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Fractional boundary value problems with $p(t)$ -Laplacian operator

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Abstract

The purpose of this paper is to discuss boundary value problems of fractional differential equations with $p(t)$ -Laplacian operator. Some new existence, uniqueness, and multiplicity results were acquired by employing some fixed point theorems. Moreover, some examples are supplied to verify our main results.

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1 Introduction

Fractional differential equations have been applied in many research fields in recent years (see [1–4]). Leszczynski and Blaszczyk [2] studied the following fractional mathematical model, which can be used to describe the height of granular material decreasing over time in a silo:

$${}^C D_{T-}^{\alpha} D_{a+}^{\alpha} h^*(t) + \beta h^*(t) = 0, \quad t \in [0, T],$$

where ${}^C D_{T-}^{\alpha}$ and D_{a+}^{α} are respectively the right Caputo and left Riemann-Liouville fractional derivatives of order $\alpha \in (0, 1)$. Moreover, many valuable results related to boundary value problems (BVPs) or initial problems for fractional differential equations have been achieved by some scholars (see [5–11]).

Bai and Lü [5] considered the fractional BVP

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1) = 0, \end{cases} \quad (1.1)$$

and obtained the existence and multiplicity of positive solutions by taking advantage of methods in cone. Here, D_{0+}^{α} is the left Riemann-Liouville fractional derivative of order $\alpha \in (1, 2]$, and $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

Recently, many scholars have focused on fractional BVPs with p -Laplacian operator (see [12–14]). Chen and Liu [13] dealt with the following BVP involving p -Laplacian operator:

$$\begin{cases} D_{0+}^{\beta} \varphi_p(D_{0+}^{\alpha} x(t)) = f(t, x(t)), & t \in [0, 1], \\ x(0) = -x(1), \quad D_{0+}^{\alpha} x(0) = -D_{0+}^{\alpha} x(1), \end{cases} \quad (1.2)$$

where $0 < \beta, \alpha \leq 1, 1 < \alpha + \beta \leq 2, D_{0+}^\alpha$ is the Caputo fractional derivative, $\varphi_p(\cdot)$ is the p -Laplacian operator, and $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

However, many papers focused on the existence and multiplicity of solutions for fractional BVPs. The uniqueness of solution for fractional BVPs with $p(t)$ -Laplacian operator has not been yet investigated. Thus, we deal with the following fractional BVP with $p(t)$ -Laplacian operator and obtain the uniqueness of its solution by the method in cone (see Theorem 3.2):

$$\begin{cases} D_{0+}^\beta \varphi_{p(t)}(D_{0+}^\alpha x(t)) + f(t, x(t)) = 0, & t \in [0, 1], \\ x'(0) = x(1) = x''(0) = 0, & D_{0+}^\alpha x(0) = 0, \end{cases} \tag{1.3}$$

where D_{0+}^α is the Caputo fractional derivative, $2 < \alpha < 3, 0 < \beta < 1, \varphi_{p(t)}(\cdot)$ is the $p(t)$ -Laplacian operator with $p(t) \in C^1[0, 1]$ such that $p(t) > 1$. Moreover, f does not need to satisfy the Lipschitz condition, so the problem becomes more complicated. An iterative scheme is shown to approximate it.

Furthermore, we also discuss the existence and multiplicity of solutions for the BVP (1.3) (see Theorem 3.1 and Theorem 3.3). Noting that when $p(t) = p$, it becomes the well-known p -Laplacian operator, our results extend and enrich some existing papers. For the problems of integer differential equations with $p(t)$ -Laplacian or p -Laplacian operator, we refer the readers to [15–19]).

2 Preliminaries

Let $E = C[0, 1]$ with norm $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$, and P be a cone of E , where $P = \{x \in E | x(t) \geq 0\}$. Moreover, we define the partial ordering \leq with respect to P by $x \leq y$ if only if $y - x \in P$. For $u_0 \in P$ such that $u_0 > \theta$ (i.e., $u_0(t)$ is unequal to zero identically), we denote

$$P_{u_0} = \{x | x \in E, \exists \lambda(x) > 0, \mu(x) > 0 \text{ s.t. } \lambda(x)u_0 \leq x \leq \mu(x)u_0\}.$$

Lemma 2.1 ([16]) *For any $(t, x) \in [0, 1] \times \mathbb{R}, \varphi_{p(t)}(x) = |x|^{p(t)-2}x$ is a homeomorphism from \mathbb{R} to \mathbb{R} and strictly monotone increasing for any fixed t . Moreover, $\varphi_{p(t)}^{-1}(\cdot)$ is continuous, sends bounded sets to bounded sets, and is defined by*

$$\varphi_{p(t)}^{-1}(x) = |x|^{\frac{2-p(t)}{p(t)-1}}x \quad \text{for } x \in \mathbb{R} \setminus \{0\}, \quad \varphi_{p(t)}^{-1}(0) = 0 \quad \text{for } x = 0.$$

Definition 2.1 ([20]) Let E be a Banach space. A cone $P \subset E$ is called normal if there is a constant $N > 0$ such that $\theta \leq x \leq y$ and $\|x\| \leq N\|y\|$ for all $x, y \in E$.

Lemma 2.2 ([21, 22]) *Let $P \subset E$ be a normal cone. Suppose that $A : P_{u_0} \rightarrow P_{u_0}$ is increasing and for any $t \in (0, 1)$, there exists $\eta(t) > 0$ such that*

$$A(tx) \geq t(1 + \eta(t))Ax, \quad x \in P_{u_0}.$$

Then, A has a unique fixed point x^ if and only if there exist $w_0, v_0 \in P_{u_0}$ such that $w_0 \leq Aw_0 \leq Av_0 \leq v_0$. Moreover, for any $x_0 \in [w_0, v_0]$, letting $x_n = Ax_{n-1}$ ($n = 1, 2, \dots$), we have $x_n \rightarrow x^*$.*

Define $P(\theta, b, d) = \{x \in P | b \leq \theta(x) \text{ and } \|x\| \leq d\}$ and $P_c = \{x \in P | \|x\| \leq c\}$, where $b, c, d > 0$.

Lemma 2.3 ([23]) *Let P be a cone of Banach space E , and $T : P_c \rightarrow P_c$ be a completely continuous map. Suppose that there exists a nonnegative continuous concave functional θ such that $\theta(x) \leq \|x\|$ for $x \in P$ and numbers $0 < a < b < d \leq c$ satisfying the following conditions:*

- (i) $\{x \in P(\theta, b, d) | \theta(x) > b\} \neq \emptyset$ and $\theta(Tx) > b$ for $x \in P(\theta, b, d)$.
- (ii) $\|Tx\| < a$ for $x \in P_a$.
- (iii) $\theta(Tx) > b$ for $x \in P(\theta, b, c)$ with $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, x_3 in P_c .

Definition 2.2 ([1]) The Riemann-Liouville fractional integral operator of order α of a function x is given by

$$I_{0^+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided that the right-hand side are pointwise defined on $[0, 1]$.

Definition 2.3 ([1]) The Caputo fractional derivative of order α of a function x is given by

$$D_{0^+}^\alpha x(t) = I_{0^+}^{n-\alpha} \frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $[0, 1]$.

Lemma 2.4 ([1]) *The general solution of the Caputo fractional differential equation*

$$D_{0^+}^\alpha x(t) = 0$$

is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

Lemma 2.5 ([1]) *Assume that $D_{0^+}^\alpha x(t) \in C[0, 1]$. Then*

$$I_{0^+}^\alpha D_{0^+}^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

where $c_i = -\frac{x^{(i)}(0)}{i!}, i = 0, 1, 2, \dots, n-1, n = [\alpha] + 1$.

Lemma 2.6 *If $y(t) \in C[0, 1]$, then the unique solution of*

$$\begin{cases} D_{0^+}^\beta \varphi_{p(t)}(D_{0^+}^\alpha x(t)) + y(t) = 0, & t \in [0, 1], \\ x'(0) = x(1) = x''(0) = 0, & D_{0^+}^\alpha x(0) = 0, \end{cases}$$

can be expressed as the integral

$$x(t) = \int_0^1 G(t, s) \varphi_{p(s)}^{-1}(I_{0+}^\beta y(s)) ds, \tag{2.1}$$

where

$$G(t, s) = \begin{cases} \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof Based on Lemma 2.5, we have

$$\varphi_{p(t)}(D_{0+}^\alpha x(t)) = -I_{0+}^\beta y(t) + c, \quad c \in \mathbb{R}.$$

Applying the operator $\varphi_{p(t)}^{-1}$ to both sides of this equality, we have

$$D_{0+}^\alpha x(t) = \varphi_{p(t)}^{-1}(-I_{0+}^\beta y(t) + c).$$

Combining this with $D_{0+}^\alpha x(0) = 0$, for fixed $t = 0$, we have $\varphi_{p(0)}^{-1}(c) = 0$. By Lemma 2.1 we have that $c = 0$ and

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta y(s)) ds + c_1 + c_2 t + c_3 t^2,$$

where $c_i \in \mathbb{R}, i = 1, 2, 3$. Since $x'(0) = x(1) = x''(0) = 0$, we get that

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta y(s)) ds$$

and $c_2 = c_3 = 0$. Thus, we have

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta y(s)) ds.$$

Therefore, (2.1) holds. The proof is complete. □

Lemma 2.7 $G(t, s)$ satisfies the following conditions:

- (D₁) $G(t, s) \in C([0, 1] \times [0, 1]), G(t, s) \geq 0$ for $t, s \in [0, 1]$.
- (D₂) $\frac{1}{\Gamma(\alpha)}(1 - t^{\alpha-1})(1 - s)^{\alpha-1} \leq G(t, s) \leq \frac{1}{\Gamma(\alpha)}(1 - t^{\alpha-1})$ for $t, s \in [0, 1]$.

Proof Clearly, (D₁) is satisfied. For $s \leq t$ and $2 < \alpha < 3$, we have

$$\begin{aligned} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} &\geq (1-s)^{\alpha-1} - (t-ts)^{\alpha-1} \\ &= (1-t^{\alpha-1})(1-s)^{\alpha-1}. \end{aligned}$$

For given t , we have

$$\frac{\partial G(t, s)}{\partial s} = \frac{1}{\Gamma(\alpha-1)}((t-s)^{\alpha-2} - (1-s)^{\alpha-2}) \leq 0.$$

Thus, $G(t, s) \leq \frac{1}{\Gamma(\alpha)}(1 - t^{\alpha-1})$.

For $t \leq s$, it is easy to get that

$$\begin{aligned} (1-s)^{\alpha-1} &\geq (1-t^{\alpha-1})(1-s)^{\alpha-1}, \\ (1-s)^{\alpha-1} &\leq (1-t)^{\alpha-1} \leq 1-t \leq 1-t^{\alpha-1}. \end{aligned}$$

Thus, (D_2) is satisfied. □

3 Main result

Define the nonnegative continuous concave functional θ by

$$\theta(Ax) = \min_{t \in [\tau, 1-\tau]} Ax(t), \quad \tau \in \left(0, \frac{1}{2}\right), x \in P,$$

where

$$Ax(t) = \int_0^1 G(t,s)\varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s,x(s))) ds.$$

Let

$$\begin{aligned} \Lambda_1 &:= \left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1} \left(\frac{1}{\Gamma(\beta+1)} s^\beta\right) ds\right)^{-1}, \\ \Lambda_2 &:= \left(\frac{1-(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1} \left(\frac{1}{\Gamma(\beta+1)} s^\beta\right) ds\right)^{-1}. \end{aligned}$$

Theorem 3.1 *Assume that $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous and there exist numbers $0 < a < b < d = c$ satisfying the following conditions:*

- (I_1) $f(t, x) < \varphi_{p(t)}(\Lambda_1 a)$ for $[0, 1] \times [0, a]$.
- (I_2) $f(t, x) > \varphi_{p(t)}(\Lambda_2 b)$ for $[\tau, 1-\tau] \times [b, c]$.
- (I_3) $f(t, x) \leq \varphi_{p(t)}(\Lambda_1 c)$ for $[0, 1] \times [0, c]$.

Then BVP (1.3) has three positive solutions.

Proof To begin with, for $u \in P_c$, we will prove that $A : P_c \rightarrow P_c$. By I_3 we have

$$\begin{aligned} \|Ax\| &= \max_{t \in [0,1]} \left| \int_0^1 G(t,s)\varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s,x(s))) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s,x(s))) ds \\ &\leq \Lambda_1 c \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1} \left(\frac{1}{\Gamma(\beta+1)} s^\beta\right) ds = c. \end{aligned}$$

Thus, $AP_c \subset P_c$. It is easy to get that A is continuous by the continuity of f . Let Ω be any bounded open subset of P_c . Since $\varphi_{p(t)}^{-1}(\cdot)$ and f are continuous, there exists a constant $B > 0$ such that $|\varphi_{p(t)}^{-1}(I_{0^+}^\beta f(t, x(t)))| \leq B$ on $[0, 1] \times \bar{\Omega}$. Thus, we have

$$\|Ax\| = \max_{t \in [0,1]} |Ax| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} B ds = \frac{B}{\Gamma(\alpha+1)}.$$

Thus, $A\bar{\Omega}$ is uniformly bounded. On the other hand, for all $t_1, t_2 \in [0, 1]$ such that $t_1 \leq t_2$ and for all $x \in \bar{\Omega}$, we have

$$\begin{aligned} |Ax(t_2) - Ax(t_1)| &= \left| \int_0^1 G(t_2, s)\varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, x(s))) ds - \int_0^1 G(t_1, s)\varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, x(s))) ds \right| \\ &\leq B \int_0^1 |G(t_2, s) - G(t_1, s)| ds \\ &\leq \frac{B}{\Gamma(\alpha + 1)}(t_2^\alpha - t_1^\alpha). \end{aligned}$$

Thus, we have

$$|Ax(t_2) - Ax(t_1)| \rightarrow 0 \quad \text{uniformly as } t_1 \rightarrow t_2.$$

Therefore, A is equicontinuous on $\bar{\Omega}$, so that $A : P_c \rightarrow P_c$ is completely continuous by the Arzelà-Ascoli theorem. Similarly, by I_1 we obtain that $\|Ax\| < a$ for $x \in P_a$. So, condition (ii) of Lemma 2.3 is satisfied. Let $x_0(t) = \frac{b+c}{2}$. Clearly, $\|x_0\| \leq c$ and $\theta(x_0) > b$. Thus,

$$\{x \in P(\theta, b, d) | \theta(x) > b\} \neq \emptyset.$$

For $x \in P(\theta, b, d)$, by (I_2) we have $b \leq x(t) \leq c, t \in [\tau, 1 - \tau]$, and

$$\begin{aligned} \theta(Ax) &= \min_{t \in [\tau, 1-\tau]} Ax(t) \\ &\geq \frac{1}{\Gamma(\alpha)} \min_{t \in [\tau, 1-\tau]} \int_0^1 (1-t^{\alpha-1})(1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, x(s))) ds \\ &> b\Lambda_2 \frac{1-(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}\left(\frac{1}{\Gamma(\beta+1)}s^\beta\right) ds = b. \end{aligned}$$

Thus, condition (i) of Lemma 2.3 holds. When $d = c$, condition (i) implies (iii) in Lemma 2.2. Then BVP (1.3) has three positive solutions. \square

Example 3.1 Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{1}{2}} \varphi_3(D_{0^+}^{\frac{8}{3}} x(t)) + f(t, x(t)) = 0, & t \in [0, 1], \\ x'(0) = x(1) = x''(0) = 0, & D_{0^+}^{\frac{8}{3}} x(0) = 0. \end{cases} \tag{3.1}$$

Choose $\alpha = \frac{8}{3}, \beta = \frac{1}{2}, p(t) = 3, \tau = \frac{1}{4}$,

$$f(t, x(t)) = \begin{cases} t + 45x^3, & x \leq 10, \\ t + \frac{9}{200}x^3 + 44,955, & x > 10, \end{cases}$$

$a = \frac{1}{4}, b = 10, c = 50$. By simple calculation we have

$$\Lambda_1 = \frac{(\Gamma(\frac{3}{2}))^{\frac{1}{2}} \Gamma(\frac{47}{12})}{\Gamma(\frac{5}{4})} \approx 5.29, \quad \Lambda_1 = \frac{(\Gamma(\frac{3}{2}))^{\frac{1}{2}} \Gamma(\frac{47}{12})}{\Gamma(\frac{5}{4})} \cdot \frac{\Gamma(\frac{8}{3})}{(1 - (\frac{3}{4}))^{\frac{5}{3}}} \approx 20.89.$$

So, we have

- (I₁) $f(t, x(t)) \leq 1.71 < 1.75 \approx \varphi_3(\Lambda_1 a)$ for $[0, 1] \times [0, \frac{1}{4}]$;
- (I₂) $f(t, x(t)) \geq 45,000.25 > 43,639.21 = \varphi_3(\Lambda_2 b)$ for $[\frac{1}{4}, \frac{3}{4}] \times [10, 50]$;
- (I₃) $f(t, x(t)) \leq 50,580 < 69,960.25 = \varphi_3(\Lambda_1 c)$ for $[0, 1] \times [0, 50]$.

Thus, Example 3.1 has three positive solutions.

Theorem 3.2 *Assume that $f : [0, 1] \times [0, \infty) \rightarrow (0, \infty)$ is continuous and the following conditions hold:*

- (I₄) $f(t, x)$ is increasing in x .
- (I₅) There exists $r \in (0, 1)$ such that

$$f(t, rx) \geq r^{\frac{p(t)-1}{2}} f(t, x), \quad \text{which implies } f\left(t, \frac{1}{r}x\right) \leq r^{-\frac{p(t)-1}{2}} f(t, x).$$

Then BVP (1.3) has a unique positive solution.

Proof Let $\Theta(t) = 1 - t^{\alpha-1}$, $\lambda(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, x(s))) ds$, $\mu(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi_{p(s)}^{-1} \times (I_{0+}^\beta f(s, x(s))) ds$. For $u \in P_\Theta$, we will prove that $A : P_\Theta \rightarrow P_\Theta$. Indeed, by Lemma 2.7, we have

$$\begin{aligned} Ax(t) &\geq \Theta(t) \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, x(s))) ds, \\ Ax(t) &\leq \Theta(t) \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, x(s))) ds. \end{aligned}$$

Clearly, we have $\lambda(x)\Theta(t) \leq Ax \leq \mu(x)\Theta(t)$. Thus, we that $A : P_\Theta \rightarrow P_\Theta$, together with (I₄) and the monotone increasing of $\varphi_{p(t)}^{-1}(\cdot)$, yields that A is an increasing operator. In view of (I₅), we have

$$\begin{aligned} A(rx) &\geq r^{\frac{1}{2}} \int_0^1 G(t, s) \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, x(s))) ds \\ &= r(1 + r^{-\frac{1}{2}} - 1) \int_0^1 G(t, s) \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, x(s))) ds. \end{aligned}$$

We have $\eta(r) := r^{-\frac{1}{2}} - 1 > 0$. Let

$$\begin{aligned} b(t) &= \int_0^1 G(t, s) \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, \Theta(s))) ds, \\ b_1 &= \min \left\{ 1, \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, \Theta(s))) ds \right\}, \\ b_2 &= \max \left\{ 1, \frac{1}{\Gamma(\alpha)} \int_0^1 \varphi_{p(s)}^{-1}(I_{0+}^\beta f(s, \Theta(s))) ds \right\}. \end{aligned}$$

Thus, we have $b_1\Theta(t) \leq b(t) \leq b_2\Theta(t)$. Let $a_1 = \min\{\frac{1}{2}, b_1\}$, $a_2 > 1$, $w_0(t) = a_1b(t)$, $v_0(t) = a_2b(t)$. Since A is an increasing operator, we have $Aw_0 \leq Av_0$. On one hand,

$$f(t, w_0) = f(t, a_1b(t)) \geq f(t, a_1b_1\Theta(t)) \geq (a_1b_1)^{\frac{p(t)-1}{2}} f(t, \Theta(t)).$$

For $a_1 \leq b_1$, we have $a_1 \leq (a_1 b_1)^{\frac{1}{2}}$. Therefore,

$$\begin{aligned} Aw_0(t) &= \int_0^1 G(t,s) \int_0^1 \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, w_0(s))) ds \\ &\geq (a_1 b_1)^{\frac{1}{2}} \int_0^1 G(t,s) \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, \Theta(s))) ds \\ &\geq a_1 \int_0^1 G(t,s) \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, \Theta(s))) ds \\ &= w_0(t). \end{aligned}$$

On the other hand,

$$f(t, v_0) = f(t, a_2 b(t)) \leq f(t, a_2 b_2 \Theta(t)) \leq \left(\frac{1}{a_2 b_2}\right)^{\frac{p(t)-1}{2}} f(t, \Theta(t)).$$

Hence,

$$\begin{aligned} Av_0(t) &= \int_0^1 G(t,s) \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, v_0(s))) ds \\ &\leq \left(\frac{1}{a_2 b_2}\right)^{\frac{1}{2}} \int_0^1 G(t,s) \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, \Theta(s))) ds \\ &\leq a_2 \int_0^1 G(t,s) \varphi_{p(s)}^{-1}(I_{0^+}^\beta f(s, \Theta(s))) ds \\ &= v_0(t). \end{aligned}$$

Thus, we have $w_0 \leq Aw_0 \leq Av_0 \leq v_0$. In view of Lemma 2.2, BVP (1.3) has a unique positive solution. Moreover, for any $u_0 \in [w_0, v_0]$, letting $u_n = Au_{n-1}$ ($n = 1, 2, \dots$), we have $u_n \rightarrow u^*$. □

Example 3.2 Consider the following BVP:

$$\begin{cases} D_{0^+}^{\frac{4}{5}} \varphi_{4t^2+9}(D_{0^+}^{\frac{12}{5}} x(t)) + x^2(t) \sin t = 0, & t \in [0, 1], \\ x'(0) = x(1) = x''(0) = 0, & D_{0^+}^{\frac{12}{5}} x(0) = 0, \end{cases} \tag{3.2}$$

where $\alpha = \frac{12}{5}, \beta = \frac{4}{5}, p(t) = 4t^2 + 9, r = \frac{1}{2}, f(t, x(t)) = x^2(t) \sin t$. It is easy to verify that (I_1) is satisfied. Moreover,

$$\left(\frac{1}{2}x\right)^2 \sin t = \frac{1}{4}x^2 \sin t \geq \left(\frac{1}{4}\right)^{t^2+2} x^2 \sin t, \quad (2x)^2 \sin t = 4x^2 \sin t \leq 4^{t^2+2} x^2 \sin t,$$

which yields that (I_2) is satisfied. Thus, Example 3.2 has a unique positive solution.

In order to state Theorem 3.3, let $P_L := \min_{t \in [0,1]} p(t), P_M := \max_{t \in [0,1]} p(t)$.

Theorem 3.3 Assume that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and the following conditions hold.

(I₆) There exist constants $l_1, l_2 > 0$ such that

$$|f(t, x)| \leq l_1 + l_2|x|^{r-1}, \quad 1 < r \leq P_L.$$

$$(I_7) \frac{K2^{\frac{1}{P_L-1}}}{\Gamma(\alpha)(\Gamma(\beta+1))^{\frac{1}{P_M-1}}} < 1, \quad K = \max\{l_2^{\frac{1}{P_L-1}}, l_2^{\frac{1}{P_M-1}}\}.$$

Then BVP (1.3) has a solution.

Proof In the same way as in the proof of Theorem 3.2, it is easy to prove that $A : C[0, 1] \rightarrow C[0, 1]$ is completely continuous. Denote

$$V = \{x \in X | x = \lambda Ax, \lambda \in (0, 1)\}.$$

According to Schaefer’s fixed point theorem, we just need to prove that V is bounded. For $x \in V$, we have

$$\begin{aligned} |I_{0+}^\beta f(t, x(t))| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s, x(s))| ds \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^1 (t-s)^{\beta-1} (l_1 + l_2|x(s)|^{r-1}) ds \\ &\leq \frac{1}{\Gamma(\beta+1)} (l_1 + l_2\|x\|_\infty^{r-1}). \end{aligned}$$

By the inequality $(x + y)^p \leq 2^p(x^p + y^p)$ for $x, y, p > 0$ we have

$$\begin{aligned} |x(t)| &= \lambda |Ax(t)| \\ &\leq \int_0^1 G(t, s) \varphi_{p(s)}^{-1} (I_{0+}^\beta f(s, x(s))) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left(\frac{2^{\frac{1}{p(s)-1}}}{(\Gamma(\beta+1))^{\frac{1}{p(s)-1}}} (l_1^{\frac{1}{p(s)-1}} + l_2^{\frac{1}{p(s)-1}} \|x\|_\infty^{\frac{r-1}{p(s)-1}}) \right) ds. \end{aligned}$$

Since $\frac{r-1}{p(s)-1} \in (0, 1]$, by the inequality $x^\alpha \leq x + 1$ for $x > 0$ and $\alpha \in (0, 1]$, we have

$$\|x\|_\infty \leq \frac{2^{\frac{1}{P_L-1}}}{\Gamma(\alpha)(\Gamma(\beta+1))^{\frac{1}{P_M-1}}} \int_0^1 (1-s)^{\alpha-1} (l_1^{\frac{1}{p(s)-1}} + l_2^{\frac{1}{p(s)-1}} (\|x\|_\infty + 1)) ds.$$

By (I₇) there exists a constant $\mu > 0$ such that $\|x\|_\infty \leq \mu$. Thus, the operator has a fixed point, which implies that BVP (1.3) has a solution. □

Example 3.3 Consider the following BVP:

$$\begin{cases} D_{0+}^{\frac{3}{4}} \varphi_{t^2+2} (D_{0+}^{\frac{5}{2}} x(t)) + 1 + \frac{1}{4}x^2(t) = 0, & t \in [0, 1], \\ x'(0) = x(1) = x''(0) = 0, & D_{0+}^{\frac{5}{2}} x(0) = 0, \end{cases} \tag{3.3}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{3}{4}$, $p(t) = t^2 + 2$, $r = 2$, $f(t, x(t)) = 1 + \frac{1}{4}x^2(t)$, $l_1 = 1$, $l_2 = \frac{1}{4}$. Clearly, (I_6) holds. Moreover,

$$\frac{1}{\Gamma(\frac{5}{2})(\Gamma(\frac{7}{4}))^{\frac{1}{2}}} < 1,$$

which implies that (I_7) is satisfied. Thus, BVP (3.3) has a solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally in this article and approved the final manuscript.

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