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On certain triple q-integral equations involving the third Jackson q-Bessel functions as kernel

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Abstract

In this paper, we consider two different systems of triple q-integral equations, where the kernel is a third Jackson q-Bessel function. The solution of the first system is reduced to two simultaneous Fredholm q-integral equations of the second kind. We solve the second system by using the solution of specific dual q-integral equations. Some examples are included.

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1 Introduction

In the past years, several authors have described various methods to solve triple integral equations especially of the form

$$\begin{aligned} &\int_0^\infty A(u)K(u,x)\,du = f(x), \quad 0 < x < a,\\ &\int_0^\infty w(u)A(u)K(u,x)\,du = g(x), \quad a < x < b,\\ &\int_0^\infty A(u)K(u,x)\,du = h(x), \quad b < x < \infty, \end{aligned}$$

where $w(u)$ is the weight function, $K(u,x)$ is the kernel function (see for example [1–10]).

Some mixed boundary value problems of the mathematical theory of elasticity are solved by reducing them to multiple integral equations. For example, the axisymmetric problem of a torsion of an elastic space, weakened by a conical crack, under the assumption that on the boundaries of the crack, the tangential displacements of the shear stress are prescribed, is solved by the application of dual integral equations (see [11]). Harmonic shear oscillations of a rigid stamp with a plane base coupled to an elastic half-space were studied in [12] and reduced to dual integral equations. For more applications see [13, 14]. This type of equations can be solved by a Fredholm integral equation of the second kind by using the modified Hankel transform operator and the Erdélyi-Kober fractional integration operator.

In this paper, we consider triple q-integral equation where the kernel is the third Jackson q-Bessel function and the q-integral is a Jackson q-integral. It is worth mentioning that different approaches for solving a dual q-integral equation are in [15]. Also, solutions for dual and triple sequence involving q-orthogonal polynomials are in [16].

The paper is organized in the following manner. The next section includes the main notations and some results we need in our investigations. In Section 3, we solve the triple q-integral equations by reducing the system to two simultaneous Fredholm q-integral equation of the second kind, we shall use a method due to Singh *et al.* [7]. The approach depends on fractional q-calculus. Furthermore, we will conclude the solutions of two dual q-integral equations to be special cases of the solution of the triple q-integral equation, and we show that this coincides with the results in [15]. In the last section, we use a result from [15] for a solution of dual q^2-integral equations to solve triple q^2-integral equations. The result of this section is a q-analog of the results introduced by Cooke in [2].

2 q-Notations and results

Throughout this paper, we will assume that q is a positive number less than one and we follow Gasper and Rahman [17] for the definitions of the q-shifted factorial, multiple q-shifted factorials, basic hypergeometric series, Jackson q-integrals, the q-gamma, and beta functions. We also follow Annaby and Mansour [18] for the definition of the q-derivative at zero.

For $t > 0$, let $A_{q,t}$, $B_{q,t}$, and $\mathbb{R}_{q,t,+}$ be the sets defined by

$$A_{q,t} := \{tq^n : n \in \mathbb{N}_0\}, \qquad B_{q,t} := \{tq^{-n} : n \in \mathbb{N}\},$$
$$\mathbb{R}_{q,t,+} := \{tq^k : k \in \mathbb{Z}\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and $\mathbb{N} := \{1, 2, \ldots\}$. Notice, if $t = 1$ we write A_q, B_q, and $\mathbb{R}_{q,+}$, and we define the following spaces:

$$L_{q,\eta}(\mathbb{R}_{q,+}) := \left\{f : \|f\|_{q,\eta} := \int_0^\infty \left|t^\eta f(t)\right| d_qt < \infty\right\},$$
$$L_{q,\eta}(A_q) := \left\{f : \|f\|_{A_q,\eta} := \int_0^1 \left|t^\eta f(t)\right| d_qt < \infty\right\},$$
$$L_{q,\eta}(B_q) := \left\{f : \|f\|_{B_q,\eta} := \int_1^\infty \left|t^\eta f(t)\right| d_qt < \infty\right\},$$

where $\eta \in \mathbb{C}$ and $L_{q,\eta}(\mathbb{R}_{q,+}) = L_{q,\eta}(A_q) \cap L_{q,\eta}(B_q)$.

Koornwinder and Swarttouw [19] introduced the following inverse pair of q-Hankel integral transforms under the side condition $f, g \in L_q^2(\mathbb{R}_{q,+})$:

$$g(\lambda) = \int_0^\infty xf(x)J_\nu(\lambda x; q^2)\, d_qx; \qquad f(x) = \int_0^\infty \lambda g(\lambda)J_\nu(\lambda x; q^2)\, d_q\lambda, \tag{2.1}$$

where $\lambda, x \in \mathbb{R}_{q,+}$.

Now we recall some definitions and results which will be needed in the sequel. Let $\alpha \in \mathbb{C}$, the q-binomial coefficient is defined by

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \begin{cases} 1, & k = 0, \\ \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(q;q)_k}, & k \in \mathbb{N}. \end{cases}$$

The third Jackson q-Bessel function $J_\nu^{(3)}(z;q)$, see [20] and [21], is defined by

$$J_\nu(z;q) = J_\nu^{(3)}(z;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}z^{2n+\nu}}{(q;q)_n(q^{\nu+1};q)_n}, \quad z\in\mathbb{C}, \tag{2.2}$$

and it satisfies the following relations (see [22]):

$$D_q\big[(\cdot)^{-\nu}J_\nu\big(\cdot;q^2\big)\big](z) = -\frac{q^{1-\nu}z^{-\nu}}{1-q}J_{\nu+1}\big(qz;q^2\big), \tag{2.3}$$

$$D_q\big[(\cdot)^{\nu}J_\nu\big(\cdot;q^2\big)\big](z) = \frac{z^{\nu}}{1-q}J_{\nu-1}\big(z;q^2\big). \tag{2.4}$$

Also, for $\Re(\nu) > -1$, the q-Bessel function $J_\nu(\cdot;q^2)$ satisfies (see [19])

$$\big|J_\nu\big(q^n;q^2\big)\big| \le \frac{(-q^2;q^2)_\infty(-q^{2\nu+2};q^2)_\infty}{(q^2;q^2)_\infty}\begin{cases} q^{n\nu}, & n\in\mathbb{N}_0,\\ q^{n^2-(\nu+1)n}, & n\in\mathbb{N}.\end{cases} \tag{2.5}$$

The functions $\cos(z;q)$ and $\sin(z;q)$ are defined by

$$\cos(z;q) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\big(zq^{-\frac{1}{2}}(1-q)\big)^{\frac{1}{2}}J_{-\frac{1}{2}}\big(z(1-q)/\sqrt{q};q^2\big),$$

$$\sin(z;q) := \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\big(z(1-q)\big)^{\frac{1}{2}}J_{\frac{1}{2}}\big(z(1-q);q^2\big), \quad z\in\mathbb{C}.$$

We need the following results from [18].

Proposition 2.1 *Let $\alpha,\beta\in\mathbb{C}$, $\rho,t\in\mathbb{R}_{q,+}$. Then, for $\Re(\beta) > \Re(\alpha) > -1$,*

$$\int_0^\infty t^{\alpha-\beta+1}J_\alpha\big(\xi t;q^2\big)J_\beta\big(\rho t;q^2\big)\,d_qt = \begin{cases} 0, & \xi>\rho,\\ \frac{(1-q)(1-q^2)^{1-\beta+\alpha}}{\Gamma_{q^2}(\beta-\alpha)}\xi^\alpha\rho^{(\beta-2\alpha-2)}(q^2\xi^2/\rho^2;q^2)_{\beta-\alpha-1}, & \xi\le\rho.\end{cases}$$

Proposition 2.2 *Let ν and α be complex numbers such that $\Re(\nu) > -1$. Then, for $\rho,u\in\mathbb{R}_{q,+}$,*

$$\int_\rho^\infty x^{2\alpha-\nu-1}\big(\rho^2/x^2;q^2\big)_{\alpha-1}J_\nu\big(ux;q^2\big)\,d_qx = \frac{(1-q)\Gamma_{q^2}(\alpha)}{(1-q^2)^{1-\alpha}}\rho^{\alpha-\nu}u^{-\alpha}q^\alpha J_{\nu-\alpha}\big(u\rho/q;q^2\big).$$

Proposition 2.3 *Let x, ν, and γ be complex numbers and $u\in\mathbb{R}_{q,+}$. Then, for $\Re(\gamma) > -1$ and $\Re(\nu) > -1$, the following identity holds:*

$$\int_0^x \rho^{\nu+1}\big(q^2\rho^2/x^2;q^2\big)_\gamma J_\nu\big(u\rho;q^2\big)\,d_q\rho = x^{\nu-\gamma+1}u^{-\gamma-1}(1-q)\big(1-q^2\big)^\gamma\Gamma_{q^2}(\gamma+1)J_{\gamma+\nu+1}\big(ux;q^2\big). \tag{2.6}$$

Moreover, if $\Re(\gamma) > 0$ *and* $\Re(\nu) > -1$, *then*

$$\int_x^\infty \rho^{2\gamma-\nu-1}\bigl(x^2/\rho^2;q^2\bigr)_{\gamma-1}J_\nu\bigl(u\rho;q^2\bigr)\,d_q\rho = x^{\gamma-\nu}u^{-\gamma}(1-q)q^\gamma\frac{(q^2;q^2)_\infty}{(q^{2\gamma};q^2)_\infty}J_{\nu-\gamma}\biggl(\frac{ux}{q};q^2\biggr). \tag{2.7}$$

The following are consequences of the above results.

Lemma 2.4 *Let* x, u, *and* α *be complex numbers such that* $u \in \mathbb{R}_{q,+}$, $\Re(\alpha) > -1$, *and* $\Re(\nu) > -1$. *Then*

$$u^\alpha J_{\nu-\alpha}\bigl(ux;q^2\bigr) = \frac{(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}x^{\alpha-\nu-1}D_{q,x}\biggl[x^{-2\alpha}\int_0^x \rho^{\nu+1}\bigl(q^2\rho^2/x^2;q^2\bigr)_{-\alpha}J_\nu\bigl(u\rho;q^2\bigr)\,d_q\rho\biggr]. \tag{2.8}$$

Proof Applying (2.6) with $\gamma = -\alpha$, we have

$$\int_0^x \rho^{\nu+1}\bigl(q^2\rho^2/x^2;q^2\bigr)_{-\alpha}J_\nu\bigl(u\rho;q^2\bigr)\,d_q\rho = x^{\nu+\alpha+1}u^{\alpha-1}(1-q)\bigl(1-q^2\bigr)^{-\alpha}\Gamma_{q^2}(1-\alpha)J_{\nu-\alpha+1}\bigl(ux;q^2\bigr). \tag{2.9}$$

Multiplying both sides of equation (2.9) by $x^{-2\alpha}$, and then calculating the q-derivative of the two sides with respect to x and using (2.4), we get the required result. □

Similarly, by using (2.9) we obtain the following result.

Lemma 2.5 *Let* x, u, *and* α *be complex numbers such that* $u \in \mathbb{R}_{q,+}$, $\Re(\alpha) > 0$, *and* $\Re(\nu) > -1$. *Then*

$$u^\alpha J_{\nu+\alpha}\bigl(ux;q^2\bigr) = -\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2}x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\int_x^\infty \rho^{-2\alpha-\nu+1}\bigl(x^2/\rho^2;q^2\bigr)_{-\alpha}J_\nu\bigl(u\rho;q^2\bigr)\,d_q\rho. \tag{2.10}$$

We end this section by introducing some q-fractional operators that we use in solving the triple q-integral equations under consideration. The technique of using fractional operators in solving dual and triple integral equations is not new. See for example [15, 23, 24].

A q-analog of the Riemann-Liouville fractional integral operator is introduced in [25] by Al-Salam through

$$I_q^\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f(t)\,d_qt, \quad \alpha \notin \{-1,-2,\ldots\}.$$

In [26], Agarwal defined the q-fractional derivative to be

$$D_q^\alpha f(x) := I_q^{-\alpha}f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)}\int_0^x (qt/x;q)_{-\alpha-1}f(t)\,d_qt.$$

Also, we have (see [18], Lemma 4.17)

$$I_q^\alpha D_q^\alpha f(x) = f(x) - I_q^{1-\alpha} f(0) \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}, \quad 0 < \alpha < 1. \tag{2.11}$$

In [25], Al-Salam defined a two parameter q-fractional operator by

$$K_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (x/t;q)_{\alpha-1} t^{-\eta-1} \phi\bigl(tq^{1-\alpha}\bigr)\, d_q t,$$

$\alpha \neq -1, -2, \dots$. This is a q-analog of the Erdélyi and Sneddon fractional operator, *cf.* [27, 28],

$$K^{\eta,\alpha} f(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\eta-1} f(t)\, dt.$$

In [15], the authors introduced a slight modification of the operator $K_q^{\eta,\alpha}$. This operator is denoted by $\mathcal{K}_q^{\eta,\alpha}$ and defined by

$$\mathcal{K}_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty (x/t;q)_{\alpha-1} t^{-\eta-1} \phi(qt)\, d_q t, \tag{2.12}$$

where $\alpha \neq -1, -2, \dots$. In case of $\eta = -\alpha$, we set

$$\begin{aligned} \mathcal{K}_q^\alpha f(x) &:= q^{-\alpha} x^\alpha q^{\alpha(\alpha-1)/2} \mathcal{K}_q^{-\alpha,\alpha} f(x) \\ &= \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} \int_x^\infty t^{\alpha-1} (x/t;q)_{\alpha-1} f(qt)\, d_q t. \end{aligned} \tag{2.13}$$

Note that this operator satisfies the following semigroup identity:

$$\mathcal{K}_q^\alpha \mathcal{K}_q^\beta \phi(x) = \mathcal{K}_q^{\alpha+\beta} \phi(x) \quad \text{for all } \alpha \text{ and } \beta. \tag{2.14}$$

The proof of (2.14) is completely similar to the proof of Theorem 5.13 in [18] and is omitted.

Lemma 2.6 *Let $\alpha \in \mathbb{C}$, $x \in B_q$. If $\Phi \in L_{q,\alpha-1}(B_q)$ and $G(x) = D_{q,x} \mathcal{K}_q^\alpha \Phi(x)$, then*

$$\Phi(x) = -q^{\alpha-1} \mathcal{K}_q^{1-\alpha} G\left(\frac{x}{q}\right).$$

Proof According to (2.13), we have

$$\begin{aligned} G(x) &= \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} D_{q,x} \int_x^\infty t^{\alpha-1} (x/t;q)_{\alpha-1} \Phi(qt)\, d_q t \\ &= \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} \left[\int_x^\infty t^{\alpha-1} \bigl(D_{q,x}(x/t;q)_{\alpha-1}\bigr) \Phi(qt)\, d_q t - x^{\alpha-1} (q;q)_{\alpha-1} \Phi(qx) \right]. \end{aligned} \tag{2.15}$$

Note that

$$D_{q,x}(x/t;q)_{\alpha-1} = -\frac{(1-q^{\alpha-1})}{t(1-q)} (qx/t;q)_{\alpha-2} = -\frac{1}{t}[\alpha-1](qx/t;q)_{\alpha-2}$$

and

$$\int_{qx}^{\infty} g(t)\,d_qt = \int_{x}^{\infty} g(t)\,d_qt + x(1-q)g(x).$$

Hence,

$$\begin{aligned} G(x) &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}[\alpha-1]\left[\int_x^{\infty} t^{\alpha-2}(qx/t;q)_{\alpha-2}\Phi(qt)\,d_qt - x^{\alpha-1}(q;q)_{\alpha-1}\Phi(qx)\right] \\ &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}[\alpha-1]\int_{qx}^{\infty} t^{\alpha-2}(qx/t;q)_{\alpha-2}\Phi(qt)\,d_qt \\ &= -\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha-1)}\int_{qx}^{\infty} t^{\alpha-2}(qx/t;q)_{\alpha-2}\Phi(qt)\,d_qt = -q^{1-\alpha}\mathcal{K}_q^{(\alpha-1)}\Phi(qx). \end{aligned}$$

This implies

$$\mathcal{K}_q^{(\alpha-1)}\Phi(x) = -q^{\alpha-1}G(x/q).$$

Using (2.14), we obtain the result and completes the proof. □

3 A system of triple q-integral equations

The goal of this section is to solve the following triple q-integral equations:

$$\int_0^{\infty} \psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu = f_1(\rho), \quad \rho \in A_{q,a}, \tag{3.1}$$

$$\int_0^{\infty} u^{-2\alpha}\psi(u)\big[1+w(u)\big]J_\nu\big(u\rho;q^2\big)\,d_qu = f_2(\rho), \quad \rho \in A_{q,b}\cap B_{q,a}, \tag{3.2}$$

$$\int_0^{\infty} \psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu = f_3(\rho), \quad \rho \in B_{q,b}, \tag{3.3}$$

where $0 < a < b < \infty$, and α, ν are complex numbers satisfying

$$\Re(\nu) > -1 \quad \text{and} \quad 0 < \Re(\alpha) < 1.$$

ψ is an unknown function to be determined, f_i $(i = 1, 2, 3)$ are known functions, and w is a non-negative bounded function defined on $\mathbb{R}_{q,+}$.

Clearly from (2.5), a sufficient condition for the convergence of the q-integrals on the left-hand side of (3.1)-(3.2) is that

$$\psi \in L_{q,\nu}(\mathbb{R}_{q,+}) \cap L_{q,\nu-2\alpha}(\mathbb{R}_{q,+}). \tag{3.4}$$

For getting the solution of the triple q-integral equations (3.1)-(3.3), we define a function C by

$$C(u) := u^{-2\alpha}\psi(u)\big[1+w(u)\big], \quad u \in \mathbb{R}_{q,+}.$$

This implies

$$\psi(u) = u^{2\alpha}C(u) - u^{2\alpha}C(u)\left[\frac{w(u)}{1+w(u)}\right],$$

and the triple q-integral equations (3.1)-(3.3) can be represented as

$$\int_0^\infty u^{2\alpha} C(u) J_\nu\left(u\rho;q^2\right) d_q u - \int_0^\infty u^{2\alpha} C(u)\left[\frac{w(u)}{1+w(u)}\right] J_\nu\left(u\rho;q^2\right) d_q u$$

$$= f_1(\rho), \quad \rho \in A_{q,a}, \tag{3.5}$$

$$\int_0^\infty C(u) J_\nu\left(u\rho;q^2\right) d_q u = f_2(\rho), \quad \rho \in A_{q,b} \cap B_{q,a}, \tag{3.6}$$

$$\int_0^\infty u^{2\alpha} C(u) J_\nu\left(u\rho;q^2\right) d_q u - \int_0^\infty \frac{w(u)}{1+w(u)} u^{2\alpha} C(u) J_\nu\left(u\rho;q^2\right) d_q u$$

$$= f_3(\rho), \quad \rho \in B_{q,b}. \tag{3.7}$$

Since equation (3.6) is linear in C, we may assume that $C := C_1 + C_2$ and

$$f_2 = g_1 + g_2, \quad \text{on } A_{q,b} \cap B_{q,a},$$

where g_1 defined on $A_{q,b}$ and g_2 defined on $B_{q,a}$. Therefore,

$$\int_0^\infty C_1(u) J_\nu\left(u\rho;q^2\right) d_q u = g_1(\rho), \quad \rho \in A_{q,b},$$

$$\int_0^\infty C_2(u) J_\nu\left(u\rho;q^2\right) d_q u = g_2(\rho), \quad \rho \in B_{q,a}.$$

So, the triple q-integral equations (3.5)-(3.7) can be rewritten in the following form:

$$\int_0^\infty u^{2\alpha}\left[C_1(u) + C_2(u)\right] J_\nu\left(u\rho;q^2\right) d_q u$$

$$- \int_0^\infty u^{2\alpha}\left[C_1(u) + C_2(u)\right] \frac{w(u)}{1+w(u)} J_\nu\left(u\rho;q^2\right) d_q u$$

$$= f_1(\rho), \quad \rho \in A_{q,a}, \tag{3.8}$$

$$\int_0^\infty C_1(u) J_\nu\left(u\rho;q^2\right) d_q u = g_1(\rho), \quad \rho \in A_{q,b}, \tag{3.9}$$

$$\int_0^\infty C_2(u) J_\nu\left(u\rho;q^2\right) d_q u = g_2(\rho), \quad \rho \in B_{q,a}, \tag{3.10}$$

$$\int_0^\infty u^{2\alpha}\left[C_1(u) + C_2(u)\right] J_\nu\left(u\rho;q^2\right) d_q u$$

$$- \int_0^\infty \frac{w(u)}{1+w(u)} u^{2\alpha}\left[C_1(u) + C_2(u)\right] J_\nu\left(u\rho;q^2\right) d_q u$$

$$= f_3(\rho), \quad \rho \in B_{q,b}. \tag{3.11}$$

Proposition 3.1 *Let ψ_1, ψ_2 be the functions defined by*

$$\psi_1(x) := \int_0^\infty u^\alpha C_1(u) J_{\nu-\alpha}\left(ux;q^2\right) d_q u, \quad x \in B_{q,b}, \tag{3.12}$$

$$\psi_2(x) := \int_0^\infty u^\alpha C_2(u) J_{\nu+\alpha}\left(ux;q^2\right) d_q u, \quad x \in A_{q,a}, \tag{3.13}$$

provided that $0 < \Re(\alpha) < 1$, $\Re(\nu) > -1$, $\Re(\nu + \alpha) > 0$, *and* $C_1 \in L_{q,\nu}(\mathbb{R}_{q,+})$, $C_2 \in L_{q,-t}(\mathbb{R}_{q,+})$ *where*

$$\Re(\nu) + 2 > \Re(t) > -\Re(\nu) + 2\Re(1-\alpha).$$

Then, for $u \in \mathbb{R}_{q,+}$, *we have*

$$C_1(u) = u^{1-\alpha}\left[\int_0^b x\Phi_1(x)J_{\nu-\alpha}\left(ux;q^2\right)d_qx + \int_b^\infty x\psi_1(x)J_{\nu-\alpha}\left(ux;q^2\right)d_qx\right], \tag{3.14}$$

$$C_2(u) = u^{1-\alpha}\left[\int_0^a x\psi_2(x)J_{\nu+\alpha}\left(ux;q^2\right)d_qx + \int_a^\infty x\Phi_2(x)J_{\nu+\alpha}\left(ux;q^2\right)d_qx\right], \tag{3.15}$$

where

$$\begin{aligned}\Phi_1(x) &= \frac{(1-q^2)^\alpha x^{\alpha-\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\left[x^{-2\alpha}\int_0^x g_1(\rho)\rho^{\nu+1}\left(q^2\rho^2/x^2;q^2\right)_{-\alpha}d_q\rho\right]\\ &= \left(1-q^2\right)^\alpha x^{\alpha-\nu-1}D^\alpha_{q^2,x}\left((\cdot)^{\nu/2}g_1(\sqrt{\cdot})\right)(x), \quad x \in A_{q,b},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Phi_2(x) &= -\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2}x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\int_x^\infty g_2(\rho)\rho^{1-2\alpha-\nu}\left(x^2/\rho^2;q^2\right)_{-\alpha}d_q\rho\\ &= -q^{\frac{\alpha(1-\alpha)}{2}}\left(1-q^2\right)^\alpha x^{\alpha+\nu-1}D_{q^2,x}\mathcal{K}^{(1-\alpha)}_{q^2}\left[(\cdot)^{-\nu/2}g_2(\sqrt{\cdot})\right]\left(\frac{x}{q^2}\right), \quad x \in B_{q,a}.\end{aligned} \tag{3.17}$$

Proof We start with proving (3.16). Let $x \in A_{q,b}$. Multiplying both sides of (3.9) by $x^{-2\alpha}\rho^{\nu+1}(q^2\rho^2/x^2;q^2)_{-\alpha}$ and then integrating with respect to ρ from 0 to x, we get

$$\begin{aligned}&\int_0^x x^{-2\alpha}\rho^{\nu+1}\left(q^2\rho^2/x^2;q^2\right)_{-\alpha}\int_0^\infty C_1(u)J_\nu\left(u\rho;q^2\right)d_qu\,d_q\rho\\ &\quad= \int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}\left(q^2\rho^2/x^2;q^2\right)_{-\alpha}d_q\rho.\end{aligned} \tag{3.18}$$

Notice that the double q-integral on the left-hand side of (3.18) is absolutely convergent for $0 < \Re(\alpha) < 1$ and for $\Re(\nu) > -1$ provided that $C_1 \in L_{q,\nu}(\mathbb{R}_{q,+})$. So, we can interchange the order of the q-integrations to obtain

$$\begin{aligned}&\int_0^\infty C_1(u)x^{-2\alpha}\int_0^x \rho^{\nu+1}\left(\frac{q^2\rho^2}{x^2};q^2\right)_{-\alpha}J_\nu\left(u\rho;q^2\right)d_q\rho\,d_qu\\ &\quad= \int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}\left(\frac{q^2\rho^2}{x^2};q^2\right)_{-\alpha}d_q\rho.\end{aligned} \tag{3.19}$$

By calculating the q-derivative of the two sides of (3.19) with respect to x and using (2.8), we get

$$\int_0^\infty u^\alpha C_1(u)J_{\nu-\alpha}\left(ux;q^2\right)d_qu = \Phi_1(x), \quad x \in A_{q,b}, \tag{3.20}$$

where

$$\Phi_1(x) = \frac{(1-q^2)^\alpha x^{\alpha-\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\left[x^{-2\alpha}\int_0^x g_1(\rho)x^{-2\alpha}\rho^{\nu+1}\left(\frac{q^2\rho^2}{x^2};q^2\right)_{-\alpha}d_q\rho\right].$$

To prove (3.17), let $x \in B_{q,a}$. Multiplying both sides of (3.10) by $\rho^{-2\alpha-\nu+1}(x^2/\rho^2;q^2)_{-\alpha}$ and q-integrating with respect to ρ from x to ∞, we get

$$\begin{aligned}&\int_x^\infty \rho^{1-2\alpha-\nu}\big(x^2/\rho^2;q^2\big)_{-\alpha}\int_0^\infty C_2(u)J_\nu\big(u\rho;q^2\big)\,d_qu\,d_q\rho\\&\quad=\int_x^\infty g_2(\rho)\rho^{-2\alpha-\nu+1}\big(x^2/\rho^2;q^2\big)_{-\alpha}\,d_q\rho.\end{aligned}\tag{3.21}$$

From (2.5), we can prove that $u^tJ_\nu(u;q^2)$ is bounded on $\mathbb{R}_{q,+}$ provided that $\Re(t+\nu) > -1$. So, if we take t such that $\Re(\nu)+2 > \Re(t) > -\Re(\nu)+2\Re(1-\alpha)$, we can prove that the double q-integral

$$\int_x^\infty \rho^{1-2\alpha-\nu}\big(x^2/\rho^2;q^2\big)_{-\alpha}\int_0^\infty C_2(u)J_\nu\big(u\rho;q^2\big)\,d_qu\,d_q\rho$$

is absolutely convergent and then we can interchange the order of the q-integration to obtain

$$\begin{aligned}&\int_0^\infty C_2(u)\int_x^\infty \rho^{1-2\alpha-\nu}\big(x^2/\rho^2;q^2\big)_{-\alpha}J_\nu\big(u\rho;q^2\big)\,d_q\rho\,d_qu\\&\quad=\int_x^\infty g_2(\rho)\rho^{-2\alpha-\nu+1}\big(x^2/\rho^2;q^2\big)_{-\alpha}\,d_q\rho.\end{aligned}\tag{3.22}$$

Calculating the q-derivative of the two sides of (3.22) with respect to x and using (2.10) yields

$$\int_0^\infty u^\alpha C_2(u)J_{\nu+\alpha}\big(ux;q^2\big)\,d_qu = \Phi_2(x),\quad x\in B_{q,a},\tag{3.23}$$

where

$$\Phi_2(x) = -\frac{(1-q^2)^\alpha q^{2\alpha+\nu-2}x^{\alpha+\nu-1}}{\Gamma_{q^2}(1-\alpha)}D_{q,x}\int_x^\infty g_2(\rho)\rho^{1-2\alpha-\nu}\big(x^2/\rho^2;q^2\big)_{-\alpha}\,d_q\rho.$$

By the above argument, if we assume that ψ_1 and ψ_2 are given by (3.12) and (3.13), then

$$\int_0^\infty u^\alpha C_1(u)J_{\nu-\alpha}\big(ux;q^2\big)\,d_qx = \begin{cases}\phi_1(x), & x\in A_{q,b},\\ \psi_1(x), & x\in B_{q,b}\end{cases}\tag{3.24}$$

and

$$\int_0^\infty u^\alpha C_2(u)J_{\nu+\alpha}\big(ux;q^2\big)\,d_qx = \begin{cases}\phi_2(x), & x\in B_{q,a},\\ \psi_2(x), & x\in A_{q,a}.\end{cases}\tag{3.25}$$

Hence, (3.14) and (3.15) follow by applying the inverse pair of q-Hankel transforms (2.1) on (3.24) and (3.25). This completes the proof. □

Remark 3.2 From the definitions of ψ_i and ϕ_i, $i=1,2$, in Proposition 3.1, one can verify that $x^{-\nu-\alpha}\phi_2$ is a bounded function in $B_{q,a}$ and $x^{-\nu-\alpha}\psi_2$ is bounded in $A_{q,a}$. Also, $x^{-\nu+\alpha}\phi_1$ is bounded in $A_{q,b}$ and $x^{-\nu+\alpha}\psi_1$ is bounded in $B_{q,b}$.

Proposition 3.3 *For $\rho \in B_{q,b}$, $\psi_1(\rho)$ satisfies the Fredholm q-integral equation of the form*

$$\psi_1(\rho) = \tilde{F}_1(\rho) + \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2}\int_b^\infty x\psi_1(x)K_1(\rho,x)\,d_qx, \tag{3.26}$$

where

$$K_1(\rho,x) = \int_0^\infty \frac{uw(u)}{1+w(u)}J_{\nu-\alpha}\big(ux;q^2\big)J_{\nu-\alpha}\big(u\rho;q^2\big)\,d_qu,$$

$$\tilde{F}_1(\rho) = F_1(\rho) - \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2}\int_0^a x\psi_2(x)\int_0^\infty \frac{u}{1+w(u)}J_{\nu+\alpha}\big(ux;q^2\big)J_{\nu-\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx,$$

and

$$\begin{aligned} F_1(\rho) = {}& \rho^{\nu-\alpha}\frac{q^{-2\alpha^2-\alpha+\nu}(1+q)(1-q^2)^{-\alpha}}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_\rho^\infty x^{2\alpha-\nu-1}f_3(qx)\big(\rho^2/x^2;q^2\big)_{\alpha-1}\,d_qx \\ & - \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2}\biggl[\int_a^\infty x\Phi_2(x)\int_0^\infty \frac{u}{1+w(u)}J_{\nu+\alpha}\big(ux;q^2\big)J_{\nu-\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx \\ & + \int_0^b x\Phi_1(x)\int_0^\infty \frac{uw(u)}{1+w(u)}J_{\nu-\alpha}\big(ux;q^2\big)J_{\nu-\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx\biggr]. \end{aligned}$$

Proof Equation (3.11) can be written in the following form:

$$\int_0^\infty u^{2\alpha}C_1(u)J_\nu\big(u\rho;q^2\big)\,d_qu = G(\rho), \quad \rho \in B_{q,b}, \tag{3.27}$$

where

$$\begin{aligned} G(\rho) = {}& f_3(\rho) - \int_0^\infty u^{2\alpha}C_2(u)\frac{1}{1+w(u)}J_\nu\big(u\rho;q^2\big)\,d_qu \\ & + \int_0^\infty u^{2\alpha}C_1(u)\frac{w(u)}{1+w(u)}J_\nu\big(u\rho;q^2\big)\,d_qu. \end{aligned} \tag{3.28}$$

By using equations (2.3) and (3.27), we get

$$G(\rho) = -(1-q)\rho^{\nu-1}q^{\nu-1}D_{q,\rho}\rho^{1-\nu}\int_0^\infty u^{2\alpha-1}C_1(u)J_{\nu-1}\big(u\rho q^{-1};q^2\big)\,d_qu. \tag{3.29}$$

Substituting the value of $C_1(u)$ from (3.14) into (3.29), we obtain

$$\begin{aligned} & D_{q,\rho}\rho^{1-\nu}\int_0^\infty u^\alpha\biggl[\int_0^b x\Phi_1(x)J_{\nu-\alpha}\big(ux;q^2\big)\,d_qx + \int_b^\infty x\psi_1(x)J_{\nu-\alpha}\big(ux;q^2\big)\,d_qx\biggr] \\ & \quad \times J_{\nu-1}\big(u\rho q^{-1};q^2\big)\,d_qu = -\frac{\rho^{1-\nu}q^{1-\nu}G(\rho)}{(1-q)}, \quad \rho \in B_{q,b}. \end{aligned} \tag{3.30}$$

From (2.5), there exists $M > 0$ such that

$$\big|J_{\nu-\alpha}\big(ux;q^2\big)\big| \le M(ux)^{\Re(\nu-\alpha)} \quad \text{for all } u,x \in \mathbb{R}_{q^2,b,+}.$$

Hence, from Remark 3.2, the double q-integration is absolutely convergent and we can interchange the order of the q-integrations to obtain

$$\begin{aligned} G(\rho) &= -(1-q)\rho^{\nu-1}q^{\nu-1}\left[\int_0^b x\Phi_1(x)\,d_qx + \int_b^\infty x\psi_1(x)\,d_qx\right] \\ &\quad \times D_{q,\rho}\rho^{1-\nu}\int_0^\infty u^\alpha J_{\nu-1}\big(u\rho q^{-1};q^2\big)J_{\nu-\alpha}\big(ux;q^2\big)\,d_qu, \quad \rho\in B_{q,b}. \end{aligned} \tag{3.31}$$

Therefore, applying Proposition 2.1 with $\Re(\nu-\alpha) > \Re(\nu-1) > -1$ we obtain

$$G(\rho) = \frac{-(1-q)^2(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}\rho^{\nu-1}D_{q,\rho}\int_\rho^\infty x^{1-\nu-\alpha}\psi_1(x)\big(\rho^2/x^2;q^2\big)_{-\alpha}\,d_qx. \tag{3.32}$$

By using

$$\int_x^\infty f(t)\,d_qt = \frac{1}{1+q}\int_{x^2}^\infty \frac{f(\sqrt{t})}{\sqrt{t}}\,d_{q^2}t, \qquad D_{q,\rho}\big(f\big(\rho^2\big)\big) = \rho(1+q)(D_{q^2}f)\big(\rho^2\big), \tag{3.33}$$

we obtain

$$\begin{aligned} G(\rho) &= \frac{-(1-q)^2(1-q^2)^\alpha}{\Gamma_{q^2}(1-\alpha)}\rho^\nu D_{q^2,\rho^2}\int_{\rho^2}^\infty x^{\frac{-(\nu+\alpha)}{2}}\psi_1(\sqrt{x})\big(\rho^2/x;q^2\big)_{-\alpha}\,d_{q^2}x \\ &= -(1-q)^2\big(1-q^2\big)^\alpha q^{\alpha^2-2\alpha-\nu}\rho^\nu\big(D_{q^2}\mathcal{K}_{q^2}^{1-\alpha}\big((\cdot)^{-\frac{\nu+\alpha}{2}}\psi_1(\cdot)\big)\big)\big(\rho^2/q^2\big). \end{aligned}$$

Replacing ρ by $q\rho$ yields

$$\begin{aligned} &-q^{-\alpha^2+\alpha}\big(1-q^2\big)^{-\alpha}(1-q)^{-2}\big[(\cdot)^{-\nu/2}G(q\sqrt{\cdot})\big]\big(\rho^2\big) \\ &\quad = D_{q^2,\rho^2}\mathcal{K}_{q^2}^{1-\alpha}\big[(\sqrt{\cdot})^{(\alpha-\nu)}\psi_1(\sqrt{\cdot})\big]\big(\rho^2\big). \end{aligned} \tag{3.34}$$

Thus, applying Proposition 3.3 yields

$$\begin{aligned} \rho^{\alpha-\nu}\psi_1(\rho) &= q^{-\alpha^2}(1-q)^{-2}\big(1-q^2\big)^{-\alpha}\mathcal{K}_{q^2}^{\alpha}\big[(\cdot)^{-\nu/2}G(q\sqrt{\cdot})\big]\big(\rho^2/q^2\big) \\ &= \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1-q)^{-2}}{\Gamma_{q^2}(\alpha)}\int_{\rho^2}^\infty x^{-\frac{\nu}{2}+\alpha-1}G(q\sqrt{x})\big(\rho^2/x;q^2\big)_{\alpha-1}\,d_{q^2}x. \end{aligned}$$

Using $\int_{x^2}^\infty f(t)\,d_{q^2}t = (1+q)\int_x^\infty tf(t^2)\,d_qt$, we obtain

$$\rho^{\alpha-\nu}\psi_1(\rho) = \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_\rho^\infty x^{2\alpha-\nu-1}G(qx)\big(\rho^2/x^2;q^2\big)_{\alpha-1}\,d_qx.$$

From (3.28), we can write the last equation in the following form:

$$\begin{aligned} &\psi_1(\rho) + \rho^{\nu-\alpha}\frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_\rho^\infty x^{2\alpha-\nu-1}\big(\rho^2/x^2;q^2\big)_{\alpha-1} \\ &\quad \times\left[\int_0^\infty \frac{u^{2\alpha}}{1+w(u)}C_2(u)J_\nu\big(qux;q^2\big)\,d_qu\right. \end{aligned}$$

$$- \int_0^\infty \frac{w(u)}{1+w(u)} u^{2\alpha} C_1(u) J_\nu\bigl(qux;q^2\bigr)\,d_qu \biggr]\,d_qx$$
$$= \rho^{\nu-\alpha} \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)}$$
$$\times \int_\rho^\infty x^{2\alpha-\nu-1} f_3(qx)\bigl(\rho^2/x^2;q^2\bigr)_{\alpha-1}\,d_qx, \quad \rho \in B_{q,b}. \tag{3.35}$$

From the condition on the function C_2, we can prove that the double q-integration

$$\int_\rho^\infty x^{2\alpha-\nu-1}\bigl(\rho^2/x^2;q^2\bigr)_{\alpha-1} \int_0^\infty C_2(u) \frac{u^{2\alpha}}{1+w(u)} J_\nu\bigl(qux;q^2\bigr)\,d_qu\,d_qx$$

is absolutely convergent. Therefore, we can interchange the order of the q-integrations and use Proposition 2.2 to obtain

$$\psi_1(\rho) + \frac{q^{-2\alpha^2-\alpha+\nu}}{(1-q)^2}\biggl[\int_0^\infty \frac{u^\alpha}{1+w(u)} C_2(u) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu$$
$$- \int_0^\infty \frac{u^\alpha w(u)}{1+w(u)} C_1(u) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu\biggr]$$
$$= \rho^{\nu-\alpha} \frac{q^{-2\alpha^2-\alpha+\nu}(1-q^2)^{-\alpha}(1+q)}{(1-q)^2\Gamma_{q^2}(\alpha)}$$
$$\times \int_\rho^\infty x^{2\alpha-\nu-1} f_3(qx)\bigl(\rho^2/x^2;q^2\bigr)_{\alpha-1}\,d_qx, \quad \rho \in B_{q,b}. \tag{3.36}$$

Substituting the value of $C_1(u)$ and $C_2(u)$ from equations (3.15) and (3.14) into equation (3.36), and then interchanging the order of the q-integrations we get

$$\psi_1(\rho) + \frac{q^{\nu-4\alpha}}{(1-q)^2}\biggl[\int_0^a x\psi_2(x) \int_0^\infty \frac{u}{1+w(u)} J_{\nu+\alpha}\bigl(ux;q^2\bigr) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu\,d_qx$$
$$- \int_b^\infty x\psi_1(x) \int_0^\infty \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}\bigl(ux;q^2\bigr) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu\,d_qx\biggr]$$
$$= F_1(\rho), \quad \rho \in B_{q,b}, \tag{3.37}$$

where

$$F_1(\rho) = \rho^{\nu+\alpha} \frac{q^{\nu-4\alpha}(1+q)(1-q^2)^{-\alpha}}{(1-q)^2\Gamma_{q^2}(\alpha)} \int_\rho^\infty x^{2\alpha-\nu-1} f_3(qx)\bigl(\rho^2/x^2;q^2\bigr)_{\alpha-1}\,d_qx$$
$$- \frac{q^{\nu-4\alpha}}{(1-q)^2}\biggl[\int_a^\infty x\Phi_2(x) \int_0^\infty \frac{u}{1+w(u)} J_{\nu+\alpha}\bigl(ux;q^2\bigr) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu\,d_qx$$
$$+ \int_0^b x\Phi_1(x) \int_0^\infty \frac{uw(u)}{1+w(u)} J_{\nu-\alpha}\bigl(ux;q^2\bigr) J_{\nu-\alpha}\bigl(u\rho;q^2\bigr)\,d_qu\,d_qx\biggr].$$

Equation (3.37) is nothing else but the Fredholm q-integral equation of the second kind (3.26). This completes the proof. □

Proposition 3.4 *For $\rho \in A_{q,a}$, $\psi_2(\rho)$ satisfies the Fredholm q-integral equation of the form*

$$\psi_2(\rho) = \tilde{F}_2(\rho) + \frac{1}{(1-q)^2} \int_0^a xK_2(\rho,x)\psi_2(x)\,d_qx, \tag{3.38}$$

where

$$K_2(\rho,x)=\int_0^\infty \frac{uw(u)}{1+w(u)}J_{\nu+\alpha}\big(ux;q^2\big)J_{\nu+\alpha}\big(u\rho;q^2\big)\,d_qu,$$

$$\tilde{F}_2(\rho)=F_2(\rho)-\frac{1}{(1-q)^2}\int_b^\infty x\psi_1(x)\int_0^\infty \frac{u}{1+w(u)}J_{\nu-\alpha}\big(ux;q^2\big)J_{\nu+\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx,$$

and

$$\begin{aligned}F_2(\rho)={}&\frac{(1-q^2)^{-\alpha}(1+q)\rho^{\alpha-\nu-2}}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_0^\rho \big(q^2x^2/\rho^2;q^2\big)_{\alpha-1}x^{\nu+1}f_1(x)\,d_qx\\&+\frac{1}{(1-q)^2}\int_a^\infty x\Phi_2(x)\int_0^\infty \frac{uw(u)}{1+w(u)}J_{\nu+\alpha}\big(ux;q^2\big)J_{\nu+\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx\\&-\frac{1}{(1-q)^2}\int_0^b x\Phi_1(x)\int_0^\infty \frac{u}{1+w(u)}J_{\nu-\alpha}\big(ux;q^2\big)J_{\nu+\alpha}\big(u\rho;q^2\big)\,d_qu\,d_qx.\end{aligned}$$

Proof The proof is similar to the proof of Proposition 3.3 and is omitted. □

Theorem 3.5 *The solution of* (3.1)-(3.2) *is given by*

$$\psi(u)=\frac{u^{2\alpha}}{1+w(u)}\big(C_1(u)+C_2(u)\big).$$

The functions C_1, C_2, ϕ_1, and ϕ_2 are given by Proposition 3.1, *and ψ_1, ψ_2 satisfies the Fredholm q-integral equations* (3.38) *and* (3.26) *of second kind.*

Example 1 1. Take $b=aq^{-m}$ and assume that $m\to\infty$. If we assume that $f_1=f$, $f_2=f$, and $w=0$. Then the system (3.1)-(3.3) is reduced to the dual q-integral equations

$$\int_0^\infty \psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu=f(\rho),\quad \rho\in A_{q,a},\tag{3.39}$$

$$\int_0^\infty u^{-2\alpha}\psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu=0,\quad \rho\in B_{q,a}.\tag{3.40}$$

Hence, from Theorem 3.5,

$$\begin{aligned}\psi(u)&=u^{1+\alpha}\int_0^\infty x\psi_2(x)J_{\nu+\alpha}\big(ux;q^2\big)\,d_qx,\quad u\in\mathbb{R}_{q,+},\\ \psi_2(\rho)&=\frac{(1-q^2)^{-\alpha}(1+q)\rho^{\alpha-\nu-2}}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_0^\rho \big(q^2x^2/\rho^2;q^2\big)_{\alpha-1}x^{\nu+1}f(x)\,d_qx\\&=\rho^{-\alpha-\nu}\frac{(1-q^2)^{-\alpha}}{(1-q)^2}I_{q^2}^{\alpha}\big(t^{\nu/2}f(\sqrt{t})\big)\big(\rho^2\big).\end{aligned}$$

Hence,

$$\psi(u)=u^{1+\alpha}\frac{(1-q^2)^{-\alpha}}{(1-q^2)}\int_0^\infty x^{1-\alpha-\nu}I_{q^2}^{\alpha}\big(t^{\nu/2}f(\sqrt{t})\big)\big(x^2\big)J_{\nu+\alpha}\big(ux;q^2\big)\,d_qx.$$

This coincides with the result in [15], Theorem 4.1, for solutions of double q-integral equations.

2. Let $a = q^m$ and assume that $m \to \infty$. If we assume that $f_2 = 0$, and $f_3 = f$, we obtain the dual q-integral system of equations

$$\int_0^\infty u^{-2\alpha}\psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu = 0, \quad \rho \in A_{q,b}, \tag{3.41}$$

$$\int_0^\infty \psi(u)J_\nu\big(u\rho;q^2\big)\,d_qu = f, \quad \rho \in B_{q,b}. \tag{3.42}$$

Hence, from Theorem 3.5,

$$\begin{aligned}
\psi(u) &= u^{1+\alpha}\int_b^\infty x\psi_1(x)J_{\nu-\alpha}\big(ux;q^2\big)\,d_qx, \quad u \in \mathbb{R}_{q,+}, \\
\psi_1(\rho) &= -\frac{(1-q^2)^{-\alpha}q^{-2\alpha}\rho^{\alpha+\nu}}{(1-q)^2\Gamma_{q^2}(\alpha)}\int_\rho^\infty \big(\rho^2/x^2;q^2\big)_{\alpha-1}x^{2\alpha-\nu-1}f(x)\,d_qx.
\end{aligned}$$

This is a special case of Theorem 5.1 in [15].

Example 2 We consider the triple q-integral equations

$$\int_0^\infty \psi(u)J_0\big(u\rho;q^2\big)\,d_qu = 0, \quad \rho \in A_{q,a}, \tag{3.43}$$

$$\int_0^\infty u^{-1}\psi(u)J_0\big(u\rho;q^2\big)\,d_qu = 1, \quad \rho \in A_{q,b}\cap B_{q,a}, \tag{3.44}$$

$$\int_0^\infty \psi(u)J_0\big(u\rho;q^2\big)\,d_qu = 0, \quad \rho \in B_{q,b}. \tag{3.45}$$

Hence, we have $\nu = 0$, $g_1 = 1$, $g_2 = 0$, $f_1 = f_3 = 0$, $w = 0$, and $\alpha = \frac{1}{2}$.

From Theorem 3.5,

$$\psi(u) = u\big(C_1(u) + C_2(u)\big),$$

where

$$\begin{aligned}
C_1(u) &= \frac{(1-q)(1-q^2)}{\Gamma_{q^2}^2(1/2)}\frac{\sin(\frac{bu}{1-q};q)}{u} \\
&\quad + \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)}\int_b^\infty \sqrt{x}\psi_1(x)\cos\left(\frac{xu\sqrt{q}}{1-q};q^2\right)d_qx, \\
C_2(u) &= \frac{\sqrt{1-q^2}}{\Gamma_{q^2}(1/2)}\int_0^a \sqrt{x}\psi_2(x)\sin\left(\frac{xu}{1-q};q^2\right)d_qx,
\end{aligned}$$

$$\psi_1(\rho) = \frac{\sqrt{\rho}(1+q)}{q(1-q)\Gamma_{q^2}^2(1/2)}\int_0^a x^{3/2}\frac{\psi_2(x)}{q\rho^2 - x^2}\,d_qx, \quad \rho \in B_{q,b}, \tag{3.46}$$

$$\begin{aligned}
\psi_2(\rho) &= -\frac{(1+q)\sqrt{\rho}}{(1-q)\Gamma_{q^2}^2(1/2)}\int_b^\infty \frac{\sqrt{x}\psi_1(x)}{qx^2-\rho^2}\,d_qx \\
&\quad + \frac{(1+q)^{3/2}}{\sqrt{1-q}\,\Gamma_{q^2}^3(1/2)}\sqrt{\rho}\int_{\rho/q}^b \frac{d_qx}{qx^2-\rho^2}.
\end{aligned} \tag{3.47}$$

We used [19], pp.455-466 or Proposition 2.4 of [15] to calculate ψ_1 and ψ_2 in equations (3.46) and (3.47), respectively. Substituting from (3.46) into (3.47), we obtain the second order Fredholm q-integral equation

$$\begin{aligned}\psi_2(\rho) &= -\frac{q^{-1}\sqrt{\rho}(1+q)}{(1-q)^2\Gamma^3_{q^2}(1/2)}\int_0^a t^{3/2}\psi_2(t)K_2(\rho,t)\,d_qt \\ &\quad + \frac{(1+q)^{3/2}}{\sqrt{1-q}\Gamma^3_{q^2}(1/2)}\sqrt{\rho}\int_{\rho/q}^b \frac{d_qx}{qx^2-\rho^2},\end{aligned} \tag{3.48}$$

where $\rho \in A_{q,a}$ and

$$K_2(\rho,t) = \int_b^\infty \frac{x}{(t^2-qx^2)(\rho^2-qx^2)}\,d_qt.$$

4 Solving system of triple q^2-integral equations by using solutions of dual q-integral equations

In [2], Cooke solved certain triple integral equations involving Bessel functions by using a result for Noble [29] for solutions for dual integral equations with Bessel functions as kernel. In this section, we use the result, Theorem A, introduced in [15] to solve the following triple q-integral equations:

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}\psi(\rho)J_\kappa\big(\sqrt{\rho\xi};q^2\big)\,d_{q^2}\rho = f(\xi), \quad \xi \in A_{q^2}, \tag{4.1}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho)J_\mu\big(\sqrt{\rho\xi};q^2\big)\,d_{q^2}\rho = g(\xi), \quad \xi \in A_{q^2}\cap B_{q^2}, \tag{4.2}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}\psi(\rho)J_\nu\big(\sqrt{\rho\xi};q^2\big)\,d_{q^2}\rho = h(\xi), \quad \xi \in B_{q^2}, \tag{4.3}$$

where a, α, β, γ, μ, ν, and κ are complex numbers such that

$$\Re(\nu) > -1, \qquad \Re(\mu) > -1, \qquad \Re(\kappa) > -1, \quad \text{and} \quad 0 < a < 1,$$

the functions $f(\rho)$, $g(\rho)$, and $h(\rho)$ are known functions, and $\psi(u)$ is the solution function to be determined.

The following is a result from [15] that we shall use to solve the system (4.1)-(4.3).

Theorem A *Let α, β, μ, and ν be complex numbers and let $\lambda := \frac{1}{2}(\mu+\nu)-(\alpha-\beta) > -1$. Assume that*

$$\Re(\nu) > -1, \qquad \Re(\mu) > -1, \qquad \Re(\lambda) > -1, \quad \textit{and} \quad \Re(\lambda-\mu-2\alpha) > 0.$$

Let $f \in L_{q^2,\frac{\mu}{2}+\alpha}(A_{q^2})$ and $g \in L_{q^2,\frac{-\mu}{2}+\alpha-1}(B_{q^2})$. Then the dual q^2-integral equations

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}\psi(\rho)J_\mu\big(\sqrt{\rho\xi};q^2\big)\,d_{q^2}\rho = f(\xi), \quad \xi \in A_{q^2}, \tag{4.4}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}\psi(\rho)J_\nu\big(\sqrt{\rho\xi};q^2\big)\,d_{q^2}\rho = g(\xi), \quad \xi \in B_{q^2}, \tag{4.5}$$

has a solution of the form

$$\psi(\xi) = (1-q^2)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda(\sqrt{\rho\xi};q^2) I_{q^2}^{\mu/2+\alpha,\lambda-\mu} f(\rho)\,d_{q^2}\rho$$
$$+ (1-q^2)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda(\sqrt{\rho\xi};q^2)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda} g(\rho)\,d_{q^2}\rho,$$

in $L_{q^2,\frac{\mu}{2}-\alpha}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\frac{\nu}{2}-\beta}(\mathbb{R}_{q^2,+}) \cap L_{q^2,\frac{\nu}{2}-\beta-\gamma}(\mathbb{R}_{q^2,+})$, *for* γ *satisfying*

$$1 + \Re(\nu) > \Re(\gamma) > \max\{0, \Re(\nu-\lambda)\}.$$

Now we shall solve the system of triple q^2-integral equations (4.1)-(4.3). Since the function $g(\rho)$ is only defined in $A_{q^2} \cap B_{q^2}$, we can write

$$g(\xi) = g_1(\xi) + g_2(\xi),$$

g_1 and g_2 defined in A_{q^2} and B_{q^2}, respectively. So, we may assume that

$$\psi = A_1 + A_2,$$

and we solve the equations in the form

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}[A_1(\rho)+A_2(\rho)]J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f(\xi), \quad \xi \in A_{q^2}, \tag{4.6}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_1(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_1(\xi), \quad \xi \in A_{q^2}, \tag{4.7}$$

$$\xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_2(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_2(\xi), \quad \xi \in B_{q^2}, \tag{4.8}$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}[A_1(\rho)+A_2(\rho)]J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = h(\xi), \quad \xi \in B_{q^2}. \tag{4.9}$$

We rewrite the equations as two pairs of dual q-integral equations, namely

$$\begin{cases} \xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_1(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_1(\xi), & \xi \in A_{q^2}, \\ \xi^{-\beta}\int_0^\infty \rho^{-\beta}A_1(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = h(\xi) - f_2(\xi), & \xi \in B_{q^2}, \end{cases} \tag{4.10}$$

$$\begin{cases} \xi^{-\alpha}\int_0^\infty \rho^{-\alpha}A_2(\rho)J_\mu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = g_2(\xi), & \xi \in B_{q^2}, \\ \xi^{-\gamma}\int_0^\infty \rho^{-\gamma}A_2(\rho)J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f(\xi) - f_1(\xi), & \xi \in A_{q^2}, \end{cases} \tag{4.11}$$

where

$$\xi^{-\gamma}\int_0^\infty \rho^{-\gamma}A_1(\rho)J_\kappa(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f_1(\xi), \quad \xi \in A_{q^2},$$

$$\xi^{-\beta}\int_0^\infty \rho^{-\beta}A_2(\rho)J_\nu(\sqrt{\rho\xi};q^2)\,d_{q^2}\rho = f_2(\xi), \quad \xi \in B_{q^2}.$$

Then we can solve the first and second pairs by Theorem A. For the first pairs

$$\begin{aligned} A_1(\xi) &= \bigl(1-q^2\bigr)^{\lambda-\nu+2\alpha-2}\xi^{\lambda/2-\mu/2+\alpha}\int_0^1 J_\lambda\bigl(\sqrt{\rho\xi};q^2\bigr)I_{q^2}^{\mu/2+\alpha,\lambda-\mu}g_1(\rho)\,d_{q^2}\rho \\ &\quad+\bigl(1-q^2\bigr)^{\lambda-\nu-2}\xi^{\lambda/2-\mu/2+\alpha}\int_1^\infty J_\lambda\bigl(\sqrt{\rho\xi};q^2\bigr)\mathcal{K}_{q^2}^{\lambda/2-\nu/2-\beta,\nu-\lambda}\bigl[h(\rho)-f_2(\rho)\bigr]\,d_{q^2}\rho, \end{aligned}$$

where $\lambda := \frac{1}{2}(\mu+\nu)-(\alpha-\beta) > -1$.

The solution of the second pair has the form

$$\begin{aligned} A_2(\xi) &= \bigl(1-q^2\bigr)^{\lambda-\mu+2\gamma-2}\xi^{\lambda/2-\kappa/2+\gamma}\int_0^a J_\lambda\bigl(\sqrt{\rho\xi};q^2\bigr)I_{q^2}^{\kappa/2+\gamma,\lambda-\kappa}\bigl[f(\rho)-f_1(\rho)\bigr]\,d_{q^2}\rho \\ &\quad+\bigl(1-q^2\bigr)^{\lambda-\mu-2}\xi^{\lambda/2-\kappa/2+\gamma}\int_a^\infty J_\lambda\bigl(\sqrt{\rho\xi};q^2\bigr)\mathcal{K}_{q^2}^{\lambda/2-\mu/2-\alpha,\mu-\lambda}g_2(\rho)\,d_{q^2}\rho, \end{aligned}$$

where $\lambda := \frac{1}{2}(\mu+\kappa)-(\gamma-\alpha) > -1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made equal and significant contributions in writing this paper. They read and approved the final manuscript.

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