-2\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 K_2 \int_{-2\tau_1}^{-\tau_1} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} \int_{t+s}^{t} \big| |x_1(r) - x_1^*(r)| + |x_2(r) - x_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_1 L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} \int_{t+s}^{t} \big| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} \int_{t+s}^{t} \big| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_4 \int_{-2\tau_2}^{-\tau_2} \int_{t+s}^{t} \big| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \big| \, dr \, ds$$

$$+ \lambda_2 L_5 \int_{-2\tau_4}^{-\tau_4} \int_{t+s}^{t} \left| |y_1(r) - y_1^*(r)| + |y_2(r) - y_2^*(r)| \right| dr\, ds$$

$$+ \lambda_2 M_1 \int_{t-\tau_3}^{t} \left| |x_1(s) - x_1^*(s)| + |x_2(s) - x_2^*(s)| \right| ds$$

$$+ \lambda_1 M_2 \int_{t-\tau_4}^{t} \left| |y_1(s) - y_1^*(s)| + |y_2(s) - y_2^*(s)| \right| ds,$$

which yields

$$\left| V(t, \bar{\Phi}, \bar{\Psi}) - V(t, \hat{\Phi}, \hat{\Psi}) \right|$$

$$\leq \bar{\lambda} \sum_{i=1}^{2} \sup_{s \in [-\tau, 0]} \left[|x_{it}(s) - x_{it}^*(s)| + |y_{it}(s) - y_{it}^*(s)| \right]$$

$$= \bar{\lambda} \left(\| \bar{\Phi} - \hat{\Phi} \| + \| \bar{\Psi} - \hat{\Psi} \| \right). \tag{40}$$

This means the second condition of Lemma 2.4 is satisfied.

Finally, we will prove the last condition of Lemma 2.4.

In fact, calculating the right derivative $D^+ V_1(t)$ of $V_1(t)$ along the solutions of system (30) we have

$$D^+ V_1(t) = \left(x_1'(t) - x_2'(t) \right) \operatorname{sgn} \left(x_1(t) - x_2(t) \right) + \left(y_1'(t) - y_2'(t) \right) \operatorname{sgn} \left(y_1(t) - y_2(t) \right). \tag{41}$$

It follows from the mean-value theorem and the product system (30) that

$$x_1'(t) - x_2'(t)$$

$$= D_1(t) e^{x_2(t-\tau_1)} - D_1(t) e^{x_1(t-\tau_1)} + \frac{C_1(t) e^{y_2(t-\tau_4)}}{(1 + A(t) e^{x_2(t-\tau_3)})(1 + B(t) e^{y_2(t-\tau_4)})}$$

$$- \frac{C_1(t) e^{y_1(t-\tau_4)}}{(1 + A(t) e^{x_1(t-\tau_3)})(1 + B(t) e^{y_1(t-\tau_4)})}.$$

Then

$$x_1'(t) - x_2'(t)$$

$$= -D_1(t) e^{\theta_1(t)} \left[x_1(t - \tau_1) - x_2(t - \tau_1) \right]$$

$$+ \frac{A(t) C_1(t) e^{\theta_2(t) + \theta_3(t)}}{(1 + A(t) e^{\theta_2(t)})^2 (1 + B(t) e^{\theta_3(t)})} \left[x_1(t - \tau_3) - x_2(t - \tau_3) \right]$$

$$- \frac{C_1(t) e^{\theta_3(t)}}{(1 + A(t) e^{\theta_2(t)})(1 + B(t) e^{\theta_3(t)})^2} \left[y_1(t - \tau_4) - y_2(t - \tau_4) \right], \tag{42}$$

where $\theta_1(t)$ lie between $x_1(t - \tau_1)$ and $x_2(t - \tau_1)$, $\theta_2(t)$ lie between $x_1(t - \tau_3)$ and $x_2(t - \tau_3)$, $\theta_3(t)$ lie between $y_1(t - \tau_4)$ and $y_2(t - \tau_4)$.

That is,

$$x_1'(t) - x_2'(t)$$

$$= -D_1(t) e^{\theta_1(t)} \left[x_1(t) - x_2(t) \right] + D_1(t) e^{\theta_1(t)} \int_{t-\tau_1}^{t} \left[x_1'(s) - x_2'(s) \right] ds$$

$$+ \frac{A(t)C_1(t)e^{\theta_2(t)+\theta_3(t)}}{(1+A(t)e^{\theta_2(t)})^2(1+B(t)e^{\theta_3(t)})}[x_1(t)-x_2(t)]$$

$$- \frac{A(t)C_1(t)e^{\theta_2(t)+\theta_3(t)}}{(1+A(t)e^{\theta_2(t)})^2(1+B(t)e^{\theta_3(t)})}\int_{t-\tau_3}^{t}[x_1'(s)-x_2'(s)]\,ds$$

$$- \frac{C_1(t)e^{\theta_3(t)}}{(1+A(t)e^{\theta_2(t)})(1+B(t)e^{\theta_3(t)})^2}[y_1(t-\tau_4)-y_2(t-\tau_4)]. \tag{43}$$

Integrating both sides of (42) on the interval $[t-\tau_1,t]$, we have

$$\int_{t-\tau_1}^{t}[x_1'(s)-x_2'(s)]\,ds = -\int_{t-\tau_1}^{t}D_1(s)e^{\theta_1(s)}[x_1(s-\tau_1)-x_2(s-\tau_1)]\,ds$$

$$+ \int_{t-\tau_1}^{t}\frac{A(s)C_1(s)e^{\theta_2(s)+\theta_3(s)}}{(1+A(s)e^{\theta_2(s)})^2(1+B(s)e^{\theta_3(s)})}[x_1(s-\tau_3)-x_2(s-\tau_3)]\,ds$$

$$- \int_{t-\tau_1}^{t}\frac{C_1(s)e^{\theta_3(s)}}{(1+A(s)e^{\theta_2(s)})(1+B(s)e^{\theta_3(s)})^2}[y_1(s-\tau_4)-y_2(s-\tau_4)]\,ds$$

$$= -\int_{-2\tau_1}^{-\tau_1}D_1(t+s+\tau_1)e^{\theta_1(t+s+\tau_1)}[x_1(t+s)-x_2(t+s)]\,ds$$

$$+ \int_{-\tau_1-\tau_3}^{-\tau_3}\frac{A(t+s+\tau_3)C_1(t+s+\tau_3)e^{\theta_2(t+s+\tau_3)+\theta_3(t+s+\tau_3)}}{(1+A(t+s+\tau_3)e^{\theta_2(s)})^2(1+B(t+s+\tau_3)e^{\theta_3(t+s+\tau_3)})}$$

$$\times [x_1(t+s)-x_2(t+s)]\,ds$$

$$- \int_{-\tau_1-\tau_4}^{-\tau_4}\frac{C_1(t+s+\tau_4)e^{\theta_3(t+s+\tau_4)}}{(1+A(t+s+\tau_4)e^{\theta_2(t+s+\tau_4)})(1+B(t+s+\tau_4)e^{\theta_3(t+s+\tau_4)})^2}$$

$$\times [y_1(t+s)-y_2(t+s)]\,ds,$$

which implies that

$$\left|\int_{t-\tau_1}^{t}[x_1'(s)-x_2'(s)]\,ds\right| \le \sqrt{K_2}\int_{-2\tau_1}^{-\tau_1}|x_1(t+s)-x_2(t+s)|\,ds$$

$$+ \sqrt{K_1}\int_{-\tau_1-\tau_3}^{-\tau_3}|x_1(t+s)-x_2(t+s)|\,ds$$

$$+ M_2\int_{-\tau_1-\tau_4}^{-\tau_4}|y_1(t+s)-y_2(t+s)|\,ds. \tag{44}$$

Similarly, we can get

$$\left|\int_{t-\tau_3}^{t}[x_1'(s)-x_2'(s)]\,ds\right| \le \sqrt{K_2}\int_{-\tau_1-\tau_3}^{-\tau_1}|x_1(t+s)-x_2(t+s)|\,ds$$

$$+ \sqrt{K_1}\int_{-2\tau_3}^{-\tau_3}|x_1(t+s)-x_2(t+s)|\,ds$$

$$+ M_2\int_{-\tau_3-\tau_4}^{-\tau_4}|y_1(t+s)-y_2(t+s)|\,ds. \tag{45}$$

Thus,

$$(x_1'(t)-x_2'(t))\,\mathrm{sgn}(x_1(t)-x_2(t))$$

$$\le -(D_1^l u_* - \sqrt{K_1})|x_1(t)-x_2(t)|$$

$$+ K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} \big| x_1(t+s) - x_2(t+s) \big| \, ds$$

$$+ K_2 \int_{-2\tau_1}^{-\tau_1} \big| x_1(t+s) - x_2(t+s) \big| \, ds + L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} \big| y_1(t+s) - y_2(t+s) \big| \, ds$$

$$+ K_1 \int_{-2\tau_3}^{-\tau_3} \big| x_1(t+s) - x_2(t+s) \big| \, ds + K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} \big| x_1(t+s) - x_2(t+s) \big| \, ds$$

$$+ L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} \big| y_1(t+s) - y_2(t+s) \big| \, ds + M_2 \big| y_1(t-\tau_4) - y_2(t-\tau_4) \big|. \tag{46}$$

By the same argument, from the second and the fourth equation of system (30) we can derive that

$$y_1'(t) - y_2'(t)$$

$$= D_2(t) e^{y_2(t-\tau_2)} - D_2(t) e^{y_1(t-\tau_2)} + \frac{C_2(t) e^{x_1(t-\tau_3)}}{(1 + A(t) e^{x_1(t-\tau_3)})(1 + B(t) e^{y_1(t-\tau_4)})}$$

$$- \frac{C_2(t) e^{x_2(t-\tau_3)}}{(1 + A(t) e^{x_2(t-\tau_3)})(1 + B(t) e^{y_2(t-\tau_4)})}$$

$$= -D_2(t) e^{\theta_4(t)} \big[y_1(t-\tau_2) - y_2(t-\tau_2) \big]$$

$$+ \frac{C_2(t) e^{\theta_5(t)}}{(1 + A(t) e^{\theta_5(t)})^2 (1 + B(t) e^{\theta_6(t)})} \big[x_1(t-\tau_3) - x_2(t-\tau_3) \big]$$

$$- \frac{B(t) C_2(t) e^{\theta_5(t)+\theta_6(t)}}{(1 + A(t) e^{\theta_5(t)})(1 + B(t) e^{\theta_6(t)})^2} \big[y_1(t-\tau_4) - y_2(t-\tau_4) \big], \tag{47}$$

where $\theta_4(t)$ lie between $y_1(t-\tau_2)$ and $y_2(t-\tau_2)$, $\theta_5(t)$ lie between $x_1(t-\tau_3)$ and $x_2(t-\tau_3)$, $\theta_6(t)$ lie between $y_1(t-\tau_4)$ and $y_2(t-\tau_4)$.

Integrating both sides of (47) on the interval $[t-\tau_2, t]$, we have

$$\int_{t-\tau_2}^t \big[y_1'(s) - y_2'(s) \big] \, ds$$

$$= \int_{t-\tau_2}^t D_2(s) e^{\theta_4(s)} \big[y_1(s-\tau_2) - y_2(s-\tau_2) \big] \, ds$$

$$+ \int_{t-\tau_2}^t \frac{C_2(s) e^{\theta_5(s)}}{(1 + A(s) e^{\theta_5(s)})^2 (1 + B(s) e^{\theta_6(s)})} \big[x_1(s-\tau_3) - x_2(s-\tau_3) \big] \, ds$$

$$- \int_{t-\tau_2}^t \frac{B(s) C_2(s) e^{\theta_5(s)+\theta_5(s)}}{(1 + A(s) e^{\theta_5(s)})(1 + B(s) e^{\theta_6(s)})^2} \big[y_1(s-\tau_4) - y_2(s-\tau_4) \big] \, ds$$

$$= - \int_{-2\tau_2}^{-\tau_2} D_2(t+s+\tau_2) e^{\theta_4(t+s+\tau_2)} \big[y_1(t+s) - y_2(t+s) \big] \, ds$$

$$+ \int_{-\tau_2-\tau_3}^{-\tau_3} \frac{C_2(t+s+\tau_3) e^{\theta_5(t+s+\tau_3)} [x_1(t+s) - x_2(t+s)]}{(1 + A(t+s+\tau_3) e^{\theta_5(t+s+\tau_3)})^2 (1 + B(t+s+\tau_3) e^{\theta_6(t+s+\tau_3)})} \, ds$$

$$- \int_{-\tau_2-\tau_4}^{-\tau_4} \frac{B(t+s+\tau_4) C_2(t+s+\tau_4) e^{\theta_5(t+s+\tau_4)+\theta_5(t+s+\tau_4)}}{(1 + A(t+s+\tau_4) e^{\theta_5(t+s+\tau_4)})(1 + B(s) e^{\theta_6(t+s+\tau_4)})^2}$$

$$\times \big[y_1(t+s) - y_2(t+s) \big] \, ds, \tag{48}$$

which leads to

$$\left| \int_{t-\tau_2}^{t} \left[y_1'(s) - y_2'(s) \right] ds \right|$$

$$\leq \sqrt{L_4} \int_{-2\tau_2}^{-\tau_2} \left| y_1(t+s) - y_2(t+s) \right| ds + \sqrt{L_5} \int_{-\tau_2-\tau_4}^{-\tau_4} \left| y_1(t+s) - y_2(t+s) \right| ds$$

$$+ M_1 \int_{-\tau_2-\tau_3}^{-\tau_3} \left| x_1(t+s) - x_2(t+s) \right| ds. \tag{49}$$

By the same argument, we can obtain

$$\left| \int_{t-\tau_4}^{t} \left[y_1'(s) - y_2'(s) \right] ds \right|$$

$$\leq \sqrt{L_4} \int_{-\tau_2-\tau_4}^{-\tau_2} \left| y_1(t+s) - y_2(t+s) \right| ds + \sqrt{L_5} \int_{-2\tau_4}^{-\tau_4} \left| y_1(t+s) - y_2(t+s) \right| ds$$

$$+ M_1 \int_{-\tau_3-\tau_4}^{-\tau_3} \left| x_1(t+s) - x_2(t+s) \right| ds. \tag{50}$$

On the other hand, it follows from (47) that

$$y_1'(t) - y_2'(t)$$

$$= D_2(t) e^{\theta_4(t)} \int_{t-\tau_2}^{t} \left[y_1(s) - y_2(s) \right] ds$$

$$- D_2(t) e^{\theta_4(t)} \left[y_1(t) - y_2(t) \right] - \frac{B(t) C_2(t) e^{\theta_5(t)+\theta_6(t)}}{(1+A(t) e^{\theta_5(t)})(1+B(t) e^{\theta_6(t)})^2} \left[y_1(t) - y_2(t) \right]$$

$$+ \frac{B(t) C_2(t) e^{\theta_5(t)+\theta_6(t)}}{(1+A(t) e^{\theta_5(t)})(1+B(t) e^{\theta_6(t)})^2} \int_{t-\tau_4}^{t} \left[y_1(s) - y_2(s) \right] ds$$

$$- \frac{C_2(t) e^{\theta_5(t)}}{(1+A(t) e^{\theta_5(t)})^2(1+B(t) e^{\theta_6(t)})} \left[x_1(t-\tau_3) - x_2(t-\tau_3) \right]. \tag{51}$$

Thus, combined (51) with (49)-(50), one can deduce that

$$\left(y_1'(t) - y_2'(t) \right) \operatorname{sgn}\left(y_1(t) - y_2(t) \right)$$

$$\leq -\left(D_2^l \nu_* + P \right) \left| y_1(t) - y_2(t) \right| + L_4 \int_{-2\tau_2}^{-\tau_2} \left| y_1(t+s) - y_2(t+s) \right| ds$$

$$+ L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} \left| y_1(t+s) - y_2(t+s) \right| ds + K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} \left| x_1(t+s) - x_2(t+s) \right| ds$$

$$+ L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} \left| y_1(t+s) - y_2(t+s) \right| ds + L_5 \int_{-2\tau_4}^{-\tau_4} \left| y_1(t+s) - y_2(t+s) \right| ds$$

$$+ K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} \left| x_1(t+s) - x_2(t+s) \right| ds + M_1 \left| x_1(t-\tau_3) - x_2(t-\tau_3) \right|. \tag{52}$$

Therefore,

$$D^+ V_1(t) \leq -\lambda_1 \left(D_1^l u_* - \sqrt{K_1} \right) \left| x_1(t) - x_2(t) \right| - \lambda_2 \left(D_2^l \nu_* + P \right) \left| y_1(t) - y_2(t) \right|$$

$$+ \lambda_1 K_1 \int_{-2\tau_3}^{-\tau_3} \left| x_1(t+s) - x_2(t+s) \right| ds + \lambda_1 K_2 \int_{-2\tau_1}^{-\tau_1} \left| x_1(t+s) - x_2(t+s) \right| ds$$

$$+ \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} |x_1(t+s) - x_2(t+s)| \, ds + \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds$$

$$+ \lambda_2 K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds + \lambda_2 K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds$$

$$+ \lambda_1 L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds + \lambda_1 L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds$$

$$+ \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} |y_1(t+s) - y_2(t+s)| \, ds + \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds$$

$$+ \lambda_2 L_4 \int_{-2\tau_2}^{-\tau_2} |y_1(t+s) - y_2(t+s)| \, ds + \lambda_2 L_5 \int_{-2\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds$$

$$+ \lambda_2 M_1 |x_1(t-\tau_3) - x_2(t-\tau_3)| + \lambda_1 M_2 |y_1(t-\tau_4) - y_2(t-\tau_4)|. \tag{53}$$

On the other hand, according to the structure of $V_i(t)$, $i = 2, 3, 4$, we can easily calculate the right derivatives $D^+V_2(t)$, $D^+V_3(t)$, and $D^+V_4(t)$ along the solution of system (30) as follows:

$$D^+V_2(t) = (\lambda_1\tau_3 K_1 + \lambda_1\tau_1 K_2 + \lambda_1\tau_3 K_3 + \lambda_1\tau_1 K_3 + \lambda_2\tau_2 K_4 + \lambda_2\tau_4 K_5) \times |x_1(t) - x_2(t)|$$

$$- \lambda_1 K_1 \int_{-2\tau_3}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds - \lambda_1 K_2 \int_{-2\tau_1}^{-\tau_1} |x_1(t+s) - x_2(t+s)| \, ds$$

$$- \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_1} |x_1(t+s) - x_2(t+s)| \, ds - \lambda_1 K_3 \int_{-\tau_1-\tau_3}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds$$

$$- \lambda_2 K_4 \int_{-\tau_2-\tau_3}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds$$

$$- \lambda_2 K_5 \int_{-\tau_3-\tau_4}^{-\tau_3} |x_1(t+s) - x_2(t+s)| \, ds; \tag{54}$$

$$D^+V_3(t) = (\lambda_1\tau_1 L_1 + \lambda_1\tau_3 L_2 + \lambda_2\tau_4 L_3 + \lambda_2\tau_2 L_3 + \lambda_2\tau_2 L_4 + \lambda_2\tau_4 L_5) \times |y_1(t) - y_2(t)|$$

$$- \lambda_1 L_1 \int_{-\tau_1-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds - \lambda_1 L_2 \int_{-\tau_3-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds$$

$$- \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_2} |y_1(t+s) - y_2(t+s)| \, ds - \lambda_2 L_3 \int_{-\tau_2-\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds$$

$$- \lambda_2 L_4 \int_{-2\tau_2}^{-\tau_2} |y_1(t+s) - y_2(t+s)| \, ds \, ds$$

$$- \lambda_2 L_5 \int_{-2\tau_4}^{-\tau_4} |y_1(t+s) - y_2(t+s)| \, ds; \tag{55}$$

$$D^+V_4(t) = \lambda_2 M_1 \big[|x_1(t) - x_2(t)| - |x_1(t-\tau_3) - x_2(t-\tau_3)| \big]$$

$$+ \lambda_1 M_2 \big[|y_1(t) - y_2(t)| - |y_1(t-\tau_4) - y_2(t-\tau_4)| \big]. \tag{56}$$

From (53)-(56), by a simple reduction we can easily give the following estimations for the right derivatives of $V(t)$:

$$D^+V(t) \leq -\big\{ \lambda_1 \big(D_1^l u_* - \sqrt{K_1} \big) - (\lambda_1\tau_3 K_1 + \lambda_1\tau_1 K_2 + \lambda_1\tau_3 K_3 + \lambda_1\tau_1 K_3$$

$$+ \lambda_2\tau_2 K_4 + \lambda_2\tau_4 K_5) \big\} \times |x_1(t) - x_2(t)| - \big\{ \lambda_2 \big(D_2^l v_* + P \big) - (\lambda_1\tau_1 L_1$$

$$+ \lambda_1 \tau_3 L_2 + \lambda_2 \tau_4 L_3 + \lambda_2 \tau_2 L_3 + \lambda_2 \tau_2 L_4 + \lambda_2 \tau_4 L_5 \big\} \times \big| y_1(t) - y_2(t) \big|$$

$$= -(\lambda_1 \beta_1 - \lambda_2 \alpha_1) \big| x_1(t) - x_2(t) \big| - (\lambda_2 \alpha_2 - \lambda_1 \beta_2) \big| y_1(t) - y_2(t) \big|. \tag{57}$$

If we denote $\delta = \min\{\beta_1 - \frac{\lambda_2}{\lambda_1}\alpha_1, \alpha_2 - \frac{\lambda_1}{\lambda_2}\beta_2\}$, then $\delta > 0$ by the condition (C4).
Thus, it follows from equalities (37) and (38) that

$$
\begin{aligned}
D^+ V(t) &= -\left(\beta_1 - \frac{\lambda_2}{\lambda_1}\alpha_1\right)\lambda_1 \big| x_1(t) - x_2(t) \big| \\
&\quad - \left(\alpha_2 - \frac{\lambda_1}{\lambda_2}\beta_2\right)\lambda_2 \big| y_1(t) - y_2(t) \big| \\
&\leq -\delta\big[\lambda_1 \big| x_1(t) - x_2(t) \big| + \lambda_2 \big| y_1(t) - y_2(t) \big|\big] \\
&= -\delta V_1(t) = -\delta \frac{V_1(t)}{V(t)} V(t) \\
&\leq -\frac{\lambda |\Phi(0) - \Psi(0)|_0}{M} \delta V(t).
\end{aligned}
\tag{58}
$$

Let $\gamma = \frac{\lambda |\Phi(0) - \Psi(0)|_0}{M}\delta$, then $\gamma > 0$ and

$$D^+ V(t) \leq -\gamma V(t), \quad t \in R^+. \tag{59}$$

This means the last condition of Lemma 2.4 is satisfied. By Lemma 2.4, the system admits a unique uniformly asymptotically stable almost periodic solution $(x(t), y(t))^T$.

Thus, by the transformation (28), we can conclude that system (6) admits a unique uniformly asymptotically stable almost periodic solution $(u(t), v(t))^T = (e^{x(t)}, e^{y(t)})^T$.

Finally, we will explain that system (4) has a unique uniformly asymptotically stable almost periodic solution.

In fact, from Lemma 2.2, we know that

$$\big(x(t), y(t)\big)^T = \left(\prod_{0 < t_k < t} (1 + h_{1k}) u(t), \prod_{0 < t_k < t} (1 + h_{2k}) v(t) \right)^T$$

is a solution of system (4). Since the condition (C2) holds, similar to the proofs of Lemma 31 and Theorem 79 in [28], we can prove that $(x(t), y(t))^T$ is almost periodic. Therefore, $(x(t), y(t))^T$ is the unique uniformly asymptotically stable almost periodic solution of system (4) because of the uniqueness and the uniformly asymptotical stability of $(u(t), v(t))^T$.

This completes the proof of this theorem. □

4 Numerical simulations and discussions

In this section, we will give a numerical example to illustrate the feasibility of our analytical results, then some discussions of the effects of impulsive perturbations and time delays to the system are referred to in the end of the paper.

Example 4.1 Consider the following nonautonomous prey-predator system with impulsive effects and delays:

$$
\begin{cases}
\begin{aligned}
\frac{dx}{dt} &= x(t)[1.04 + 0.02\sin(2t) - (0.90 + 0.02\cos(3t))x(t-0.005) \\
&\quad - \frac{(0.03+0.01\sin(3t))y(t-0.001)}{(1+x(t-0.002))(1+y(t-0.001))}], \\
\frac{dy}{dt} &= y(t)[1.19 + 0.01\sin(3t) - (1.92 + 0.02\sin(2t))y(t-0.001) \\
&\quad + \frac{(0.005+0.001\cos(2t))x(t-0.002)}{(1+x(t-0.002))(1+y(t-0.001))}], \\
\end{aligned} & t \neq t_k, \\
\left.\begin{aligned}
x(t_k^+) &= (1 + h_{1k})x(t_k), \\
y(t_k^+) &= (1 + h_{2k})y(t_k),
\end{aligned}\right\} & t = t_k, k \in N,
\end{cases}
\tag{60}
$$

where $r_1(t) = 1.04 + 0.02\sin(2t)$, $r_2(t) = 1.19 + 0.01\sin(3t)$, $d_1(t) = 0.90 + 0.02\cos(3t)$, $d_2(t) = 1.92 + 0.02\sin(2t)$, $c_1(t) = 0.03 + 0.01\sin(3t)$, $c_2(t) = 0.005 + 0.001\cos(2t)$, $\alpha(t) = \beta(t) = 1$ are all positive, bounded and almost periodic functions which satisfy the condition (C1) in the paper.

If we can set $h_{1k} = -0.05$, $h_{2k} = -0.06$, $t_k = k$, $t \in [0, 20]$, and choose $H_1^l = 0.3584$, $H_2^l = 0.2901$, $H_1^u = 1$, $H_2^u = 1$, then it is not difficult to verify

$$
H_i^l \leq H_i(t) = \prod_{0 < t_k < t = 20} (1 + h_{ik}) \leq H_u^l \quad (i = 1, 2),
$$

which satisfies the condition (C2).

Furthermore, according to the notations in Section 3 one can make some simple calculations as follows:

$$u^* \approx 3.3779, \qquad u_* \approx 0.9482, \qquad v^* \approx 2.2101, \qquad v_* \approx 0.6064;$$

$$A^l \approx 0.3585, \qquad A^u = 1, \qquad B^l \approx 0.2901, \qquad B^u = 1;$$

$$r_1^l = 1.02, \qquad r_1^u = 1.06, \qquad r_2^l = 1.18, \qquad r_2^u = 1.20;$$

$$C_1^l \approx 0.0072, \qquad C_1^u = 0.04, \qquad C_2^l \approx 0.0012, \qquad C_2^u = 0.006,$$

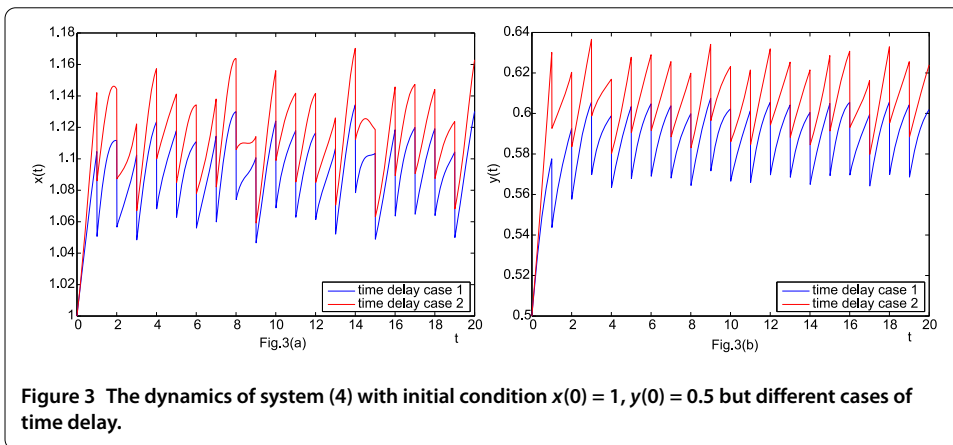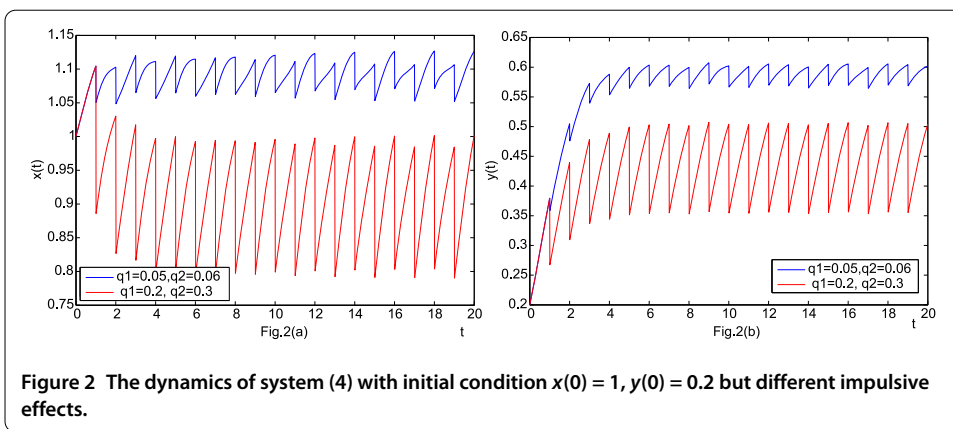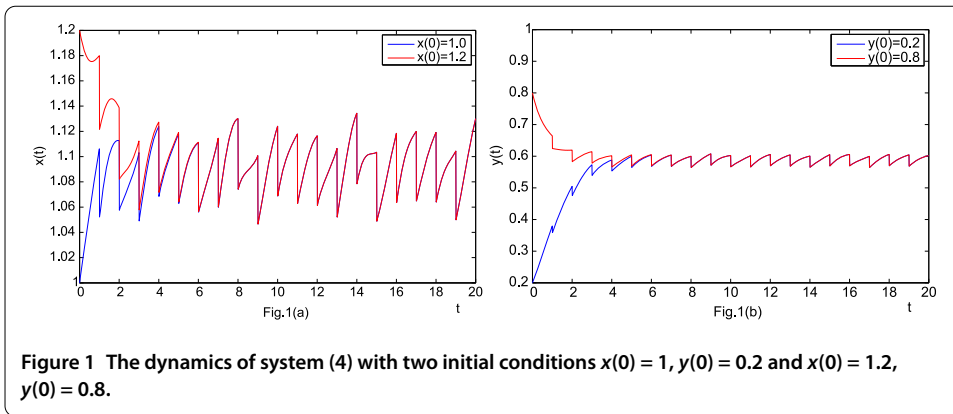$$D_1^l \approx 0.3155, \qquad D_1^u = 0.92, \qquad D_2^l \approx 0.5512, \qquad D_2^u = 1.94.$$

Then it is obvious that $C_1^u \approx 0.04 < B^l r_1^l \approx 0.2959$, which satisfies the condition (C3) in Theorems 3.2 and 3.3. Meanwhile, with the help of mathematical software such as Maple, we can calculate that

$$\alpha_1 \approx 0.00004, \qquad \beta_1 \approx 0.1081, \qquad \alpha_2 \approx 0.3156, \qquad \beta_2 \approx 0.00075.$$

Then we can choose positive constants $\lambda_1 = \lambda_2 = 1$, which satisfy

$$\frac{\alpha_1}{\beta_1} \approx 0.0004 < \frac{\lambda_1}{\lambda_2} = 1 \ll \frac{\alpha_2}{\beta_2} \approx 418.1291.$$

That is to say, the condition (C4) in Theorem 3.3 is also satisfied. Thus, all the conditions of Theorems 3.2 and 3.3 hold, according to the theoretical analysis, system (4) should be permanent and admits a unique uniformly asymptotically stable almost periodic solution. In order to verify this point by numerical simulations, we plot the time-series of the predator $x(t)$ (see Figure 1(a)) and the prey $y(t)$ (see Figure 1(b)). One can see that the population

Figure 1 The dynamics of system (4) with two initial conditions $x(0) = 1$, $y(0) = 0.2$ and $x(0) = 1.2$, $y(0) = 0.8$.

Figure 2 The dynamics of system (4) with initial condition $x(0) = 1$, $y(0) = 0.2$ but different impulsive effects.

Figure 3 The dynamics of system (4) with initial condition $x(0) = 1$, $y(0) = 0.5$ but different cases of time delay.

density of the predator oscillates around $x(t) = 1.09$ and the population density of the prey oscillates around $y(t) = 0.59$, this indicates the permanence of the system. The almost periodic phenomenon is also easily observed from the figure. Moreover, we can see that the solution $U_1(t) = (x_1(t), y_1(t))^T$ with initial condition $x(0) = 1.2$, $y(0) = 0.8$ gradually approximate the other solution of $U_2(t) = (x_2(t), y_2(t))^T$ with initial condition $x(0) = 1$, $y(0) = 0.2$, and it proved that the almost periodic case is asymptotically stable.

On the other hand, when the system is permanent, we will show the difference with two different impulsive effects (see Figure 2) and different cases of time delays (see Figure 2).

From Figure 2, we can see that both the mean density population of the predator and the prey are decreasing with stronger impulsive effects, such as enhancing the harvesting rate, the efficiency of the insecticides for the species, and so on. In addition, in order to show the effects of the delays to the dynamics of the species, we consider two different cases of time delays, Case 1 (with $\tau_1 = 0.005$, $\tau_2 = 0.001$, $\tau_3 = 0.002$, $\tau_4 = 0.001$ in Figure 3(a)) and Case 2 (with $\tau_1 = 0.5$, $\tau_2 = 1$, $\tau_3 = 0.2$, $\tau_4 = 0.1$ in Figure 3(b)) with the same initial condition $x(0) = 1$, $y(0) = 0.5$. We can see that the mean population density of the bigger time delays (Case 2) is bigger that the smaller ones (Case 1). This suggests that a longer digest delay in the predation may be conductive and could increase the possibility of the permanence of the population.

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