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State estimation for discrete-time systems with generalized Lipschitz nonlinear dynamics

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Abstract

This paper considers the state estimation problem for a class of discrete-time systems with generalized Lipschitz nonlinear dynamics. Under the assumption that the system nonlinearities satisfy a quadratically inner-boundedness condition, we design both the full-order observer and the reduced-order observer for the discrete-time nonlinear system. Sufficient conditions ensuring the existence of full-order observers as well as reduced-order observers for such systems are established and formulated in terms of linear matrix inequality (LMI). Compared with some existing results, we remove the one-sided Lipschitz restrict and extend the classical Lipschitz observer design to a larger class of discrete-time nonlinear systems. A numerical example is included to illustrate the effectiveness of the proposed design.

Keywords: observer design; quadratically inner-boundedness; Lipschitz condition; linear matrix inequality (LMI); discrete-time nonlinear systems

1 Introduction

During the past two decades, the state estimation or observer design problem for nonlinear dynamic systems has received extensively research attention; see [1–14] and the references therein. This is partly due to the fact that knowledge of the state of a dynamic system plays a key role in many control problems. It is well known that state estimation can be used for control design, diagnosis or synchronization and unknown input recovery. However, designing a state observer for a general nonlinear system is not easy or even impossible. Many current research efforts are focused on some specialized classes of nonlinear systems. For instance, Arcak *et al.* [1, 2] developed a circle-criterion approach to design observer for sector nonlinear systems. For Lipschitz nonlinear systems, the existence conditions of the full-order as well as the reduce-order observers were established in Rajamani [3] and Zhu and Han [4], respectively. Robust observers for Lipschitz nonlinear systems subject to disturbances were proposed in [5, 6]. Nonlinear observer for neutral uncertain time-delay systems was addressed in [7]. Very recently, the classical Lipschitz nonlinear observer design has been extended to the one-sided Lipschitz case; see *e.g.* [8–14].

It should be noted that most of the above-mentioned works are concerned on continuous-time nonlinear systems. Generally, the state estimation problem for discrete-time nonlinear systems has received little attention. Moreover, in the existing literature there have been several useful observer design approaches for some specialized classes of discrete-time nonlinear systems [15–23]. For example, Ibrir [15] proposed the circle-criterion approach to discrete-time nonlinear observer design. In [16] and [17], the authors considered the observer design for discrete-time Lipschitz nonlinear systems. Motivated by the Arcak-type observer design [1, 2], Zemouche and Boutayeb [18] provided a unified observer design method for discrete-time Lipschitz systems and extended it to H_∞ synchronization and unknown input recovery. An LMI approach was proposed by Wang *et al.* [19] to design state observer for discrete-time Lipschitz descriptor systems. In [20] the authors considered an observer design for discrete-time epidemic models. A new reduced-order observer normal form for nonlinear discrete-time systems was provided in [21].

Very recently, several authors have considered the observer design for one-sided Lipschitz nonlinear systems in the discrete-time case. Both full-order and reduced-order observer designs were studied in Benallouch *et al.* [22]. In fact, they have developed an LMI-based design approach to deal with the state estimation problem of one-sided Lipschitz discrete-time systems. Zhang *et al.* [23] investigated the same problem and proposed a simple observer synthesis condition to ensure the asymptotic convergence. It should be emphasized that the systems considered by Benallouch *et al.* [22] and Zhang *et al.* [23] are actually a subset of one-sided Lipschitz nonlinear systems (see Figure 1 below). More precisely, the systems are assumed to simultaneously satisfy the one-sided Lipschitz condition and the quadratically inner-bounded condition. This assumption may lead to more conservative results and bring additional restrictions on the system model. How to reduce the conservatism in the existing results of observer design of nonlinear systems is still an open problem. This motivates our present research.

In this paper, we focus on state observer design for a general class of nonlinear discrete-time systems that satisfies the quadratically inner-bounded condition only. The main contributions of this paper are two folds. First, we remove the one-sided Lipschitz restriction and only need the assumption of quadratically inner-bounded condition. Note that the quadratically inner-bounded condition includes the classical Lipschitz condition as a special case; see *e.g.* Figure 1 below. Therefore, we extend the state observer design to a larger class of discrete-time nonlinear systems. Second, some simple stability conditions are obtained for both full-order and reduced-order observer designs. In our approach, the observer designs are formulated as an LMI feasible problem, which is easily solved by standard convex optimization algorithms. An example on the single-link flexible joint robot is given to illustrate the effectiveness of the proposed design.

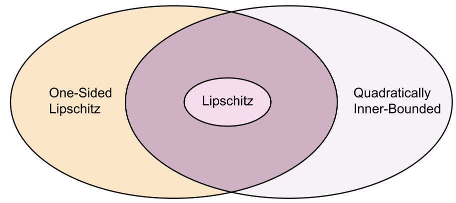
Notations: $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space. $\langle\cdot,\cdot\rangle$ represents the inner product in $\mathbb{R}^n$, *i.e.*, for given $x, y \in \mathbb{R}^n$, then $\langle x, y\rangle = x^T y$, where x^T is the transpose of the column vector $x \in \mathbb{R}^n$. $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. For a symmetric matrix P, $P > 0$ ($P < 0$) means that the matrix is positive definite (negative definite). In symmetric block matrices, we use an asterisk $*$ to represent a term induced by symmetry. I represents an identity matrix with appropriate dimension.

2 Problem statement and preliminaries

In this paper, we consider the class of discrete-time nonlinear systems described by

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + f(x(k), y(k)), \\ y(k) = Cx(k), \end{cases} \tag{1}$$

Figure 1 The Lipschitz, one-sided Lipschitz, and quadratically inner-bounded function sets [24].



where $x(k) \in \mathbb{R}^n$ is the state vector and $y(k) \in \mathbb{R}^p$ is the linear measured output. A, B and C are constant matrices of appropriate dimensions. $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a real nonlinear vector field, which is assumed to satisfy the following quadratically inner-bonded condition [22].

Assumption 1 (see *e.g.* [9, 22]) f is *quadratically inner-bounded* with respect to $x(k)$, *i.e.*, for all $x, \hat{x} \in \mathbb{R}^n$, there exist $\beta, \gamma \in \mathbb{R}$ such that

$$\left\| f\left(x, y(k)\right) - f\left(\hat{x}, y(k)\right) \right\|^2 \leq \beta \|x - \hat{x}\|^2 + \gamma \left\langle x - \hat{x}, f\left(x, y(k)\right) - f\left(\hat{x}, y(k)\right) \right\rangle. \tag{2}$$

It is clear that any Lipschitz function is also quadratically inner-bounded corresponding to $\beta > 0$ and $\gamma = 0$. Consequently, Lipschitz continuity implies quadratic inner-boundedness, but the converse is not true [8, 9]. It should be emphasized that β and γ in (2) can be any real number and are not necessarily positive. Therefore, the system considered in the paper includes the well-known Lipschitz nonlinear system as a special case (see Figure 1).

For the purpose of comparison, we introduce the following two assumptions, which are commonly used in the recent literature for observer design of nonlinear systems. For further details, we refer the interested reader to [8, 9, 22].

Assumption 2 (see *e.g.* [9]) f is *Lipschitz* with respect to $x(k)$, *i.e.*, for all $x, \hat{x} \in \mathbb{R}^n$, there exists a scalar $\lambda > 0$ such that

$$\left\| f\left(x, y(k)\right) - f\left(\hat{x}, y(k)\right) \right\| \leq \lambda \|x - \hat{x}\|. \tag{3}$$

Assumption 3 (see *e.g.* [9, 22]) f is *one-sided Lipschitz* with respect to $x(k)$, *i.e.*, for all $x, \hat{x} \in \mathbb{R}^n$, there exists a scalar $\rho \in \mathbb{R}$ such that

$$\left\langle x - \hat{x}, f\left(x, y(k)\right) - f\left(\hat{x}, y(k)\right) \right\rangle \leq \rho \|x - \hat{x}\|^2. \tag{4}$$

Notice that Assumption 2 is the well-known Lipschitz condition, while Assumption 3 is the so-called *one-sided Lipschitz* condition. It is worth mentioning that the one-sided Lipschitz condition has been frequently employed in the study of synchronization of complex networks [25, 26]. Moreover, as shown in [8] and [9], the one-sided Lipschitz condition implies the Lipschitz condition but the converse is not true. Figure 1 shows the relation between the Lipschitz, one-sided Lipschitz, and quadratically inner-bounded function sets [24].

We end this section by introducing a useful lemma.

Lemma 1 (The Schur complement lemma; see *e.g.* [27]) *For a real symmetric matrix* Σ, *the following assertions are equivalent*:

(1) $\Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} < 0$.

(2) $\Sigma_{11} < 0$, *and* $\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} < 0$.

(3) $\Sigma_{22} < 0$, *and* $\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T < 0$.

3 Full-order observer design

In this section, we consider the full-order observer design for system (1) under Assumption 1. As usual, we consider a Luenberger-like observer for system (1) in the form of

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + f(\hat{x}(k), u(k)) + L(y(k) - \hat{y}(k)), \\ \hat{y}(k) = C\hat{x}(k), \end{cases} \tag{5}$$

where $\hat{x}(k)$ denotes the estimate of the state $x(k)$. Our design goal is to find a gain matrix L such that the estimation error $e(k) := x(k) - \hat{x}(k)$ converges asymptotically toward zero. From (1) and (5), the dynamics of the estimation error is governed by

$$e(k+1) = (A - LC)e(k) + \Delta f_k, \tag{6}$$

where $\Delta f_k := f(x(k), y(k)) - f(\hat{x}(k), y(k))$.

Now, we have the following conclusion.

Theorem 1 *Suppose that system* (1) *satisfies Assumption* 1 *and the observer has the form of* (5). *Then the error dynamics is asymptotically stable if there exist matrices* $P > 0$ *and* R *with appropriate dimensions and a scalar* $\omega > 0$ *such that the following LMI is feasible*:

$$\begin{bmatrix} -P + 2\omega\beta I & A^T P - C^T R + \omega\gamma I & A^T P - C^T R \\ * & P - 2\omega I & 0 \\ * & * & -P \end{bmatrix} < 0. \tag{7}$$

The resulting observer gain matrix L *is given by* $L = P^{-1}R^T$.

Proof For the estimation error dynamics (6), let us consider the Lyapunov function candidate $V(k) = e^T(k)Pe(k)$. Then the difference of $V(k)$ along the trajectories of (6) is given by

$$\Delta V_k := V(k+1) - V(k) = e^T(k+1)Pe(k+1) - e^T(k)Pe(k). \tag{8}$$

By Assumption 1,

$$\Delta f_k^T \Delta f_k \leq \beta e^T(k)e(k) + \gamma e^T(k)\Delta f_k. \tag{9}$$

It then follows from (9) that

$$2\omega\beta e^T(k)e(k) + 2\omega\gamma e^T(k)\Delta f_k - 2\omega\Delta f_k^T \Delta f_k \geq 0, \tag{10}$$

where $\omega > 0$ is a scalar. Adding the left-hand side of (10) to ΔV_k yields

$$\begin{aligned}\Delta V_k &\le e^T(k+1)Pe(k+1) - e^T(k)Pe(k) + 2\omega\beta e^T(k)e(k) \\ &\quad + 2\omega\gamma e^T(k)\Delta f_k - 2\omega\Delta f_k^T\Delta f_k \\ &= \xi_k^T\Omega\xi_k,\end{aligned} \tag{11}$$

where $\xi_k^T = [e(k)\ \Delta f_k]^T$ and

$$\Omega = \begin{bmatrix} (A-LC)^TP(A-LC) - P + 2\omega\beta I & (A-LC)^TP + \omega\gamma I \\ P(A-LC) + \omega\gamma I & P - 2\omega I \end{bmatrix}.$$

Applying Lemma 1, $\Omega < 0$ is equivalent to

$$\Gamma = \begin{bmatrix} -P + 2\omega\beta I & (A-LC)^TP + \omega\gamma I & (A-LC)^TP \\ * & P - 2\omega I & 0 \\ * & * & -P \end{bmatrix} < 0. \tag{12}$$

By denoting $R = L^TP$, the condition (7) implies $\Gamma < 0$. Therefore, we have $\Delta V_k < 0$ for all $e(k) \neq 0$ if (7) is satisfied. This completes the proof. □

Since the quadratically inner-bounded condition include the Lipschitz condition as a special case, we immediately have Corollary 1.

Corollary 1 *Suppose that system* (1) *satisfies Assumption* 2 *and the observer has the form of* (5). *Then the error dynamics is asymptotically stable if there exist matrices* $P > 0$ *and* R *with appropriate dimensions and a scalar* $\omega > 0$ *such that the following LMI is feasible*:

$$\begin{bmatrix} -P + \omega\lambda^2 I & A^TP - C^TR & A^TP - C^TR \\ * & P - \omega I & 0 \\ * & * & -P \end{bmatrix} < 0. \tag{13}$$

The resulting observer gain matrix L *is given by* $L = P^{-1}R^T$.

4 Reduced-order observer design

In this section, we address the reduced-order observer design problem for system (1) under Assumption 1. Note that our design is inspired by the approach developed in [17] and [22], but we remove the one-sided Lipschitz restriction and provide a simple observer synthesis condition. Let $\xi(k)$ denote the reduced state vector to be estimated. Without loss of generality, assume

$$z(k) = Hx(k), \tag{14}$$

where $H \in \mathbb{R}^{(n-p)\times n}$ is a matrix so that $\left[\begin{smallmatrix} H \\ C \end{smallmatrix}\right]$ is nonsingular with

$$\begin{bmatrix} H \\ C \end{bmatrix}^{-1} = \begin{bmatrix} N & M \end{bmatrix}. \tag{15}$$

We then have

$$x(k) = Nz(k) + My(k). \tag{16}$$

From (1), (14), and (16), we obtain the following nonlinear reduced form:

$$\begin{aligned} z(k+1) &= Hx(k+1) \\ &= H\bigl(Ax(k) + Bu(k) + f\bigl(x(k), y(k)\bigr)\bigr) \\ &= A_Z z(k) + HBu(k) + Hg\bigl(z(k), y(k)\bigr) + B_Z y(k), \end{aligned} \tag{17}$$

where $A_Z := HAN$, $B_Z := HAM$, and $g(z(k), y(k)) := f(Nz(k) + My(k), y(k))$.

Inspired by [22], we design a reduced-order observer corresponding to (17) as follows:

$$\begin{cases} \hat{z}(k+1) = A_Z\hat{z}(k) + HBu(k) + B_Z y(k) + Hg(\hat{z}(k), y(k)) + K(y(k+1) - C\zeta(k)), \\ \zeta(k) = AN\hat{z}(k) + AMy(k) + g(\hat{z}(k), y(k)), \\ \hat{x}(k) = N\hat{z}(k) + My(k). \end{cases} \tag{18}$$

Denoting the estimator error by $\varepsilon(k) := z(k) - \hat{z}(k)$ and letting $C_Z := CAN$, we have

$$\begin{aligned} K\bigl(y(k+1) - C\zeta(k)\bigr) &= KC\bigl(x(k+1) - \zeta(k)\bigr) \\ &= KC\bigl(Ax(k) + f\bigl(x(k), y(k)\bigr) - \zeta(k)\bigr) \\ &= KC\bigl(ANz(k) + AMy(k) + f\bigl(Nz(k) + My(k), y(k)\bigr) - \zeta(k)\bigr) \\ &= KC_Z\varepsilon(k) + KC\Delta g_k, \end{aligned} \tag{19}$$

where $\Delta g_k := g(z(k), y(k)) - g(\hat{z}(k), y(k))$.

From (17)-(19), we know that the dynamics of the estimation error is governed by

$$\varepsilon(k+1) = (A_Z - KC_Z)\varepsilon(k) + (H - KC)\Delta g_k. \tag{20}$$

Now, we have the following theorem.

Theorem 2 *Under Assumption* 1, *the proposed reduced-order observer* (18) *is an asymptotic observer for system* (1) *if there exist matrices* $P > 0$ *and* K *of appropriate dimensions and a scalar* $\omega > 0$ *such that the following matrix inequality is feasible*:

$$\begin{bmatrix} -N^TPN + 2\omega\beta N^TN & \omega\gamma N^T & (A_Z - KC_Z)^TN^TP \\ * & -2\omega I & (H - KC)^TN^TP \\ * & * & -P \end{bmatrix} < 0. \tag{21}$$

Proof Notice that $e(k) = N\varepsilon(k)$. For the error dynamics (20), we also consider the Lyapunov function candidate $V(k) = e^T(k)Pe(k)$, *i.e.*, $V(k) = \varepsilon^T(k)N^TPN\varepsilon(k)$. Then the difference of $V(k)$ along the trajectories of (20) is given by

$$\Delta V_k := V(k+1) - V(k) = \varepsilon^T(k+1)N^TPN\varepsilon(k+1) - \varepsilon^T(k)N^TPN\varepsilon(k). \tag{22}$$

By Assumption 1, the nonlinear function $f(x(k), y(k))$ is quadratically inner-bounded, then also the function $g(z(k), y(k))$ is quadratically inner-bounded with constants β_g and γ_g. In fact, from Assumption 1, we can deduce

$$2\omega\beta e^T(k)e(k) + 2\omega\gamma e^T(k)\Delta f_k - 2\omega\Delta f_k^T\Delta f_k \geq 0, \tag{23}$$

where $\omega > 0$ is a scalar. Note that $e(k) = N\varepsilon(k)$ and $\Delta f_k = \Delta g_k$. It follows from (23) that

$$2\omega\beta\varepsilon^T(k)N^TN\varepsilon(k) + 2\omega\gamma\varepsilon^T(k)N^T\Delta g_k - 2\omega\Delta g_k^T\Delta g_k \geq 0. \tag{24}$$

Adding the left-hand side of (24) to ΔV_k yields

$$\begin{aligned}\Delta V_k &\leq \varepsilon^T(k+1)N^TPN\varepsilon(k+1) - \varepsilon^T(k)N^TPN\varepsilon(k) + 2\omega\beta\varepsilon^T(k)N^TN\varepsilon(k) \\ &\quad + 2\omega\gamma\varepsilon^T(k)N^T\Delta g_k - 2\omega\Delta g_k^T\Delta g_k \\ &= \chi_k^T\Pi\chi_k,\end{aligned} \tag{25}$$

where $\chi_k^T = [\varepsilon(k)\ \Delta g_k]^T$, and

$$\begin{aligned}\Pi &= \begin{bmatrix}(A_Z - KC_Z)^T \\ (H - KC)^T\end{bmatrix} N^TPN\begin{bmatrix}A_Z - KC_Z & H - KC\end{bmatrix} \\ &\quad + \begin{bmatrix}-N^TPN + 2\omega\beta N^TN & \omega\gamma N^T \\ & -2\omega I\end{bmatrix}.\end{aligned}$$

Note that $\Delta V_k < 0$ if $\Pi < 0$. Using Lemma 1, $\Pi < 0$ is equivalent to

$$\begin{bmatrix}-N^TPN + 2\omega\beta N^TN & \omega\gamma N^T & (A_Z - KC_Z)^TN^TP \\ * & -2\omega I & (H - KC)^TN^TP \\ * & * & -P\end{bmatrix} < 0.$$

Therefore, if the matrix inequality (21) has a feasible solution, we have $\Delta V_k < 0$ for all $\varepsilon(k) \neq 0$. By the standard Lyapunov theorem, we know that the estimation error system is asymptotically stable, which means (18) is an asymptotic reduced-order observer for system (1). This completes the proof. □

Remark 1 Compared with the full-order or the reduced-order observer design in [22], the paper removes the one-sided Lipschitz restriction, which significantly reduces the conservatism and complexity of the designs. In fact, in Theorems 1 and 2, we only assume that f satisfies the quadratically inner-bounded condition (2) and do not employ the one-sided Lipschitz condition (3).

Remark 2 It should be noted that (21) is not an LMI. To make it more tractable, we can formulate it into an LMI by letting $P = \alpha I$ for a prior given scalar $\alpha > 0$. In this case, (21) becomes

$$\begin{bmatrix}-\alpha N^TN + 2\omega\beta N^TN & \omega\gamma N^T & \alpha(A_Z - KC_Z)^TN^T \\ * & -2\omega I & \alpha(H - KC)^TN^T \\ * & * & -\alpha I\end{bmatrix} < 0. \tag{26}$$

Similarly, we have Corollary 2, since the quadratically inner-bounded condition includes the Lipschitz condition as a special case.

Corollary 2 *Under Assumption* 2, *the proposed reduced-order observer* (18) *is an asymptotic observer for system* (1) *if there exist matrices* $P > 0$ *and* K *of appropriate dimensions and a scalar* $\omega > 0$ *such that the following matrix inequality is feasible*:

$$\begin{bmatrix} -N^T PN + 2\omega\lambda^2 N^T N & 0 & (A_Z - KC_Z)^T N^T P \\ * & -2\omega I & (H - KC)^T N^T P \\ * & * & -P \end{bmatrix} < 0. \tag{27}$$

5 Illustrative example

In this section gives a numerical example to illustrate the applications of the proposed observer design. For convenience, we take the well-known single-link flexible joint robotic system as an example [3, 4]. The continuous-time model of the system is described by

$$\begin{cases} \dot{x}(t) = A_c x(t) + B_c u(t) + f_c(x(t), y(t)), \\ y(t) = C_c x(t), \end{cases} \tag{28}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \qquad C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$f_c(x, y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.333 \sin(x_1) \end{bmatrix}.$$

Let T_e be the sample time. Then by using the Euler discretized approach on system (28), we can derive the following discrete-time system model:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + f(x(k), y(k)), \\ y(k) = Cx(k), \end{cases} \tag{29}$$

where

$$A = I_4 + T_e A_c, \qquad B = T_e B_c, \qquad C = C_c, \qquad f\bigl(x(k), y(k)\bigr) = T_e f_c\bigl(x(t), y(t)\bigr).$$

It is easy to verify that $f(x(k), y(k))$ is quadratically inner-bounded with $\beta = (0.333T_e)^2$ and $\gamma = 0$. Let the sample time $T_e = 0.1[s]$. To design the full-order observer, we need to solve the LMI (7). By using the Matlab LMI tool, we get

$$P = \begin{bmatrix} 16.3522 & 0 & 0 & 0 \\ 0 & 1.4394 & -4.0400 & 1.9195 \\ 0 & -4.0400 & 14.8752 & -10.0169 \\ 0 & 1.9195 & -10.0169 & 15.9081 \end{bmatrix},$$

$$R = \begin{bmatrix} 16.3522 & -6.6211 & 17.6813 & -6.2265 \\ 1.6352 & 1.9279 & -4.0321 & 0.9475 \end{bmatrix}, \qquad \omega = 16.7916.$$

Therefore, the full-order observer gain matrix L is given by

$$L = P^{-1}R^T = \begin{bmatrix} 1.0000 & 0.1000 \\ -4.8600 & 2.4927 \\ 0 & 0.4228 \\ 0.1950 & 0.0250 \end{bmatrix}.$$

On the other hand, the reduced-order observer can be designed by using Theorem 2. With

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we have

$$N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $\alpha = 0.001$. By solving the LMI (21), we obtain the reduced-order observer gain matrix

$$K = \begin{bmatrix} 0 & 0.2057 \\ 0 & -0.0401 \end{bmatrix}.$$

6 Conclusion

We have addressed the state estimation problem for a general class of nonlinear discrete-time systems that satisfies the quadratically inner-bounded condition. The system under consideration need not satisfy the one-sided Lipschitz restriction, which is a common assumption in some recent literature on observer design for nonlinear discrete-time systems. We considered both the full-order and the reduced-order observer designs and formulated the observer synthesis condition as an LMI formulation. Finally, we used an example on the single-link flexible joint robotic system to illustrate the effectiveness of the proposed design.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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