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A universal difference method for time-space fractional Black-Scholes equation

Yang Xiaozhong, Wu Lifei[*], Sun Shuzhen and Zhang Xue

[*]Correspondence:
wulf@ncepu.edu.cn
School of Mathematics and Physics,
North China Electric Power
University, Beijing, 102206, P.R. China

Abstract

The fractional Black-Scholes (B-S) equation is an important mathematical model in finance engineering, and the study of its numerical methods has very significant practical applications. This paper constructs a new kind of universal difference method to solve the time-space fractional B-S equation. The universal difference method is analyzed to be stable, convergent, and uniquely solvable. Furthermore, it is proved that with numerical experiments the universal difference method is valid and efficient for solving the time-space fractional B-S equation. At the same time, numerical experiments indicate that the time-space fractional B-S equation is more consistent with the actual financial market.

Keywords: time-space fractional Black-Scholes equation; universal difference method; stability; convergence; numerical experiments

1 Introduction

The Black-Scholes (B-S) equation is an important mathematical model in option pricing theory of finance engineering. In the financial market, the extensive application of B-S option pricing model has been driven by the rapid development of the financial derivatives market [1, 2]. However, as we know the classical B-S model was established under some strict assumptions. According to the research on the stock market, the hypothesis of the traditional B-S equation is so idealistic that it is not completely consistent with the actual stock movement. Some extensions of the B-S model are obtained by weakening these assumptions, such as the fractional B-S model [3–8], the B-S model with transactions costs [9], the jump-diffusion model [10] *etc.*

During the past few decades, many important phenomena in electromagnetics, acoustics, viscoelasticity, and material science could be well described by fractional differential equations [11–13]. This is due to the fact that a realistic model of a physical phenomenon has a dependence not only on the time instant, but also the previous time history can be successfully described by using fractional calculus [14–18].

In recent years, progress has been made in the study of the fractional B-S equation. Wyss first deduced the fractional B-S equation with a time fractional derivative to price European call option [3]. Later, Jumarie applied the fractional Taylor formula to derive the fractional B-S equation based on the classical B-S equation [4]. Jumarie promoted the previous work and gave an optimal fractional Merton's portfolio, which has wider applications in the actual financial market [5]. Cartea and del-Castillo-Negrete obtained several frac-

tional diffusion models of option prices in markets with jumps and priced barrier option using the fractional partial differential equation [6].

The time-space fractional option pricing model - the time-space fractional B-S equation [3, 5] takes the form

$$P_t^{(\alpha)}(S, t) = \left(\frac{r}{\Gamma(2-\alpha)} P - rS^\alpha P_S^{(\alpha)} \right) t^{1-\alpha} - \frac{\Gamma^3(1+\alpha)}{\Gamma(1+2\alpha)} \Gamma^2(2-\alpha)\sigma^2 S^{2\alpha} P_S^{(2\alpha)}. \tag{1}$$

Here $t > 0$, $0 < \alpha \leq 1$ and $P(S, t)$ is the European call option price at asset price S and time t, r is the risk free interest rate. σ represents the volatility of underlying asset, and α denotes the fractional order. The fractional derivatives $P_t^{(\alpha)}(S, t)$, $P_S^{(\alpha)}$, and $P_S^{(2\alpha)}$ are Riemann-Liouville time fractional derivatives.

There exists no perfect analytic solution of the time-space fractional B-S equation, so it is important to study its numerical solutions. At present, there are a few achievements on the numerical methods for solving the fractional B-S equation [19–24]. Kumar *et al.* provided analytic solution of the fractional B-S option pricing equation by homotopy perturbation method with coupling of the Laplace transform [19]. In 2014, they also presented a numerical algorithm for the time fractional B-S equation with boundary condition by homotopy perturbation method and homotopy analysis method [20]. Song and Wang employed implicit finite difference method to solve the time fractional B-S equation together with the conditions satisfied by the standard put options [22]. Yang *et al.* proposed an Implicit-Explicit and Explicit-Implicit difference scheme for the time fractional B-S equation [23]. However, up to now there is no research on the numerical methods of the time-space fractional B-S equation.

Based on the existing problems, this paper mainly studies the numerical methods of the time-space fractional B-S option pricing model in the actual financial market. We combine the call options to construct the universal difference scheme for solving the time-space fractional B-S equation. The existence and uniqueness of a numerical solution, computational stability, and convergence of the universal difference scheme are analyzed. Finally, numerical experiments demonstrate the effectiveness of the universal difference scheme for solving the time-space fractional B-S equation.

2 Universal difference scheme of time-space fractional B-S equation

2.1 Time-space fractional B-S equation

In order to get the value of a European call option, equation (1) must be integrated with boundary conditions for numerical solutions. There are three boundary conditions:

(1) $P(S, T) = \max\{S - K, 0\}$. This condition is quite clear, the profit and loss when the option expires is its price. Here, K is the exercise price.

(2) $S \to \infty$, $P(S, t) \sim S - Ke^{-r(T-t)}$. This condition means when S is sufficiently great, the option price is close to $S - Ke^{-r(T-t)}$. T is the due date of the options.

(3) $P(0, t) = 0$. This condition means when S is zero, the option price is approximant to zero.

Therefore, the European call option pricing is to solve the following equation:

$$\begin{cases} P_t^{(\alpha)} = (\frac{r}{\Gamma(2-\alpha)} P - rS^\alpha P_S^{(\alpha)}) t^{1-\alpha} - \frac{\Gamma^3(1+\alpha)}{\Gamma(1+2\alpha)} \Gamma^2(2-\alpha)\sigma^2 S^{2\alpha} P_S^{(2\alpha)}, \\ P(S, T) = \max(S - K, 0). \end{cases} \tag{2}$$

Equation (2) is an anti-variable coefficient parabolic equation.

Boundary conditions:

$$P(0, t) = 0, \qquad \lim_{S \to +\infty} P(S, t) = S - K e^{-r(T-t)}.$$

Solution region:

$$\Sigma = \{0 \le S \le \infty, 0 \le t \le T\}.$$

In order to construct difference scheme, we make the following coordinate transform [5]: $S = e^x$, $t = T - \tau$, $P(S, t) = e^{-r\tau} V(x, \tau)$. Equation (2) converses into the following parabolic equation:

$$\begin{cases} V_\tau^{(\alpha)}(x, \tau) - [\gamma(\alpha) \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \sigma^2 + r(T-\tau)^{1-\alpha}] \tau^{1-\alpha} (T-\tau)^{\alpha-1} V_x(x, \tau) \\ \quad - \gamma(\alpha) \sigma^2 \tau^{1-\alpha} (T-\tau)^{\alpha-1} V_{xx}(x, \tau) = 0, \\ V(x, 0) = \max(e^x - K, 0). \end{cases} \tag{3}$$

Here, $\gamma(\alpha) = \frac{\Gamma^3(1+\alpha) \Gamma^2(2-\alpha)}{\Gamma(1+2\alpha)}$.

The solution region converses into

$$\Sigma_0 = \{-\infty \le x < +\infty, 0 \le \tau \le T\}.$$

In the theory, the price of the underlying asset will not always appear to be zero or infinity. Therefore, we provide a small enough number M^- as the lower boundary and a large enough number M^+ as the upper boundary in actual computation.

Therefore, the solution region converses into a finite domain:

$$\Sigma_1 = \{M^- \le x < M^+, 0 \le \tau \le T\}.$$

At the same time, the boundary conditions converse into

$$V(M^+, \tau) = e^{M^+ + r\tau} - K, \qquad V(M^-, \tau) = 0.$$

2.2 Universal difference scheme

h, k are defined, respectively, as a spatial step and a time step, here $h = \frac{M^+ - M^-}{M}$, $k = \frac{T}{N}$.

M and N are positive integers.

$$\begin{cases} x_i = M^- + (i-1)h, & i = 1, 2, \ldots, M+1, \\ \tau_n = (n-1)k, & n = 1, 2, \ldots, N+1. \end{cases}$$

The approximate value of equation (3) in the point (x_i, τ_n) is defined as V_i^n.

In order to construct the universal difference scheme (θ-difference scheme), we shall introduce the classic explicit scheme and implicit scheme of equation (3).

The classic explicit scheme of equation (3):

$$\frac{\partial^\alpha V(x_i, \tau_{n+1})}{\partial \tau^\alpha} = [ab + r(T - nk + k)^{1-\alpha}](nk - k)^{1-\alpha} (T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - V_{i-1}^n}{2h}$$

$$+ a(nk - k)^{1-\alpha} (T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2}. \tag{4}$$

The classic implicit scheme of equation (3):

$$\frac{\partial^\alpha V(x_i, \tau_{n+1})}{\partial \tau^\alpha} = \left[ab + r(T - nk)^{1-\alpha} \right] (nk)^{1-\alpha} (T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2h}$$
$$+ a(nk)^{1-\alpha} (T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2}. \tag{5}$$

Here, $a = \gamma(\alpha)\sigma^2$, $b = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}$, $n = 1, 2, \ldots, N$, $i = 1, 2, \ldots, M$.

Then we assume a parameter θ ($0 \leq \theta \leq 1$), and let $(1 - \theta)$ multiply equation (4), θ multiply equation (5), then add up their results,

$$\frac{\partial^\alpha V(x_i, \tau_{n+1})}{\partial \tau^\alpha}$$
$$= (1 - \theta) \left[\left(ab + r(T - nk + k)^{1-\alpha} \right)(nk - k)^{1-\alpha}(T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - V_{i-1}^n}{2h} \right.$$
$$\left. + a(nk - k)^{1-\alpha}(T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2} \right]$$
$$+ \theta \left[\left(ab + r(T - nk)^{1-\alpha} \right)(nk)^{1-\alpha}(T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2h} \right.$$
$$\left. + a(nk)^{1-\alpha}(T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} \right].$$

The discrete scheme of time fractional derivative is as follows:

$$\frac{\partial^\alpha V(x_i, \tau_{n+1})}{\partial \tau^\alpha} = \frac{k^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n \left[V(x_i, \tau_{n+2-j}) - V(x_i, \tau_{n+1-j}) \right] \left[j^{1-\alpha} - (j-1)^{1-\alpha} \right].$$

Ignoring the errors, we can get equation (6):

$$\frac{k^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n \left[V(x_i, \tau_{n+2-j}) - V(x_i, \tau_{n+1-j}) \right] \left[j^{1-\alpha} - (j-1)^{1-\alpha} \right]$$
$$= (1 - \theta) \left[\left(ab + r(T - nk + k)^{1-\alpha} \right)(nk - k)^{1-\alpha}(T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - V_{i-1}^n}{2h} \right.$$
$$\left. + a(nk - k)^{1-\alpha}(T - nk + k)^{\alpha-1} \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2} \right]$$
$$+ \theta \left[\left(ab + r(T - nk)^{1-\alpha} \right)(nk)^{1-\alpha}(T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2h} \right.$$
$$\left. + a(nk)^{1-\alpha}(T - nk)^{\alpha-1} \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} \right]. \tag{6}$$

Equation (6) is the universal difference scheme for equation (3), it can be written as follows:

$$V_i^{n+1} - V_i^n + \sum_{j=2}^n \left[V(x_i, \tau_{n+2-j}) - V(x_i, \tau_{n+1-j}) \right] l_j$$
$$= m_1 \left[(1 - \theta)(abg_n + rq_n)\left(V_{i+1}^n - V_{i-1}^n \right) + \theta(abg_{n+1} + rq_{n+1})\left(V_{i+1}^{n+1} - V_{i-1}^{n+1} \right) \right]$$
$$+ m_2 \left[(1 - \theta)g_n\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n \right) + \theta g_{n+1}\left(V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1} \right) \right].$$

Sorting out the last equation, then we can get the following equation:

$$
\begin{aligned}
&\big[-m_1(abg_{n+1} + rq_{n+1}) - m_2 g_{n+1}\big]\theta V_{i+1}^{n+1} + (1 + 2\theta m_2 g_{n+1})V_i^{n+1} \\
&\quad + \big[m_1(abg_{n+1} + rq_{n+1}) - m_2 g_{n+1}\big]\theta V_{i-1}^{n+1} \\
&= (1-\theta)\big[m_1(abg_n + rq_n) + m_2 g_n\big]V_{i+1}^n + \big[1 - 2m_2 g_n(1-\theta)\big]V_i^n \\
&\quad + (1-\theta)\big[-m_1(abg_n + rq_n) + m_2 g_n\big]V_{i-1}^n - \big(2^{1-\alpha} - 1\big)V_i^n + \sum_{j=2}^{n-1} d_j V_i^{n+1-j} + l_n V_i^1 \\
&= (1-\theta)\big[m_1(abg_n + rq_n) + m_2 g_n\big]V_{i+1}^n + \big[-2m_2 g_n(1-\theta)\big]V_i^n \\
&\quad + (1-\theta)\big[-m_1(abg_n + rq_n) + m_2 g_n\big]V_{i-1}^n + \sum_{j=1}^{n-1} d_j V_i^{n+1-j} + l_n V_i^1.
\end{aligned}
\tag{7}
$$

The matrix form of universal difference scheme is as follows:

$$
\begin{cases}
G_1 V^{n+1} = (G_2 + Id_1)V^n + B, \\
B = \sum_{j=2}^{n-1} d_j V^{n+1-j} + l_n V^1 + C, \\
C = (a_n' V_1^n - a_{n+1} V_1^{n+1}, 0, \ldots, 0, c_n' V_{M+1}^n - c_{n+1} V_{M+1}^{n+1})',
\end{cases}
\qquad n = 1, 2, 3, \ldots, N.
\tag{8}
$$

The matrix (8) can be written as follows:

$$
\begin{pmatrix}
b_{n+1} & c_{n+1} & & & \\
a_{n+1} & b_{n+1} & c_{n+1} & & \\
& \ddots & \ddots & \ddots & \\
& & a_{n+1} & b_{n+1} & c_{n+1} \\
& & & a_{n+1} & b_{n+1}
\end{pmatrix}
\begin{pmatrix}
V_2^{n+1} \\ V_3^{n+1} \\ \vdots \\ V_{M-1}^{n+1} \\ V_M^{n+1}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
b_n' & c_n' & & & \\
a_n' & b_n' & c_n' & & \\
& \ddots & \ddots & \ddots & \\
& & a_n' & b_n' & c_n' \\
& & & a_n' & b_n'
\end{pmatrix}
\begin{pmatrix}
V_2^n \\ V_3^n \\ \vdots \\ V_{M-1}^n \\ V_M^n
\end{pmatrix}
+ \sum_{j=1}^{n-1} d_j
\begin{pmatrix}
V_2^{n+1-j} \\ V_3^{n+1-j} \\ \vdots \\ V_{M-1}^{n+1-j} \\ V_M^{n+1-j}
\end{pmatrix}
$$

$$
+ l_n
\begin{pmatrix}
V_2^1 \\ V_3^1 \\ \vdots \\ V_{M-1}^1 \\ V_M^1
\end{pmatrix}
+
\begin{pmatrix}
a_n' V_1^n - a_{n+1} V_1^{n+1} \\ 0 \\ \vdots \\ 0 \\ c_n' V_{M+1}^n - c_{n+1} V_{M+1}^{n+1}
\end{pmatrix},
$$

$$
\sum_{j=1}^{n-1} d_j
\begin{pmatrix}
V_2^{n+1-j} \\ V_3^{n+1-j} \\ \vdots \\ V_{M-1}^{n+1-j} \\ V_M^{n+1-j}
\end{pmatrix}
= d_1
\begin{pmatrix}
V_2^n \\ V_3^n \\ \vdots \\ V_{M-1}^n \\ V_M^n
\end{pmatrix}
+ d_2
\begin{pmatrix}
V_2^{n-1} \\ V_3^{n-1} \\ \vdots \\ V_{M-1}^{n-1} \\ V_M^{n-1}
\end{pmatrix}
+ d_3
\begin{pmatrix}
V_2^{n-2} \\ V_3^{n-2} \\ \vdots \\ V_{M-1}^{n-2} \\ V_M^{n-2}
\end{pmatrix}
+ \cdots + d_{n-1}
\begin{pmatrix}
V_2^2 \\ V_3^2 \\ \vdots \\ V_{M-1}^2 \\ V_M^2
\end{pmatrix}.
$$

Here, $m_1 = \Gamma(2-\alpha)k^\alpha/2h$, $m_2 = \gamma(\alpha)\sigma^2\Gamma(2-\alpha)k^\alpha/h^2$,

$$l_j = j^{1-\alpha} - (j-1)^{1-\alpha}, \qquad d_j = 2j^{1-\alpha} - (j+1)^{1-\alpha} - (j-1)^{1-\alpha}, \quad j = 1, 2, \ldots, n,$$

$$g_n = (nk-k)^{1-\alpha}(T - nk + k)^{\alpha-1}, \qquad q_n = (nk-k)^{1-\alpha},$$

$$a_{n+1} = \theta\big[m_1(abg_{n+1} + rq_{n+1}) - m_2 g_{n+1}\big],$$

$$b_{n+1} = 1 + 2\theta m_2 g_{n+1}, \qquad c_{n+1} = \theta\big[-m_1(abg_{n+1} + rq_{n+1}) - m_2 g_{n+1}\big],$$

$$a'_n = (1-\theta)\big[-m_1(abg_n + rq_n) + m_2 g_n\big],$$

$$b'_n = -2m_2 g_n(1-\theta), \qquad c'_n = (1-\theta)\big[m_1(abg_n + rq_n) + m_2 g_n\big].$$

3 The theoretical analysis of universal difference scheme for the time-space fractional B-S equation

3.1 Existence and uniqueness of the universal difference scheme solution

For G_1, there are $a_{n+1} < 0$, $c_{n+1} < 0$, $b_{n+1} > 0$ and $b_{n+1} - |a_{n+1} + c_{n+1}| = 1$, so the matrix G_1 is a diagonally dominant matrix. In other words, the coefficient matrix G_1 is an invertible matrix.

For G_2, there are $a'_n > 0$, $c'_n > 0$, $b'_n < 0$ and $|b'_n| - |a'_n + c'_n| = 0$, so the matrix $G_2 + Id_1$ is a diagonally dominant matrix. In other words, the coefficient matrix $G_2 + Id_1$ is an invertible matrix. Therefore, the universal difference scheme (6) has an unique solution.

Theorem 1 *The universal difference scheme* (6) *for the time-space fractional B-S equation is uniquely solvable.*

3.2 Stability and convergence of universal difference scheme

Lemma 1 *The following equations hold* [14]:

$$0 < d_n < \cdots < d_2 < d_1 < 1, \qquad d_j = l_j - l_{j+1}, \qquad \sum_{j=1}^{n-1} d_j = 1 - l_n, \qquad l_1 = 1.$$

Lemma 2 *Assume that $\tilde{V}_i^n$ is the approximate solution of universal difference scheme* (6), *and $\varepsilon_i^n = \tilde{V}_i^n - V_i^n$, $E^n = (\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{m-1}^n)$, then when $\frac{1}{2} \leq \theta \leq 1$ for any $1 \leq n \leq N+1$, one will set $\|E^n\|_\infty \leq \|E^1\|_\infty$; when $0 \leq \theta < \frac{1}{2}$ and $\frac{a\Gamma(2-\alpha)k^\alpha}{h^2}N^{\alpha-1} < 1$, one will set $\|E^n\|_\infty \leq \|E^1\|_\infty$.*

Proof Applying mathematical induction.

When $n = 1$, one will set

$$c_2 \varepsilon_{i+1}^2 + b_2 \varepsilon_i^2 + a_2 \varepsilon_{i-1}^2 = \varepsilon_i^1.$$

When $n > 1$, one will set

$$c_{n+1} \varepsilon_{i+1}^{n+1} + b_{n+1} \varepsilon_i^{n+1} + a_{n+1} \varepsilon_{i-1}^{n+1} = c'_n \varepsilon_{i+1}^n + b'_n \varepsilon_i^n + a'_n \varepsilon_{i-1}^n + \sum_{j=1}^{n-1} d_j \varepsilon_i^{n-j} + l_n \varepsilon_i^1.$$

Define $|\varepsilon_l^2| = \max_{2 \leq i \leq M} |\varepsilon_i^2|$, then one will set

$$\big|\varepsilon_l^2\big| \leq c_2\big|\varepsilon_{l+1}^2\big| + b_2\big|\varepsilon_l^2\big| + a_2\big|\varepsilon_{l-1}^2\big| \leq \big|c_2\varepsilon_{l+1}^2 + b_2\varepsilon_l^2 + a_2\varepsilon_{l-1}^2\big| = \big|\varepsilon_i^1\big| = \big\|E^1\big\|_\infty.$$

Assuming $n \leq s$, we will have $\|E^n\|_\infty \leq \|E^1\|_\infty$.

When $n = s + 1$, assume $|\varepsilon_l^{s+1}| = \max_{2 \le i \le M} |\varepsilon_i^{s+1}|$, then one will set

$$
\begin{aligned}
\left|\varepsilon_l^{s+1}\right| &\le c_{s+1}\left|\varepsilon_{l+1}^{s+1}\right| + b_{s+1}\left|\varepsilon_l^{s+1}\right| + a_{s+1}\left|\varepsilon_{l-1}^{s+1}\right| \\
&\le \left|c_{s+1}\varepsilon_{l+1}^{s+1} + b_{s+1}\varepsilon_l^{s+1} + a_{s+1}\varepsilon_{l-1}^{s+1}\right| \\
&= \left|c_s'\varepsilon_{i+1}^s + b_s'\varepsilon_i^s + a_s'\varepsilon_{i-1}^s + \sum_{j=1}^{s-1} d_j \varepsilon_i^{s+1-j} + l_s \varepsilon_i^1\right| \\
&\le c_s'\left\|E^s\right\|_\infty + b_s'\left\|E^s\right\|_\infty + a_s'\left\|E^s\right\|_\infty + d_1\left\|E^s\right\|_\infty + d_2\left\|E^s\right\|_\infty + \cdots \\
&\quad + d_{s-1}\left\|E^s\right\|_\infty + l_s\left\|E^1\right\|_\infty \\
&= d_1\left\|E^s\right\|_\infty + d_2\left\|E^s\right\|_\infty + \cdots + d_{s-1}\left\|E^s\right\|_\infty + l_s\left\|E^1\right\|_\infty \\
&\le (d_1 + d_2 + \cdots + d_{s-1} + l_s)\left\|E^1\right\|_\infty = \left\|E^1\right\|_\infty.
\end{aligned}
$$

Obviously, we have the conclusion $\|E^{s+1}\|_\infty \le \|E^1\|_\infty$. □

Therefore, we can obtain the following theorem.

Theorem 2 *When $\frac{1}{2} \le \theta \le 1$, the universal difference scheme* (6) *for the time-space fractional B-S equation is stable; when $0 \le \theta < \frac{1}{2}$ and the inequality $\frac{a\Gamma(2-\alpha)k^\alpha}{h^2}N^{\alpha-1} < 1$ holds, the universal difference scheme* (6) *for the time-space fractional B-S equation is stable.*

Lemma 3 *Assuming $V(x_i, \tau_n)$ is the exact solution of the differential equation on the mesh point (x_i, τ_n), and defining $e_i^n = V(x_i, \tau_n) - V_i^n$, $e^1 = 0$, $e^n = (e_1^n, e_2^n, \ldots, e_{m-1}^n)$, $\|e^n\|_\infty = \max_{1 \le i \le m-1} |e_i^n|$, here, $n = 1, 2, \ldots, N$, then when $\frac{1}{2} \le \theta \le 1$, one will set $\|e^n\|_\infty \le l_{n-1}^{-1} H(\tau^{1-\alpha} + \tau^\alpha h^2)$; when $0 \le \theta < \frac{1}{2}$ and the inequality $\frac{a\Gamma(2-\alpha)k^\alpha}{h^2}N^{\alpha-1} < 1$ holds, one will set $\|e^n\|_\infty \le l_{n-1}^{-1} H(\tau^{1-\alpha} + \tau^\alpha h^2)$. Here, H is a constant.*

Proof We will apply mathematical induction. Substitute $V_i^n = V(x_i, \tau_n) - e_i^n$ into the difference scheme.

When $n = 1$, one will set

$$
c_2 e_{i+1}^2 + b_2 e_i^2 + a_2 e_{i-1}^2 = R_i^1.
$$

When $n > 1$, one will set

$$
c_{n+1} e_{i+1}^{n+1} + b_{n+1} e_i^{n+1} + a_{n+1} e_{i-1}^{n+1} = c_n' e_{i+1}^n + b_n' e_i^n + a_n' e_{i-1}^n + \sum_{j=1}^{n-1} d_j e_i^{n-j} + R_i^{n+1}.
$$

Here, $\|R_i^{n+1}\| \le H(\tau^{1+\alpha} + \tau^\alpha h^2)$, H is a constant, $n = 1, 2, \ldots, N$.

When $n = 1$, assuming $|e_l^2| = \max_{2 \le i \le M} |e_i^2|$ the relation is as follows:

$$
\begin{aligned}
\left\|e^2\right\|_\infty = \left|e_l^2\right| &\le c_2\left|e_{l+1}^2\right| + b_2\left|e_l^2\right| + a_2\left|e_{l-1}^2\right| \\
&\le \left|c_2 e_{l+1}^2 + b_2 e_l^2 + a_2 e_{l-1}^2\right| \\
&= \left|R_l^1\right| \le H(\tau^{1+\alpha} + \tau^\alpha h^2) \\
&= l_1^{-1} H(\tau^{1+\alpha} + \tau^\alpha h^2).
\end{aligned}
$$

Assuming $k \leq s$, we will have $\|e^{k+1}\|_{\infty} \leq l_k^{-1} H(\tau^{1+\alpha} + \tau^{\alpha} h^2)$. We already have $l_j^{-1} \leq l_k^{-1}$, $j = 1, 2, \ldots, k$. Then, when $k = s + 1$ and assuming $|e_l^{s+1}| = \max_{2 \leq i \leq M} |e_i^{s+1}|$, we obtain the following result:

$$
\begin{aligned}
\|e^{s+1}\|_{\infty} = |e_l^{s+1}| &\leq c_{s+1}|e_{l+1}^{s+1}| + b_{s+1}|e_l^{s+1}| + a_{s+1}|e_{l-1}^{s+1}| \\
&\leq |c_{s+1}e_{l+1}^{s+1} + b_{s+1}e_l^{s+1} + a_{s+1}e_{l-1}^{s+1}| \\
&= \left| c_s' e_{l+1}^s + b_s' e_l^s + a_s' e_{l-1}^s + \sum_{j=1}^{s-1} d_j e_i^{s+1-j} + R_l^{s+1} \right| \\
&\leq c_s' \|e^s\|_{\infty} + b_s' \|e^s\|_{\infty} + a_s' \|e^s\|_{\infty} + d_1 \|e^s\|_{\infty} + d_2 \|e^{s-1}\|_{\infty} + \cdots \\
&\quad + d_{s-1} \|e^2\|_{\infty} + H(\tau^{1+\alpha} + \tau^{\alpha} h^2) \\
&= d_1 \|e^s\|_{\infty} + d_2 \|e^{s-1}\|_{\infty} + \cdots + d_{s-1} \|e^2\|_{\infty} + H(\tau^{1+\alpha} + \tau^{\alpha} h^2) \\
&\leq (d_1 l_{s-1}^{-1} + d_2 l_{s-2}^{-1} + \cdots + d_{s-1} l_1^{-1} + 1) H(\tau^{1+\alpha} + \tau^{\alpha} h^2) \\
&\leq \left(\sum_{j=1}^{s-1} l_s^{-1} d_j + 1 \right) H(\tau^{1+\alpha} + \tau^{\alpha} h^2) \\
&= l_s^{-1} \left(\sum_{j=1}^{s-1} d_j + l_s \right) H(\tau^{1+\alpha} + \tau^{\alpha} h^2) \\
&= l_s^{-1} H(\tau^{1+\alpha} + \tau^{\alpha} h^2).
\end{aligned}
$$

Because

$$
\lim_{n \to \infty} \frac{l_n^{-1}}{n^{\alpha}} = \lim_{n \to \infty} \frac{n^{-\alpha}}{n^{1-\alpha} - (n-1)^{1-\alpha}} = \lim_{n \to \infty} \frac{n^{-1}}{1 - (1 - \frac{1}{n})^{1-\alpha}} = \frac{1}{1-\alpha},
$$

there is a constant $c > 0$, by which we can obtain

$$
\|e^n\|_{\infty} \leq n^{\alpha} c(\tau^{1+\alpha} + \tau^{\alpha} h^2) = (n\tau)^{\alpha} c(\tau + h^2), \quad n = 1, 2, \ldots, N.
$$

We know $n\tau \leq T$, which is a limit number, then we can get the conclusion as follows:

$$
|V(x_i, \tau_n) - V_i^n| \leq \tilde{c}(\tau + h^2), \quad i = 2, 3, \ldots, M, n = 1, 2, \ldots, N.
$$

Here, $\tilde{c} = T^{\alpha} c$. $\qquad \square$

Theorem 3 *When $\frac{1}{2} \leq \theta \leq 1$, the universal difference scheme* (6) *for the time-space fractional B-S equation is convergent; when $0 \leq \theta < \frac{1}{2}$ and the inequality $\frac{a\Gamma(2-\alpha)k^{\alpha}}{h^2} N^{\alpha-1} < 1$ holds, the universal difference scheme* (6) *for the time-space fractional B-S equation is convergent, and the degree of convergence is first-order time and second-order space.*

4 Numerical examples

Using a Pentium (R) Dual Core CPU 3.00 GHz, we will experiment by utilizing the universal difference scheme in the Matlab 7.0 environment. In order to compare with the integer order B-S equation, we use the universal difference scheme to calculate the price of a European call option.

Table 1 The price of a European call option ($\alpha = 5/7$, $M = 200$, $N = 40$)

Time (T/month)	3	6	9	12	The stability
$\theta = 1$	45.0580	45.3705	45.7238	46.1680	stable
$\theta = 2/3$	45.2422	45.8177	46.5242	47.7146	stable
$\theta = 1/2$	45.3345	46.0423	46.9278	48.4993	stable
$\theta = 1/3$	45.4269	46.2676	47.3336	49.2917	unstable
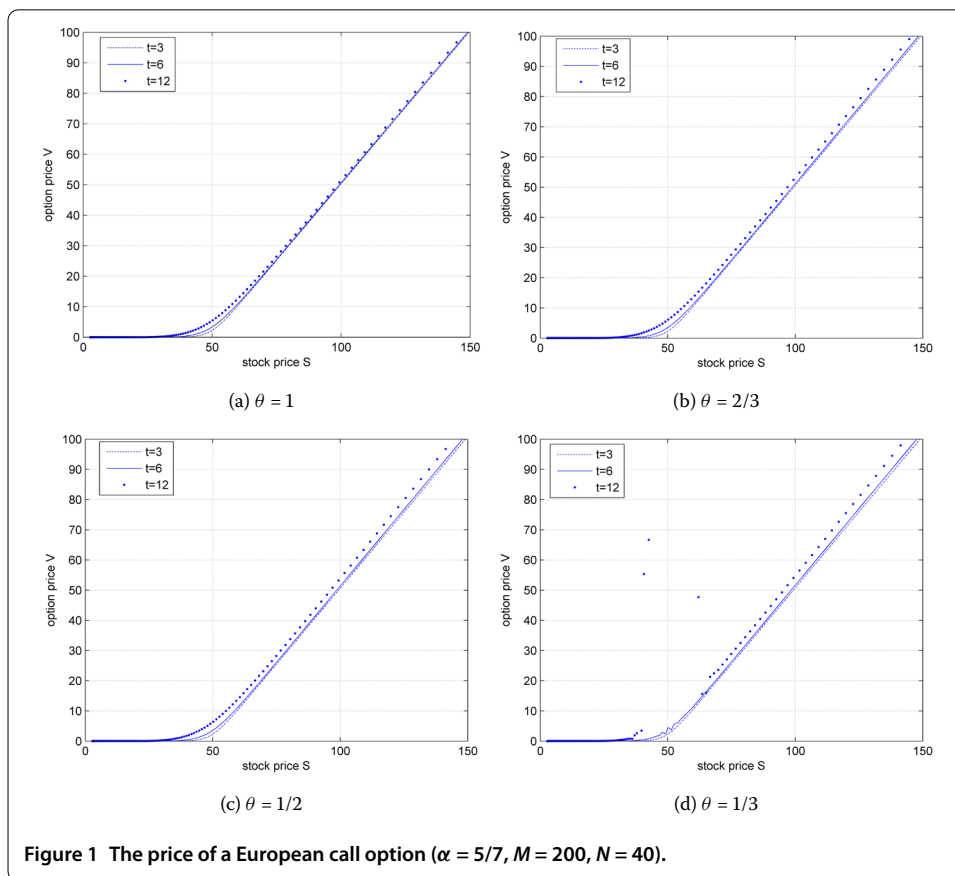
Figure 1 The price of a European call option ($\alpha = 5/7$, $M = 200$, $N = 40$).

Example Supposing a European call option, whose maturity is 3, 6, 12 months, the current price of the stock is 97\$, the strike price is 50\$, the risk free nominal interest rate is 1%, and the stock's volatility is 20%.

Solution The parameters are

$$S = 97, \quad K = 50, \quad T = 1, \quad r = 0.01, \quad \sigma = 0.2, \quad M^+ = \ln 300, \quad M^- = 1.$$

Then we take different spatial steps (see case I and case II) and temporal steps to compute the numerical solutions.
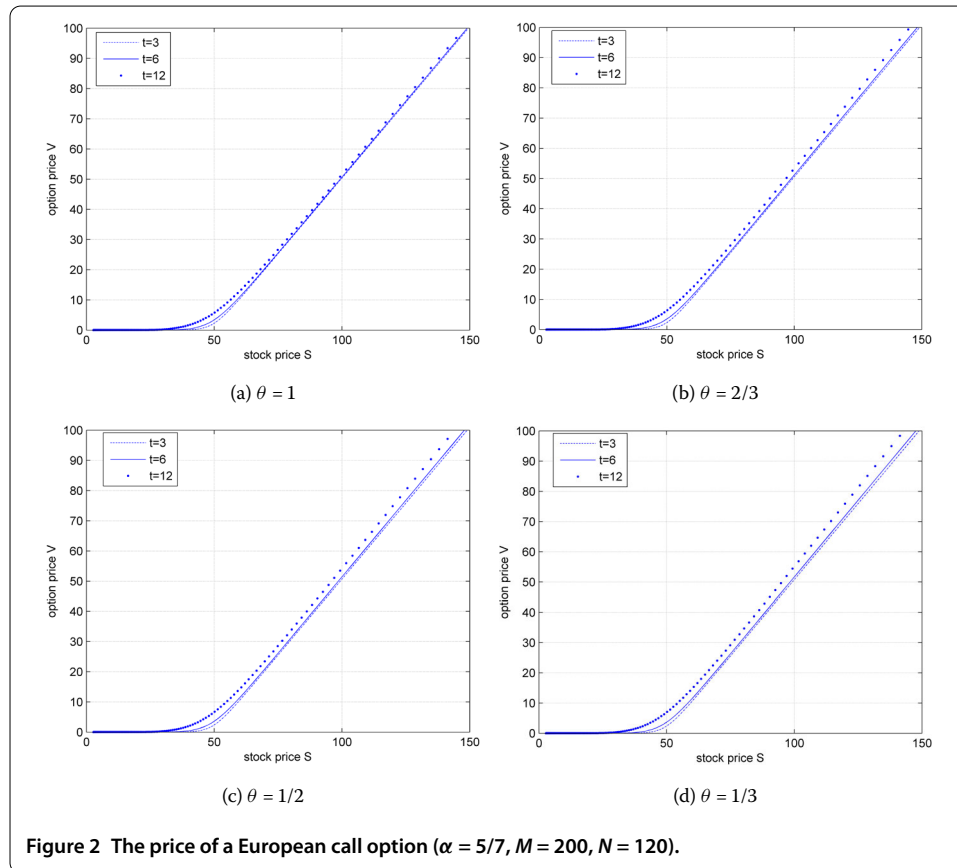
Case I:

$$\alpha = 5/7, \quad \theta = 1, 2/3, 1/2, 1/3, \quad M = 200, \quad N = 40, \quad k = 0.025, \quad h = 0.0235,$$

$$\frac{a\Gamma(2 - \alpha)k^\alpha}{h^2}N^{\alpha - 1} = 1.13 > 1.$$

We get the results in Table 1 and Figure 1.

Table 2　The price of a European call option ($\alpha = 5/7$, $M = 200$, $N = 120$)

Time (T/month)	3	6	9	12	The stability
$\theta = 1$	45.0578	45.3698	45.7219	46.2439	stable
$\theta = 2/3$	45.2641	45.8436	46.5591	47.9096	stable
$\theta = 1/2$	45.3675	46.0816	46.9812	48.7626	stable
$\theta = 1/3$	45.4709	46.3204	47.4058	49.6250	stable

Figure 2　The price of a European call option ($\alpha = 5/7$, $M = 200$, $N = 120$).

From the numerical solution in Table 1 and Figure 1, we can see that when $\theta = 1/3$, the universal difference scheme does not satisfy the stability condition, so the calculation is unstable and the numerical solutions have few references; when selecting $\theta = 1/2, 2/3, 1$, the universal difference scheme satisfies the stability condition, so the calculation is stable.

Case II:

$$\alpha = 5/7, \quad \theta = 1, 2/3, 1/2, 1/3, \quad M = 200, \quad N = 120, \quad k = 0.0083, \quad h = 0.0235,$$

$$\frac{a\Gamma(2-\alpha)k^{\alpha}}{h^2}N^{\alpha-1} = 0.37 < 1.$$

We get the results in Table 2 and Figure 2.

According to the given date in Table 2 and Figure 2 and the theoretical analysis, we can see that when selecting $\theta = 1, 2/3, 1/2, 1/3$, the universal difference scheme satisfies the stability condition, so the calculation is stable. The results above indicate that the universal difference scheme (6) for the time-space fractional B-S equation is an efficient and practical difference scheme.

Table 3 The price of a European call option ($\theta = 1$, $\alpha = 1, 2/3, 1/2, 1/3$)

Time (T/month)	3	6	9	12	CPU time (s)
$\alpha = 1$	44.9980	45.2358	45.4745	45.7157	0.45
$\alpha = 2/3$	45.1751	45.6286	46.1718	47.1348	0.43
$\alpha = 1/2$	45.4855	46.3806	47.6890	51.5890	0.45
$\alpha = 1/3$	45.6278	46.8333	49.0078	59.5237	0.46

Figure 3 The price of a European call option ($\theta = 1$, $\alpha = 1, 2/3, 1/2, 1/3$).
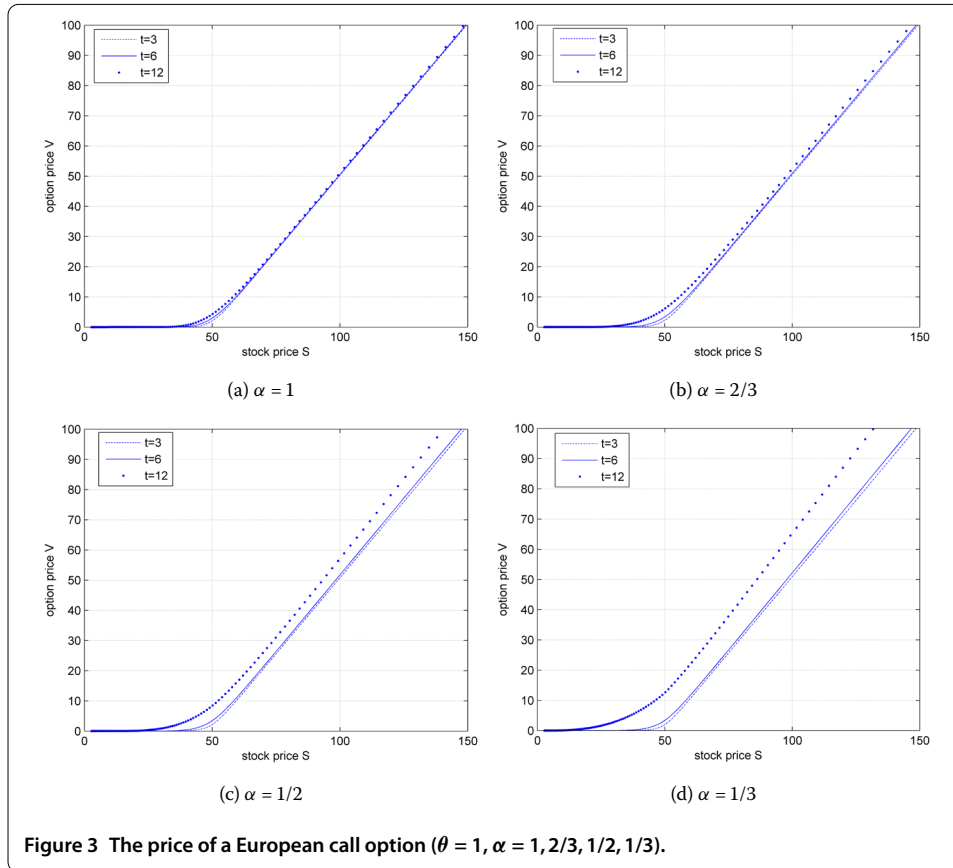
In order to further examine the effectiveness of the time-space fractional B-S option pricing modeling and the feasibility of the universal difference scheme method for solving the time-space fractional B-S equation, we will compute the numerical solutions with the condition of case III and IV. Specific plan is as follows: select an unconditional stability implicit scheme ($\theta = 1$) and the Crank-Nicolson scheme ($\theta = 1/2$), study of the effect of α value for option price, the value of α selected 2/3, 1/2, 1/3, respectively.

Case III:

$$\theta = 1, \quad \alpha = 1, 2/3, 1/2, 1/3, \quad M = 200, \quad N = 120, \quad k = 0.0083, \quad h = 0.0235.$$

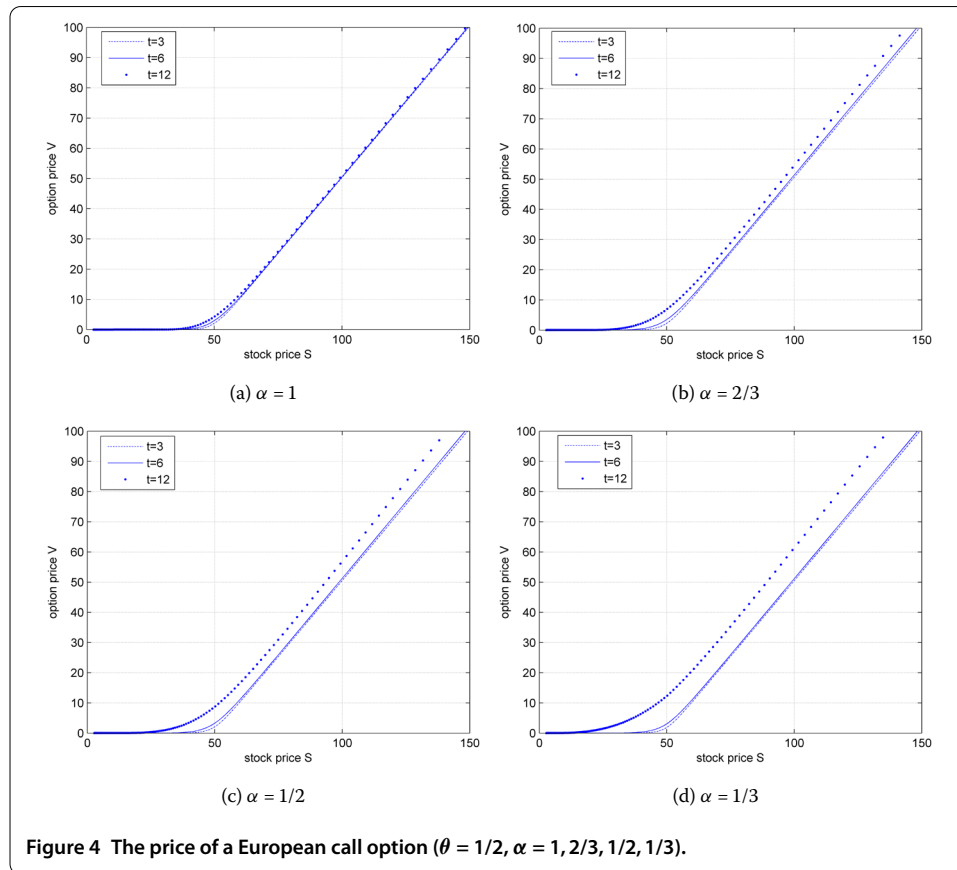We get the results in Table 3 and Figure 3.

Case IV:

$$\theta = 1/2, \quad \alpha = 1, 2/3, 1/2, 1/3, \quad M = 200, \quad N = 120, \quad k = 0.0083, \quad h = 0.0235.$$

We get the results in Table 4 and Figure 4.

Table 4 The price of a European call option ($\theta = 1/2$, $\alpha = 1, 2/3, 1/2, 1/3$)

Time (T/month)	3	6	9	12	CPU time (s)
$\alpha = 1$	44.9940	45.2318	45.4705	45.7114	0.33
$\alpha = 2/3$	45.3411	46.0406	46.9641	49.0869	0.42
$\alpha = 1/2$	45.2673	45.9415	47.0145	51.3765	0.40
$\alpha = 1/3$	45.1913	45.8369	47.1096	55.8129	0.37

(a) $\alpha = 1$　　　　　　　　　　(b) $\alpha = 2/3$

(c) $\alpha = 1/2$　　　　　　　　　　(d) $\alpha = 1/3$

Figure 4 The price of a European call option ($\theta = 1/2$, $\alpha = 1, 2/3, 1/2, 1/3$).

From Figures 3 and 4, we can see that the visible shapes and the trend of the time-space fractional B-S equation are similar to the classical European call option pricing model based on the standard B-S equation ($\alpha = 1$), which illustrates the essential characteristics of the European call options. Therefore, they illustrate that the time-space fractional B-S option pricing equation is effective and the universal difference scheme (6) is feasible for solving the time-space fractional B-S equation.

According to the numerical results in Tables 3, 4 and Figures 3 and 4, when $\frac{1}{2} \leq \alpha < 1$ for the time-space fractional B-S equation, the results of the time-space fractional B-S equation are better than the standard B-S equation. The option price of the standard B-S equation ($\alpha = 1$) is small, lower than for the actual financial market (see [24]). It affirms that the time-space fractional B-S equation is more consistent with the actual financial market.

When $0 < \alpha < \frac{1}{2}$, the influence of the option price is larger for the time-space fractional B-S equation, that is to say, the option price of 12 months is higher than the standard B-S

equation. In order to meet the actual financial market, the parameter of the time-space B-S equation should be selected properly according to the actual data.

5 Conclusions

In this work, the universal difference method is employed to solve the time-space fractional B-S equation with the boundary conditions satisfied by standard European call options. Theoretical analysis demonstrates that the universal difference method satisfies conditional stability and convergence. Numerical experiments are well in agreement with theoretical analysis. All the results illustrate that the time-space fractional B-S equation is effective and the universal difference scheme is feasible to solve the time-space fractional B-S equation.

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