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Iterative oscillation tests for differential equations with several non-monotone arguments

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Abstract
Sufficient oscillation conditions involving lim sup and lim inf for first-order differential equations with several non-monotone deviating arguments and nonnegative coefficients are obtained. The results are based on the iterative application of the Grönwall inequality. Examples illustrating the significance of the results are also given.

MSC: 34K11; 34K06

Keywords: differential equations with deviating arguments; non-monotone arguments; delay equations; advanced arguments; oscillation; Grönwall inequality

1 Introduction

In this paper we consider the differential equation with several variable deviating arguments of either delay

$$x'(t) + \sum_{i=1}^{m} p_i(t)x\bigl(\tau_i(t)\bigr) = 0, \quad t \geq 0, \tag{1.1}$$

or advanced type

$$x'(t) - \sum_{i=1}^{m} p_i(t)x\bigl(\sigma_i(t)\bigr) = 0, \quad t \geq 0. \tag{1.2}$$

Equations (1.1) and (1.2) are studied under the following assumptions: everywhere $p_i(t) \geq 0$, $1 \leq i \leq m$, $t \geq 0$, $\tau_i(t)$, $1 \leq i \leq m$, are Lebesgue measurable functions satisfying

$$\tau_i(t) \leq t, \quad \forall t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leq i \leq m, \tag{1.3}$$

and

$$\sigma_i(t) \geq t, \quad t \geq 0, 1 \leq i \leq m, \tag{1.4}$$

respectively. In addition, we consider the initial condition for (1.1)

$$x(t) = \varphi(t), \quad t \leq 0, \tag{1.5}$$

where $\varphi : (-\infty, 0] \to \mathbb{R}$ is a bounded Borel measurable function.

Definition 1 *A solution of* (1.1), (1.5) is an absolutely continuous on $[0, \infty)$ function satisfying (1.1) for almost all $t \geq 0$ and (1.5) for all $t \leq 0$. By *a solution of* (1.2) we mean an absolutely continuous on $[0, \infty)$ function satisfying (1.2) for almost all $t \geq 0$.

In the special case $m = 1$ equations (1.1) and (1.2) reduce to the form

$$x'(t) + p(t)x\bigl(\tau(t)\bigr) = 0, \quad t \geq 0, \tag{1.6}$$

and

$$x'(t) - p(t)x\bigl(\sigma(t)\bigr) = 0, \quad t \geq 0, \tag{1.7}$$

respectively.

Definition 2 A solution $x(t)$ of (1.1) (or (1.2)) is *oscillatory* if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

In the last few decades, oscillatory behavior and stability of first-order differential equations with deviating arguments have been extensively studied; see, for example, papers [1–20] and references cited therein. For the general oscillation theory of differential equations the reader is referred to the monographs [21–24].

In 1978, Ladde [13] and in 1982, Ladas and Stavroulakis [12] proved that if

$$\liminf_{t\to\infty} \int_{\tau_{\max}(t)}^{t} \sum_{i=1}^{m} p_i(s)\,ds > \frac{1}{e}, \tag{1.8}$$

where $\tau_{\max}(t) = \max_{1\leq i\leq m}\{\tau_i(t)\}$, then all solutions of (1.1) oscillate, while if

$$\liminf_{t\to\infty} \int_{t}^{\sigma_{\min}(t)} \sum_{i=1}^{m} p_i(s)\,ds > \frac{1}{e}, \tag{1.9}$$

where $\sigma_{\min}(t) = \min_{1\leq i\leq m}\{\sigma_i(t)\}$, then all solutions of (1.2) oscillate. See also [24], Theorem 2.7.1, and [6], Theorem 1′.

In 1984, Hunt and Yorke [7] proved that if $t - \tau_i(t) \leq \tau_0$ for some $\tau_0 > 0$, $1 \leq i \leq m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\bigl(t - \tau_i(t)\bigr) > \frac{1}{e}, \tag{1.10}$$

then all solutions of (1.1) oscillate.

In 1990, Zhou [20] proved that if $\sigma_i(t) - t \leq \sigma_0$ for some $\sigma_0 > 0$, $1 \leq i \leq m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\bigl(\sigma_i(t) - t\bigr) > \frac{1}{e}, \tag{1.11}$$

then all solutions of (1.2) oscillate. See also this result in the monograph [23], Corollary 2.6.12.

For differential equation (1.6) with one delay, in 2011 Braverman and Karpuz [2] established the following theorem in the case that the argument $\tau(t)$ is non-monotone and $g(t)$ is defined as

$$g(t) = \sup_{s\leq t} \tau(s), \quad t \geq 0.$$

Theorem 1 *Assume that* (1.3) *holds and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{g(t)} p(\xi)\, d\xi \right\} ds > 1. \tag{1.12}$$

Then all solutions of (1.6) *oscillate.*

In 2014, Theorem 1 was improved by Stavroulakis [16] as follows.

Theorem 2 *Assume that* (1.3) *holds*; *we have*

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds \leq \frac{1}{e}$$

and

$$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{g(t)} p(\xi)\, d\xi \right\} ds > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}.$$

Then all solutions of (1.6) *oscillate.*

In 2015, Chatzarakis and Öcalan [3] established the following theorem in the case that the arguments $\sigma_i(t)$, $1 \leq i \leq m$ are non-monotone and $\rho_i(t) = \inf_{s\geq t} \sigma_i(s)$, $t \geq 0$, $\rho(t) = \min_{1\leq i\leq m} \rho_i(t)$, $t \geq 0$.

Theorem 3 *Assume that* (1.4) *holds, and either*

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} p_i(s) \exp\left\{ \sum_{j=1}^{m} \int_{\rho_i(t)}^{\sigma_i(s)} p_j(\xi)\, d\xi \right\} ds > 1, \tag{1.13}$$

or

$$\liminf_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} p_i(s) \exp\left\{ \sum_{j=1}^{m} \int_{\rho_i(t)}^{\sigma_i(s)} p_j(\xi)\, d\xi \right\} ds > \frac{1}{e}. \tag{1.14}$$

Then all solutions of (1.2) *oscillate.*

In addition to purely mathematical interest, consideration of non-monotone arguments is important, since it approximates the natural phenomena described by equations of the type of (1.1) or (1.2). In fact, there are always natural disturbances (*e.g.* noise in communication systems) that affect all the parameters of the equation and therefore monotone arguments will generally become non-monotone. In view of this, it is interesting to consider the case where the arguments (delays and advances) are non-monotone. In the present paper we obtain sufficient oscillation conditions involving lim sup and lim inf.

2 Main results

In this section, we establish sufficient oscillation conditions for (1.1) and (1.2) satisfying (1.3) and (1.4), respectively. The method we apply is based on the iterative construction of solution estimates and repetitive application of the Grönwall inequality. It also uses some ideas of [9], where some oscillation results for a differential equation with a single delay were established.

2.1 Delay equations

Let

$$g_i(t) = \sup_{0 \le s \le t} \tau_i(s), \quad t \ge 0, \tag{2.1}$$

and

$$g(t) = \max_{1 \le i \le m} g_i(t), \quad t \ge 0. \tag{2.2}$$

As follows from their definitions, the functions $g_i(t)$, $1 \le i \le m$ and $g(t)$ are non-decreasing Lebesgue measurable functions satisfying $g(t) \le t$, $g_i(t) \le t$, $1 \le i \le m$ for all $t \ge 0$.

The following lemma provides an estimation for a rate of decay for a positive solution. Such estimates are a basis for most oscillation conditions.

Lemma 1 *Assume that $x(t)$ is a positive solution of* (1.1). *Denote*

$$a_1(t,s) := \exp\left\{ \int_s^t \sum_{i=1}^m p_i(\zeta)\, d\zeta \right\} \tag{2.3}$$

and

$$a_{r+1}(t,s) := \exp\left\{ \int_s^t \sum_{i=1}^m p_i(\zeta) a_r\bigl(\zeta, \tau_i(\zeta)\bigr)\, d\zeta \right\}, \quad r \in \mathbb{N}. \tag{2.4}$$

Then

$$x(t) a_r(t,s) \le x(s), \quad 0 \le s \le t. \tag{2.5}$$

Proof The function $x(t)$ is a positive solution of (1.1) for any t, so

$$x'(t) = -\sum_{i=1}^m p_i(t) x\bigl(\tau_i(t)\bigr) \le 0, \quad t \ge 0,$$

which means that the solution $x(t)$ is monotonically decreasing. Thus $x(\tau_i(t)) \geq x(t)$ and

$$x'(t) + x(t)\sum_{i=1}^{m} p_i(t) \leq 0, \quad t \geq 0.$$

Applying the Grönwall inequality, we obtain

$$x(t) \leq x(s)\exp\left\{-\int_s^t \sum_{i=1}^{m} p_i(\zeta)\,d\zeta\right\}, \quad 0 \leq s \leq t,$$

or

$$x(t)\exp\left\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)\,d\zeta\right\} \leq x(s), \quad 0 \leq s \leq t,$$

that is, estimate (2.5) is valid for $r = 1$.

Next, let us proceed to the induction step: assume that (2.5) holds for some $r > 1$, then

$$x(t)a_r\bigl(t, \tau_i(t)\bigr) \leq x\bigl(\tau_i(t)\bigr). \tag{2.6}$$

Substituting (2.6) into (1.1) leads to the estimate

$$x'(t) + x(t)\sum_{i=1}^{m} p_i(t)a_r\bigl(t, \tau_i(t)\bigr) \leq 0.$$

Again, applying the Grönwall inequality, we have

$$x(t) \leq x(s)\exp\left\{-\int_s^t \sum_{i=1}^{m} p_i(\zeta)a_r\bigl(\zeta, \tau_i(\zeta)\bigr)\,d\zeta\right\},$$

or

$$x(t)\exp\left\{\int_s^t \sum_{i=1}^{m} p_i(\zeta)a_r\bigl(\zeta, \tau_i(\zeta)\bigr)\,d\zeta\right\} \leq x(s),$$

that is,

$$x(t)a_{r+1}(t,s) \leq x(s),$$

which completes the induction step and the proof of the lemma. □

Let us illustrate how the estimate developed in Lemma 1 works in the case of autonomous equations. The series of estimates is evaluated using computer tools, which recently became an efficient method in computer-assisted proofs [25]. We suggest that, similarly, a computer algebra can be used to construct the estimate iterates and, ideally, the limit estimate. The example below illustrates the procedure.

Example 1 The equation

$$x'(t) + \alpha e^{-\alpha} x(t-1) = 0, \quad t \geq 0, \alpha \geq 0$$

has an exact nonoscillatory solution $e^{-\alpha t}$. For $\alpha = 0.5$ the exact rate of decay (up to the sixth digit after the decimal point) is $x(t+1) \approx 0.606531x(t)$, while $a_1^{-1}(t,t-1) \approx 0.738403$, $a_2^{-1}(t,t-1) \approx 0.663183$, $a_{10}^{-1}(t,t-1) \approx 0.606725$, $a_{18}^{-1}(t,t-1) \approx 0.606531$. The largest value of the coefficient of $1/e$ is attained at $\alpha = 1$; it is well known that it is the maximal coefficient when the equation is still nonoscillatory. The decay of the estimate $x(t+1) \leq \frac{1}{e}x(t) \approx 0.367879x(t)$ is the slowest: $a_1^{-1}(t,t-1) \approx 0.692201$, $a_2^{-1}(t,t-1) \approx 0.587744$, $a_{10}^{-1}(t,t-1) \approx 0.430949$, $a_{50}^{-1}(t,t-1) \approx 0.381994$, $a_{100}^{-1}(t,t-1) \approx 0.375068$, $a_{1,000}^{-1}(t,t-1) \approx 0.368613$.

Theorem 4 *Let* $p_i(t) \geq 0$, $1 \leq i \leq m$, *and* $g(t)$ *be defined by* (2.2), *while* $a_r(t,s)$ *by* (2.3), (2.4). *If* (1.3) *holds and for some* $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta > 1, \tag{2.7}$$

then all solutions of (1.1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Since $-x(t)$ is also a solution of (1.1), we can consider only the case when the solution $x(t)$ is eventually positive. Then there exists $t_1 > 0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$, for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -\sum_{i=1}^{m} p_i(t) x\bigl(\tau_i(t)\bigr) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually non-increasing positive function.

Integrating (1.1) from $g(t)$ to t, and using the fact that the function x is non-increasing, while the function g defined by (2.2) is non-decreasing, and taking into account that

$$\tau_i(t) \leq g(t) \quad \text{and} \quad x\bigl(\tau_i(s)\bigr) \geq x\bigl(g(t)\bigr) a_r\bigl(g(t), \tau_i(s)\bigr),$$

we obtain, for sufficiently large t,

$$\begin{aligned} x\bigl(g(t)\bigr) &= x(t) + \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) x\bigl(\tau_i(\zeta)\bigr)\, d\zeta \\ &> \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) x\bigl(\tau_i(\zeta)\bigr)\, d\zeta \\ &\geq x\bigl(g(t)\bigr) \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta. \end{aligned}$$

Hence

$$x\bigl(g(t)\bigr) \left[1 - \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \right] \geq 0,$$

which implies

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\big(g(t),\tau_i(\zeta)\big)\,d\zeta \le 1.$$

The last inequality contradicts (2.7), and the proof is complete. □

The following example illustrates the significance of the condition $\lim_{t\to\infty}\tau_i(t) = \infty$, $1 \le i \le m$, in Theorem 4.

Example 2 Consider the delay differential equation (1.6) with

$$p(t) \equiv 2, \qquad \tau(t) = \begin{cases} -1, & \text{if } t \in [2k, 2k+1), \\ t, & \text{if } t \in [2k+1, 2k+2), \end{cases} \qquad k \in \mathbb{N}_0 = \{0,1,2,\dots\}.$$

By (2.2), we find

$$g(t) = \sup_{0\le s\le t} \tau(s) = \begin{cases} [t], & \text{if } t \in [2k, 2k+1), \\ t, & \text{if } t \in [2k+1, 2k+2), \end{cases} \qquad k \in \mathbb{N}_0.$$

If $t = 2k + 0.8$, then $g(t) = [t] = 2k$ and

$$\int_{g(t)}^{t} p(\zeta)\,d\zeta = \int_{g(2k+0.8)}^{2k+0.8} p(\zeta)\,d\zeta = 2\int_{2k}^{2k+0.8} d\zeta = 1.6 > 1,$$

which means that (2.7) is satisfied for any r.

However, equation (1.6) has a nonoscillatory solution

$$x(t) = \varphi(t) = t + 1, \quad t \in [-1, 0], \qquad x(t) = \begin{cases} e^{-2[t]}, & \text{if } t \in [2k, 2k+1), \\ e^{-2(t-k-1)}, & \text{if } t \in [2k+1, 2k+2), \end{cases}$$

which illustrates the significance of the condition $\lim_{t\to\infty}\tau(t) = \infty$ in Theorem 4.

In 1992, Yu *et al.* [18] proved the following result.

Lemma 2 *In addition to the hypothesis* (1.3), *assume that* $g(t)$ *is defined by* (2.2),

$$0 < \alpha := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)\,ds \le \frac{1}{e}, \tag{2.8}$$

and $x(t)$ *is an eventually positive solution of* (1.1). *Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(g(t))} \ge \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.9}$$

Based on inequality (2.9), we establish the following theorem.

Theorem 5 *Assume that* $p_i(t) \geq 0$, $1 \leq i \leq m$, $g(t)$ *is defined by* (2.2), $a_r(t,s)$ *by* (2.4), (2.3) *and* (2.8) *holds. If for some* $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.10}$$

then all solutions of (1.1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Then, as in the proof of Theorem 4, we obtain, for sufficiently large t,

$$\begin{aligned} x\bigl(g(t)\bigr) &= x(t) + \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) x\bigl(\tau_i(\zeta)\bigr)\, d\zeta \\ &\geq x(t) + x\bigl(g(t)\bigr) \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta. \end{aligned}$$

That is,

$$\int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \leq 1 - \frac{x(t)}{x(g(t))},$$

which gives

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \leq 1 - \liminf_{t\to\infty} \frac{x(t)}{x(g(t))}.$$

Taking into account that (2.9) holds, the last inequality leads to

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \leq 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},$$

which contradicts condition (2.10).

The proof of the theorem is complete. □

Next, let us proceed to an oscillation condition involving lim inf.

Theorem 6 *Assume that* $p_i(t) \geq 0$, $1 \leq i \leq m$, (1.3) *holds and* $a_r(t,s)$ *are defined by* (2.3), (2.4). *If for some* $r \in \mathbb{N}$

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta > \frac{1}{e}, \tag{2.11}$$

then all solutions of (1.1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Similarly to the proof of Theorem 4, we can confine our discussion only to the case of $x(t)$ being eventually positive. Then there exists $t_1 > 0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -\sum_{i=1}^{m} p_i(t)x\bigl(\tau_i(t)\bigr) \leq 0, \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually non-increasing positive function.

For $t \geq t_1$, (1.1) can be rewritten as

$$\frac{x'(t)}{x(t)} + \sum_{i=1}^{m} p_i(t)\frac{x(\tau_i(t))}{x(t)} = 0, \quad \text{for all } t \geq t_1.$$

Integrating from $g(t)$ to t gives

$$\ln\left(\frac{x(t)}{x(g(t))}\right) + \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)\frac{x(\tau_i(\zeta))}{x(\zeta)}\,d\zeta = 0 \quad \text{for all } t \geq t_2 \geq t_1.$$

Since $g(t) \geq \tau_i(\zeta)$, by Lemma 1 we have $x(\tau_i(\zeta)) \geq a_r(g(t), \tau_i(\zeta))x(g(t))$, and therefore

$$\ln\left(\frac{x(t)}{x(g(t))}\right) + \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r\bigl(g(t), \tau_i(\zeta)\bigr)\frac{x(g(t))}{x(\zeta)}\,d\zeta \leq 0.$$

In view of $x(g(t)) \geq x(\zeta)$, the last inequality becomes

$$\ln\left(\frac{x(t)}{x(g(t))}\right) + \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r\bigl(g(t), \tau_i(\zeta)\bigr)\,d\zeta \leq 0. \tag{2.12}$$

Also, from (2.11) it follows that there exists a constant $c > 0$ such that for some $t_3 \geq t_2$

$$\int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r\bigl(g(t), \tau_i(\zeta)\bigr)\,d\zeta \geq c > \frac{1}{e}, \quad t \geq t_3 \geq t_2. \tag{2.13}$$

Combining inequalities (2.12) and (2.13), we obtain

$$\ln\left(\frac{x(t)}{x(g(t))}\right) + c \leq 0, \quad t \geq t_3.$$

Thus

$$\frac{x(g(t))}{x(t)} \geq e^c \geq ec > 1,$$

which implies for some $t \geq t_4 \geq t_3$

$$(ec)x(t) \leq x\bigl(g(t)\bigr).$$

Repeating the above argument leads to a new estimate $x(g(t))/x(t) > (ec)^2$, for t large enough. Continuing by induction, we get

$$\frac{x(g(t))}{x(t)} \geq (ec)^k, \quad \text{for sufficiently large } t,$$

where $ec > 1$. As $ec > 1$, there is $k \in \mathbb{N}$ satisfying $k > 2(\ln(2) - \ln(c))/(1 + \ln(c))$ such that for t large enough

$$\frac{x(g(t))}{x(t)} \geq (ec)^k > \frac{4}{c^2}. \tag{2.14}$$

Further, integrating (1.1) from $g(t)$ to t yields

$$x\bigl(g(t)\bigr) - x(t) - \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) x\bigl(\tau_i(\zeta)\bigr)\, d\zeta = 0.$$

Inequality (2.5) in Lemma 1 used in the above equality leads to the differential inequality

$$x\bigl(g(t)\bigr) - x(t) - x\bigl(g(t)\bigr) \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \geq 0.$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$x\bigl(g(t)\bigr) \left[1 - \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \right] > 0.$$

From (2.13), for large enough t,

$$0 < c \leq \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta < 1. \tag{2.15}$$

Taking the integral on $[g(t), t]$, which is not less than c, we split the interval into two parts where integrals are not less than $c/2$, let $t_m \in (g(t), t)$ be the splitting point:

$$\int_{g(t)}^{t_m} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \geq \frac{c}{2}, \qquad \int_{t_m}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \geq \frac{c}{2}.$$

Since $g(t) \leq g(t_m)$ in the first integral, we obtain

$$\int_{g(t)}^{t_m} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t_m), \tau_i(\zeta)\bigr)\, d\zeta \geq \frac{c}{2}, \qquad \int_{t_m}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr)\, d\zeta \geq \frac{c}{2}. \tag{2.16}$$

Integrating (1.1) from t_m to t, along with incorporating the inequality $x(\tau_i(\zeta)) \geq a_r(g(t), \tau_i(\zeta))x(g(t))$, gives

$$-x(t_m) + x(t) + x\bigl(g(t)\bigr) \int_{t_m}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r\bigl(g(t), \tau_i(\zeta)\bigr) \leq 0.$$

Together with the second inequality in (2.16), this implies

$$x(t_m) \geq \frac{c}{2} x\bigl(g(t)\bigr). \tag{2.17}$$

By Lemma 1 we have $x(\tau_i(\zeta)) \geq a_r(g(t), \tau_i(\zeta))x(g(t))$.

Similarly, integration of (1.1) from $g(t)$ to t_m with a subsequent application of Lemma 1 leads to

$$x(t_m) - x\big(g(t)\big) + x\big(g(t_m)\big)\int_{g(t)}^{t_m}\sum_{i=1}^{m} p_i(\zeta)a_r\big(g(t_m),\tau_i(\zeta)\big)\,d\zeta \le 0,$$

which together with the first inequality in (2.16) yields

$$x\big(g(t)\big) \ge \frac{c}{2}x\big(g(t_m)\big). \tag{2.18}$$

Inequalities (2.17) and (2.18) imply

$$x\big(g(t_m)\big) \le \frac{2}{c}x\big(g(t)\big) \le \frac{4}{c^2}x(t_m),$$

which contradicts (2.14). Thus, all solutions of (1.1) oscillate. □

As non-oscillation of (1.1) is equivalent to the existence of a positive or a negative solution of the relevant differentiation inequalities (see, for example, [21], Theorem 2.1, p. 25), Theorems 4, 5, and 6 lead to the following result.

Theorem 7 *Assume that all the conditions of anyone of Theorems* 4, 5 *and* 6 *hold. Then*

(i) *the differential inequality*

$$x'(t) + \sum_{i=1}^{m} p_i(t)x\big(\tau_i(t)\big) \le 0, \quad t \ge 0,$$

has no eventually positive solutions;

(ii) *the differential inequality*

$$x'(t) + \sum_{i=1}^{m} p_i(t)x\big(\tau_i(t)\big) \ge 0, \quad t \ge 0,$$

has no eventually negative solutions.

2.2 Advanced equations

Similar oscillation theorems for the (dual) advanced differential equation (1.2) can be derived easily. The proofs of these theorems are omitted, since they are quite similar to the proofs for the delay equation (1.1).

Denote

$$\rho_i(t) = \inf_{s\ge t}\sigma_i(s), \quad t \ge 0, \tag{2.19}$$

and

$$\rho(t) = \min_{1\le i\le m}\rho_i(t), \quad t \ge 0. \tag{2.20}$$

Clearly, the functions $\rho(t)$, $\rho_i(t)$, $1 \le i \le m$, are Lebesgue measurable non-decreasing and $\rho(t) \ge t$, $\rho_i(t) \ge t$, $1 \le i \le m$ for all $t \ge 0$.

Theorem 8 *Assume that* $p_i(t) \ge 0$, $1 \le i \le m$, (1.4) *holds*, $\rho(t)$ *is defined by* (2.20) *and by the* $b_r(t,s)$ *we denote*

$$b_1(t,s) := \exp\left\{\int_t^s \sum_{i=1}^m p_i(\zeta)\,d\zeta\right\} \tag{2.21}$$

and

$$b_{r+1}(t,s) := \exp\left\{\int_t^s \sum_{i=1}^m p_i(\zeta) b_r\bigl(\zeta, \sigma_i(\zeta)\bigr)\,d\zeta\right\}, \quad r \in \mathbb{N}. \tag{2.22}$$

If for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m p_i(\zeta) b_r\bigl(\rho(t), \sigma_i(\zeta)\bigr)\,d\zeta > 1, \tag{2.23}$$

then all solutions of (1.2) *oscillate*.

We would like to mention that Lemma 2 can be extended to the advanced type differential equation (1.2) (*cf.* [23], Section 2.6.6).

Lemma 3 *In addition to the hypothesis* (1.4), *assume that* $\rho(t)$ *is defined by* (2.20),

$$0 < \alpha := \liminf_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m p_i(s)\,ds \le \frac{1}{e}, \tag{2.24}$$

and $x(t)$ *is an eventually positive solution of* (1.2). *Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(\rho(t))} \ge \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}.$$

Based on the above inequality, we establish the following theorem.

Theorem 9 *Assume that* $p_i(t) \ge 0$, $1 \le i \le m$, (1.4) *is satisfied*, $\rho(t)$ *is defined by* (2.20), $b_r(t,s)$ *by* (2.21) *and* (2.22), *and* (2.24) *holds. If for some* $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m p_i(\zeta) b_r\bigl(\rho(t), \sigma_i(\zeta)\bigr)\,d\zeta > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{2.25}$$

then all solutions of (1.2) *oscillate*.

Theorem 10 *Assume that* $p_i(t) \ge 0$, $1 \le i \le m$, (1.4) *holds*, $\rho(t)$ *is defined by* (2.20), $b_r(t,s)$ *are defined in* (2.21), (2.22). *If for some* $r \in \mathbb{N}$

$$\liminf_{t\to\infty} \sum_{i=1}^m \int_t^{\rho(t)} p_i(\zeta) b_r\bigl(\rho(t), \sigma_i(\zeta)\bigr)\,d\zeta > \frac{1}{e}, \tag{2.26}$$

then all solutions of (1.2) *oscillate*.

A slight modification in the proofs of Theorems 8, 9 and 10 leads to the following result as regards advanced differential inequalities.

Theorem 11 *Assume that all the conditions of any of Theorems* 8, 9, *and* 10 *hold. Then*

(i) *the differential inequality*

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) \geq 0, \quad t \geq 0,$$

has no eventually positive solutions;

(ii) *the differential inequality*

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) \leq 0, \quad t \geq 0,$$

has no eventually negative solutions.

3 Examples

In this section we provide two examples illustrating Theorems 3 and 7. Similarly, examples to illustrate the other main results of the paper can be constructed.

Example 3 Consider the delay differential equation

$$x'(t) + \frac{1}{2e}x(\tau_1(t)) + \frac{1}{2.2e}x(\tau_2(t)) = 0, \quad t \geq 1, \tag{3.1}$$

where (see Figure 1(a))

$$\tau_1(t) = \begin{cases} -t + 4k + 1, & \text{if } t \in [2k+1, 2k+2], \\ 3t - 4k - 7, & \text{if } t \in [2k+2, 2k+3], \end{cases} \quad \text{and}$$

$$\tau_2(t) = \tau_1(t) - 0.1, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

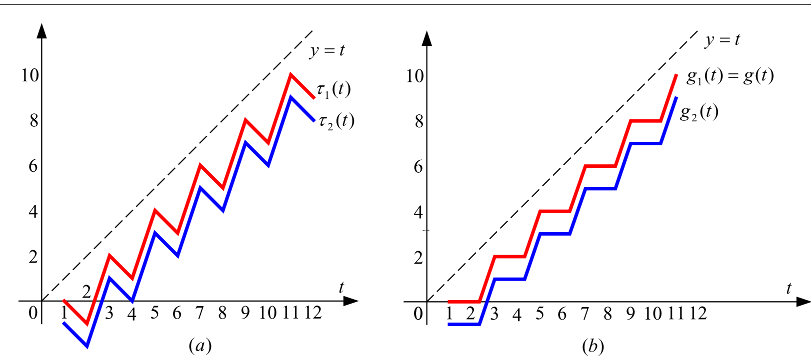

Figure 1 The graphs of (a) $\tau_i(t)$ and (b) $g_i(t)$.

By (2.1), we see (Figure 1(b)) that

$$g_1(t) = \sup_{s\le t} \tau_1(s) = \begin{cases} 2k, & \text{if } t \in [2k+1, 2k+7/3], \\ 3t-4k-7, & \text{if } t \in [2k+7/3, 2k+3], \end{cases} \quad k \in \mathbb{N}_0$$

and

$$g_2(t) = \sup_{s\le t} \tau_2(s) = g_1(t) - 0.1.$$

Therefore, in view of (2.2), we have

$$g(t) = \max_{1\le i\le 2} \{g_i(t)\} = g_1(t).$$

Define the function $f_r : [1, +\infty) \to (0, +\infty)$ as

$$f_r(t) = \int_{g(t)}^{t} \sum_{i=1}^{2} p_i(\zeta) a_r\big(g(t), \tau_i(\zeta)\big)\, d\zeta.$$

Now, at $t = 2k+3$, $k \in \mathbb{N}_0$, we have $g(t = 2k+3) = 2k+2$. Thus

$$\begin{aligned}
&f_1(t = 2k+3) \\
&\quad = \int_{2k+2}^{2k+3} \sum_{i=1}^{2} p_i(\zeta) a_1\big(2k+2, \tau_i(\zeta)\big)\, d\zeta \\
&\quad = \int_{2k+2}^{2k+3} \big[p_1(\zeta) a_1\big(2k+2, \tau_1(\zeta)\big) + p_2(\zeta) a_1\big(2k+2, \tau_2(\zeta)\big)\big]\, d\zeta \\
&\quad = \frac{1}{2e} \int_{2k+2}^{2k+3} \exp\left\{ \int_{\tau_1(\zeta)}^{2k+2} \big(p_1(\xi) + p_2(\xi)\big)\, d\xi \right\} d\zeta \\
&\qquad + \frac{1}{2.2e} \int_{2k+2}^{2k+3} \exp\left\{ \int_{\tau_2(\zeta)}^{2k+2} \big(p_1(\xi) + p_2(\xi)\big)\, d\xi \right\} d\zeta \\
&\quad = \frac{1}{2e} \int_{2k+2}^{2k+3} \exp\left\{ \frac{2.1}{2.2e} \int_{-\zeta+4k+1}^{2k+2} d\xi \right\} d\zeta + \frac{1}{2.2e} \int_{2k+2}^{2k+3} \exp\left\{ \frac{2.1}{2.2e} \int_{-\zeta+4k+0.9}^{2k+2} d\xi \right\} d\zeta \\
&\quad = \frac{1}{2e} \int_{2k+2}^{2k+3} \exp\left\{ \frac{2.1}{2.2e} (\zeta - 2k + 1) \right\} d\zeta + \frac{1}{2.2e} \int_{2k+2}^{2k+3} \exp\left\{ \frac{2.1}{2.2e} (\zeta - 2k + 1.1) \right\} d\zeta \\
&\quad = \frac{11}{21} \left[\exp\left\{ \frac{2.1}{2.2e} \cdot 4 \right\} - \exp\left\{ \frac{2.1}{2.2e} \cdot 3 \right\} \right] + \frac{10}{21} \left[\exp\left\{ \frac{2.1}{2.2e} \cdot 4.1 \right\} - \exp\left\{ \frac{2.1}{2.2e} \cdot 3.1 \right\} \right] \\
&\quad \simeq 1.22696
\end{aligned}$$

and therefore

$$\limsup_{t\to\infty} f_1(t) \gtrapprox 1.22696 > 1.$$

That is, condition (2.7) of Theorem 4 is satisfied for $r = 1$, and therefore all solutions of (3.1) oscillate.

Observe, however, that

$$\begin{aligned}\liminf_{t\to\infty}\int_{\tau_{\max}(t)}^{t}\sum_{i=1}^{m}p_i(s)\,ds &= \liminf_{t\to\infty}\int_{\tau_1(t)}^{t}\sum_{i=1}^{2}p_i(s)\,ds\\ &= \left(\frac{1}{2e}+\frac{1}{2.2e}\right)\liminf_{t\to\infty}\bigl(t-\tau_1(t)\bigr) = \frac{2.1}{2.2e}\cdot 1<\frac{1}{e},\\ \liminf_{t\to\infty}\sum_{i=1}^{m}p_i(t)\bigl(t-\tau_i(t)\bigr) &= \liminf_{t\to\infty}\left[\frac{1}{2e}\bigl(t-\tau_1(t)\bigr)+\frac{1}{2.2e}\bigl(t-\tau_2(t)\bigr)\right]\\ &= \frac{1}{2e}\cdot 1+\frac{1}{2.2e}\cdot 1.1=\frac{1}{e},\end{aligned}$$

and therefore none of conditions (1.8) and (1.10) is satisfied.

Example 4 Consider the advanced differential equation

$$x'(t)-\frac{7}{40}x\bigl(\sigma_1(t)\bigr)-\frac{7}{40}x\bigl(\sigma_2(t)\bigr)=0,\quad t\geq 1, \tag{3.2}$$

where (see Figure 2(a))

$$\sigma_1(t)=\begin{cases}4t-6k-2, & \text{if } t\in[2k+1,2k+2],\\ -2t+6k+10, & \text{if } t\in[2k+2,2k+3],\end{cases}\quad\text{and}$$

$$\sigma_2(t)=\sigma_1(t)+0.1,\quad k\in\mathbb{N}_0.$$

By (2.19), we see (Figure 2(b)) that

$$\rho_1(t):=\inf_{t\leq s}\sigma_1(s)=\begin{cases}4t-6k-2, & \text{if } t\in[2k+1,2k+1.5],\\ 2k+4, & \text{if } t\in[2k+1.5,2k+3],\end{cases}\quad k\in\mathbb{N}_0$$

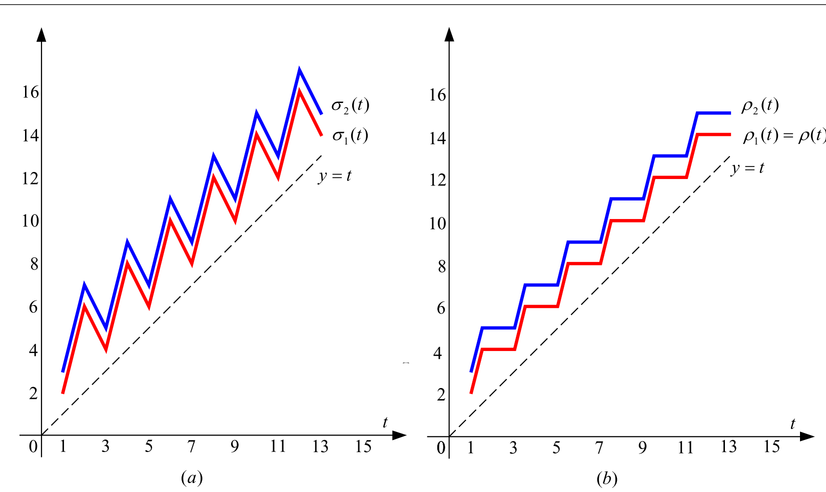

Figure 2 The graphs of (a) $\sigma_i(t)$ and (b) $\rho_i(t)$.

and

$$\rho_2(t) = \inf_{t\le s}\sigma_2(s) = \rho_1(t) + 0.1.$$

Therefore, (2.20) gives

$$\rho(t) = \min_{1\le i\le 2}\{\rho_i(t)\} = \rho_1(t) = \begin{cases} 4t-6k-2, & \text{if } t\in[2k+1,2k+1.5], \\ 2k+4, & \text{if } t\in[2k+1.5,2k+3], \end{cases} \quad k\in\mathbb{N}_0.$$

Define the function $f_r:[1,+\infty)\to(0,+\infty)$ as

$$f_r(t) = \int_t^{\rho(t)}\sum_{i=1}^{2}p_i(\zeta)b_r\bigl(\rho(t),\sigma_i(\zeta)\bigr)\,d\zeta.$$

Now, at $t=2k+1$, $k\in\mathbb{N}_0$, we have $\rho(t=2k+1)=2k+2$. Thus

$$\begin{aligned}
&f_1(t=2k+1)\\
&\quad= \int_{2k+1}^{2k+2}\sum_{i=1}^{2}p_i(\zeta)b_1\bigl(2k+2,\sigma_i(\zeta)\bigr)\,d\zeta\\
&\quad= \int_{2k+1}^{2k+2}\bigl[p_1(\zeta)b_1\bigl(2k+2,\sigma_1(\zeta)\bigr)+p_2(\zeta)b_1\bigl(2k+2,\sigma_2(\zeta)\bigr)\bigr]\,d\zeta\\
&\quad= \frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\frac{7}{20}\int_{2k+2}^{4\zeta-6k-2}d\xi\biggr\}\,d\zeta+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\frac{7}{20}\int_{2k+2}^{4\zeta-6k-1.9}d\xi\biggr\}\,d\zeta\\
&\quad= \frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\frac{7}{20}(4\zeta-8k-4)\biggr\}\,d\zeta+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\frac{7}{20}(4\zeta-8k-3.9)\biggr\}\,d\zeta\\
&\quad= \frac{1}{8}\bigl[\exp(1.4)-1\bigr]+\frac{1}{8}\bigl[\exp(1.435)-\exp(0.035)\bigr]\simeq 0.777403,
\end{aligned}$$

$$\begin{aligned}
&f_2(t=2k+1)\\
&\quad= \int_{2k+1}^{2k+2}\sum_{i=1}^{2}p_i(\zeta)b_2\bigl(2k+2,\sigma_i(\zeta)\bigr)\,d\zeta\\
&\quad= \int_{2k+1}^{2k+2}\bigl[p_1(\zeta)b_2\bigl(2k+2,\sigma_1(\zeta)\bigr)+p_2(\zeta)b_2\bigl(2k+2,\sigma_2(\zeta)\bigr)\bigr]\,d\zeta\\
&\quad= \frac{7}{40}\int_{2k+1}^{2k+2}b_2\bigl(2k+2,\sigma_1(\zeta)\bigr)\,d\zeta+\frac{7}{40}\int_{2k+1}^{2k+2}b_2\bigl(2k+2,\sigma_2(\zeta)\bigr)\,d\zeta\\
&\quad= \frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\int_{2k+2}^{\sigma_1(\zeta)}\biggl[\frac{7}{40}b_1\bigl(2k+2,\sigma_1(\xi)\bigr)+\frac{7}{40}b_1\bigl(2k+2,\sigma_2(\xi)\bigr)\biggr]\,d\xi\biggr\}\,d\zeta\\
&\qquad+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\int_{2k+2}^{\sigma_2(\zeta)}\biggl[\frac{7}{40}b_1\bigl(2k+2,\sigma_1(\xi)\bigr)+\frac{7}{40}b_1\bigl(2k+2,\sigma_2(\xi)\bigr)\biggr]\,d\xi\biggr\}\,d\zeta\\
&\quad= \frac{7}{40}\int_{2k+1}^{2k+2}\exp\biggl\{\int_{2k+2}^{\sigma_1(\zeta)}\biggl[\frac{7}{40}\exp\biggl(\frac{7}{20}\int_{2k+2}^{\sigma_1(\xi)}du\biggr)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{40}\exp\left(\frac{7}{20}\int_{2k+2}^{\sigma_2(\xi)}du\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
&+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\int_{2k+2}^{\sigma_2(\zeta)}\left[\frac{7}{40}\exp\left(\frac{7}{20}\int_{2k+2}^{\sigma_1(\xi)}du\right)\right.\right. \\
&+\frac{7}{40}\exp\left(\frac{7}{20}\int_{2k+2}^{\sigma_2(\xi)}du\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
={}&\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\int_{2k+2}^{\sigma_1(\zeta)}\left[\frac{7}{40}\exp\left(\frac{7}{20}\big(\sigma_1(\xi)-2k-2\big)\right)\right.\right. \\
&+\frac{7}{40}\exp\left(\frac{7}{20}\big(\sigma_2(\xi)-2k-2\big)\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
&+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\int_{2k+2}^{\sigma_2(\zeta)}\left[\frac{7}{40}\exp\left(\frac{7}{20}\big(\sigma_1(\xi)-2k-2\big)\right)\right.\right. \\
&+\frac{7}{40}\exp\left(\frac{7}{20}\big(\sigma_2(\xi)-2k-2\big)\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
={}&\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\int_{2k+2}^{\sigma_1(\zeta)}\left[\frac{7}{40}\exp\left(\frac{7}{20}(-2\xi+4k+8)\right)\right.\right. \\
&+\frac{7}{40}\exp\left(\frac{7}{20}(-2\xi+4k+8.1)\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
&+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\int_{2k+2}^{\sigma_2(\zeta)}\left[\frac{7}{40}\exp\left(\frac{7}{20}(-2\xi+4k+8)\right)\right.\right. \\
&+\frac{7}{40}\exp\left(\frac{7}{20}(-2\xi+4k+8.1)\right)\bigg]\,d\xi\bigg\}\,d\zeta \\
={}&\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\begin{array}{c}-\frac{1}{4}[\exp(\frac{7}{20}(-8\zeta+16k+12))-\exp(1.4)]\\ -\frac{1}{4}[\exp(\frac{7}{20}(-8\zeta+16k+12.1))-\exp(1.435)]\end{array}\right\}d\zeta \\
&+\frac{7}{40}\int_{2k+1}^{2k+2}\exp\left\{\begin{array}{c}-\frac{1}{4}[\exp(\frac{7}{20}(-8\zeta+16k+11.8))-\exp(1.4)]\\ -\frac{1}{4}[\exp(\frac{7}{20}(-8\zeta+16k+11.9))-\exp(1.435)]\end{array}\right\}d\zeta \\
\simeq{}&1.558893>1.
\end{aligned}
$$

Thus condition (2.23) of Theorem 8 is satisfied for $r = 2$, and therefore all solutions of (3.2) oscillate.

Observe, however, that

$$
\begin{aligned}
\liminf_{t\to\infty}\sum_{i=1}^{m}\int_t^{\sigma_{\min}(t)}p_i(s)\,ds &= \liminf_{t\to\infty}\sum_{i=1}^{2}\int_t^{\sigma_1(t)}p_i(s)\,ds \\
&= \left(\frac{7}{40}+\frac{7}{40}\right)\liminf_{t\to\infty}\big(\sigma_1(t)-t\big)=\frac{7}{20}<\frac{1}{e}, \\
\liminf_{t\to\infty}\sum_{i=1}^{m}p_i(t)\big(\sigma_i(t)-t\big) &= \liminf_{t\to\infty}\left[\frac{7}{40}\big(\sigma_1(t)-t\big)+\frac{7}{40}\big(\sigma_2(t)-t\big)\right] \\
&= \frac{7}{40}\cdot 1+\frac{7}{40}\cdot 1.1 = 0.3675<\frac{1}{e},
\end{aligned}
$$

and therefore none of conditions (1.9) and (1.11) is satisfied.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that they have made equal contributions to the paper.

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