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Degenerate q-Euler polynomials

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Abstract

Recently, some identities of degenerate Euler polynomials arising from p-adic fermionic integrals on $\mathbb{Z}_p$ were introduced in Kim and Kim (Integral Transforms Spec. Funct. 26(4):295-302, 2015). In this paper, we study degenerate q-Euler polynomials which are derived from p-adic q-integrals on $\mathbb{Z}_p$.

MSC: 11B68; 11S80

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation in $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-extension of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. For $f \in C(\mathbb{Z}_p) = \{f \mid, f$ is a $\mathbb{C}_p$-valued continuous function on $\mathbb{Z}_p\}$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [1, 2])}, \tag{1.1}$$

where $[x]_{-q} = \frac{1-(-q)^x}{1+q}$.

By (1.1), we easily get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0) \quad \big(f_1(x) = f(x+1)\big), \tag{1.2}$$

and

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l) \quad (n \in \mathbb{N}), \tag{1.3}$$

where $f_n(x) = f(x+n)$ (see [1–16]).

The ordinary fermionic p-adic integral on $\mathbb{Z}_p$ is defined as

$$\lim_{q\to 1} I_{-q}(f) = I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \quad \text{(see [2])}. \tag{1.4}$$

The degenerate Euler polynomials of order r ($\in \mathbb{N}$) are defined by the generating function to be

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(x \mid \lambda)\frac{t^n}{n!} \quad \text{(see [5, 6, 10])}, \tag{1.5}$$

where $\lambda, t \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$.

From (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathcal{E}_n^{(r)}(x \mid \lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{2}{e^t+1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!},\end{aligned} \tag{1.6}$$

where $E_n^{(r)}(x)$ are the higher-order Euler polynomials.

Thus, by (1.6), we get

$$\lim_{\lambda\to 0} \mathcal{E}_n^{(r)}(x \mid \lambda) = E_n^{(r)}(x) \quad (n \geq 0). \tag{1.7}$$

When $x = 0$, $\mathcal{E}_n^{(r)}(\lambda) = \mathcal{E}_n^{(r)}(0 \mid \lambda)$ are called the higher-order degenerate Euler numbers, while $\lim_{\lambda\to 0} \mathcal{E}_n^{(r)}(\lambda) = E_n^{(r)}$ are called the higher-order Euler numbers.

In [10], it was shown that

$$\mathcal{E}_n^{(r)}(x \mid \lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_r + x \mid \lambda)_n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \tag{1.8}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ and $n \in \mathbb{Z}_{\geq 0}$.

In this paper, we study q-extensions of the degenerate Euler polynomials and give some formulae and identities of those polynomials which are derived from the fermionic p-adic q-integrals on $\mathbb{Z}_p$.

2 Some identities of q-analogues of higher-order degenerate Euler polynomials

In this section, we assume that $\lambda, t \in \mathbb{Z}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. From (1.2), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{(x_1+\cdots+x_r+x)/\lambda} \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &\quad = \left(\frac{[2]_q}{q(1+\lambda t)^{1/\lambda}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}}.\end{aligned} \tag{2.1}$$

Now, we define a q-analogue of degenerate Euler polynomials of order r as follows:

$$\left(\frac{[2]_q}{q(1+\lambda t)^{1/\lambda}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(r)}(x\mid\lambda)\frac{t^n}{n!}. \tag{2.2}$$

Thus, by (2.2), we easily get

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{E}_{n,q}^{(r)}(x\mid\lambda)\frac{t^n}{n!}&=\lim_{\lambda\to 0}\left(\frac{[2]_q}{q(1+\lambda t)^{1/\lambda}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}\\&=\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}\\&=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

where $E_{n,q}^{(r)}(x)$ are called the higher-order q-Euler polynomials (see [15–17]). Thus, by (2.3), we get

$$\lim_{\lambda\to 0}\mathcal{E}_{n,q}^{(r)}(x\mid\lambda)=E_{n,q}^{(r)}(x)\quad(n\geq 0).$$

For $\lambda\in\mathbb{C}_p$ with $\lambda\neq 1$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x\mid\lambda)\frac{t^n}{n!}\quad\text{(see [3, 18])}. \tag{2.4}$$

By replacing λ by $-q^{-1}$, we get

$$\left(\frac{1+q^{-1}}{e^t+q^{-1}}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}\left(x\mid -q^{-1}\right)\frac{t^n}{n!}. \tag{2.5}$$

Now, we define the degenerate Frobenius-Euler polynomials of order r as follows:

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}h_n^{(r)}(x,u\mid\lambda)\frac{t^n}{n!}. \tag{2.6}$$

From (2.6), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}h_n^{(r)}(x,u\mid\lambda)\frac{t^n}{n!}&=\lim_{\lambda\to 0}\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}\\&=\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n(x\mid u)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Thus, by (2.7), we get

$$\lim_{\lambda\to 0} h_n^{(r)}(x,u \mid \lambda) = H_n(x \mid u) \quad (n \geq 0).$$

By (2.2) and (2.6), we get

$$\mathcal{E}_{n,q}^{(r)}(x \mid \lambda) = h_n^{(r)}\bigl(x, -q^{-1} \mid \lambda\bigr) \quad (n \geq 0). \tag{2.8}$$

From (2.1) and (2.2), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{x_1 + \cdots + x_r + x}{\lambda}\right)_n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \frac{\lambda^n t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(r)}(x \mid \lambda) \frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Now, we define

$$\begin{aligned}&(x \mid \lambda)_n = x(x-\lambda) \cdots \bigl(x - (n-1)\lambda\bigr) \quad (n > 0), \\ &(x \mid \lambda)_0 = 1.\end{aligned} \tag{2.10}$$

By (2.9) and (2.10), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_r \mid \lambda)_n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \mathcal{E}_{n,q}^{(r)}(x \mid \lambda) \quad (u \geq 0). \tag{2.11}$$

Therefore, by (2.6) and (2.11), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$\begin{aligned}\mathcal{E}_{n,q}^{(r)}(x \mid \lambda) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x \mid \lambda)_n \, d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= h_n^{(r)}\bigl(x, -q^{-1} \mid \lambda\bigr) \quad (n \geq 0),\end{aligned}$$

where $h_n^{(r)}(x,u \mid \lambda)$ are called the degenerate Frobenius-Euler polynomials of order r.

It is not difficult to show that

$$\begin{aligned}&(x_1 + \cdots + x_r + x \mid \lambda)_n \\ &\quad = (x_1 + \cdots + x_r + x)(x_1 + \cdots + x_r + x - \lambda) \cdots \bigl(x_1 + \cdots + x_r + x - (n-1)\lambda\bigr) \\ &\quad = \lambda^n \left(\frac{x_1 + \cdots + x_r + x}{\lambda}\right)_n \\ &\quad = \lambda^n \sum_{l=0}^{n} S_1(n,l) \left(\frac{x_1 + \cdots + x_r + x}{\lambda}\right)^l \\ &\quad = \sum_{l=0}^{n} \lambda^{n-l} S_1(n,l)(x_1 + \cdots + x_r + x)^l, \end{aligned} \tag{2.12}$$

where $S_1(n,l)$ is the Stirling number of the first kind.

We observe that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t}\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}. \tag{2.13}$$

Thus, by (2.13), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\frac{t^n}{n!}\\&\quad=\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

By comparing the coefficients on both sides of (2.14), we get

$$E_{n,q}^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r). \tag{2.15}$$

From Theorem 2.1, (2.12) and (2.15), we note that

$$\begin{aligned}h_n^{(r)}\left(x,-q^{-1}\mid\lambda\right)&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x\mid\lambda)_n\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\&=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^l\,d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\&=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)E_{l,q}^{(r)}(x)\\&=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)H_l^{(r)}\left(x\mid -q^{-1}\right).\end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.2 *For $n\geq 0$, we have*

$$h_n^{(r)}\left(x,-q^{-1}\mid\lambda\right)=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)H_l^{(r)}\left(x\mid -q^{-1}\right).$$

In particular,

$$\mathcal{E}_{n,q}^{(r)}(x\mid\lambda)=\sum_{l=0}^{n}\lambda^{n-l}S_1(n,l)E_{l,q}^{(r)}(x).$$

By replacing t by $(e^{\lambda t}-1)/\lambda$ in (2.2), we get

$$\begin{aligned}&\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}\\&\quad=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}^{(r)}(x\mid\lambda)\frac{1}{n!}\frac{1}{\lambda^n}\left(e^{\lambda t}-1\right)^n\end{aligned}$$

$$= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(r)}(x \mid \lambda) \frac{1}{\lambda^n} \sum_{m=n}^{\infty} S_2(m,n) \frac{\lambda^m}{m!} t^m$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \mathcal{E}_{n,q}^{(r)}(x \mid \lambda) \lambda^{m-n} S_2(m,n) \right) \frac{t^m}{m!}, \tag{2.17}$$

where $S_2(m,n)$ is the Stirling number of the second kind.

Thus, by (2.17), we obtain the following theorem.

Theorem 2.3 *For $m \geq 0$, we have*

$$H_m^{(r)}(x \mid -q^{-1}) = \sum_{n=0}^{m} h_n^{(r)}(x, -q^{-1} \mid \lambda) \lambda^{m-n} S_2(m,n).$$

In particular,

$$E_{m,q}^{(r)}(x) = \sum_{n=0}^{m} \mathcal{E}_{n,q}^{(r)}(x \mid \lambda) \lambda^{m-n} S_2(m,n).$$

When $r = 1$, $\mathcal{E}_{n,q}(x \mid \lambda) = \mathcal{E}_{n,q}^{(1)}(x \mid \lambda)$ are called the degenerate q-Euler polynomials. In particular, $x = 0$, $\mathcal{E}_{n,q}(\lambda) = \mathcal{E}_{n,q}(0 \mid \lambda)$ are called the degenerate q-Euler numbers. $h_n(x, u \mid \lambda) = h_n^{(1)}(x, u \mid \lambda)$ are called the degenerate Frobenius-Euler polynomials. When $x = 0$, $h_n(u \mid \lambda) = h_n(0, u \mid \lambda)$ are called the degenerate Frobenius-Euler numbers.

From (1.2), we have

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x_1 + x}{\lambda}} \, d\mu_{-q}(x_1)$$

$$= \left(\frac{[2]_q}{q(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \right) (1 + \lambda t)^{\frac{x}{\lambda}}$$

$$= \left(\frac{1 + q^{-1}}{(1 + \lambda t)^{\frac{1}{\lambda}} + q^{-1}} \right) (1 + \lambda t)^{\frac{x}{\lambda}}$$

$$= \sum_{n=0}^{\infty} h_n(x, -q^{-1} \mid \lambda) \frac{t^n}{n!}. \tag{2.18}$$

Thus, by (2.18), we get

$$h_n(x, -q^{-1} \mid \lambda)$$

$$= \int_{\mathbb{Z}_p} (x_1 + x \mid \lambda)_n \, d\mu_{-q}(x_1)$$

$$= \lambda^n \int_{\mathbb{Z}_p} \left(\frac{x_1 + x}{\lambda} \right)_n d\mu_{-q}(x_1)$$

$$= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} \int_{\mathbb{Z}_p} (x_1 + x)^l \, d\mu_{-q}(x_1)$$

$$= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} H_l(x \mid -q^{-1}) \tag{2.19}$$

and

$$\begin{aligned} h_n(-q^{-1} \mid \lambda) &= \int_{\mathbb{Z}_p} (x_1 \mid \lambda)_n \, d\mu_{-q}(x_1) \\ &= \lambda^n \int_{\mathbb{Z}_p} \left(\frac{x_1}{\lambda}\right)_n d\mu_{-q}(x_1) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} H_l(-q^{-1}). \end{aligned} \tag{2.20}$$

For $d \in \mathbb{N}$, by (1.3), we get

$$\begin{aligned} &q^d \int_{\mathbb{Z}_p} (x_1 + d \mid \lambda)_n \, d\mu_{-q}(x_1) + (-1)^{d-1} \int_{\mathbb{Z}_p} (x_1 \mid \lambda)_n \, d\mu_{-q}(x_1) \\ &\quad = [2]_q \sum_{l=0}^{d-1} (-1)^{d-1-l} q^l (l \mid \lambda)_n. \end{aligned} \tag{2.21}$$

Let $d \equiv 1 \pmod 2$. Then we have

$$[2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda)_n = q^d h_n(d, -q^{-1} \mid \lambda) + h_n(-q^{-1} \mid \lambda). \tag{2.22}$$

For $d \in \mathbb{N}$ with $d \equiv 0 \pmod 2$, we get

$$[2]_q \sum_{l=0}^{d-1} (-1)^{l-1} q^l (l \mid \lambda)_n = q^d h_n(d, -q^{-1} \mid \lambda) - h_n(-q^{-1} \mid \lambda). \tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.4 *Let $d \in \mathbb{N}$ and $n \geq 0$.*

(i) *For $d \equiv 1 \pmod 2$, we have*

$$q^d h_n(d, -q^{-1} \mid \lambda) + h_n(-q^{-1} \mid \lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda)_n.$$

(ii) *For $d \equiv 0 \pmod 2$, we have*

$$q^d h_n(d, -q^{-1} \mid \lambda) - h_n(-q^{-1} \mid \lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^{l-1} q^l (l \mid \lambda)_n.$$

Corollary 2.5 *Let $d \in \mathbb{N}$ and $n \geq 0$.*

(i) *For $d \equiv 1 \pmod 2$, we have*

$$q^d E_{n,q}(d \mid \lambda) + E_{n,q}(\lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^l q^l (l \mid \lambda)_n.$$

(ii) *For* $d \equiv 0 \pmod 2$, *we have*

$$q^d E_{n,q}(d \mid \lambda) - E_{n,q}(\lambda) = [2]_q \sum_{l=0}^{d-1} (-1)^{l-1} q^l (l \mid \lambda)_n.$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \frac{[2]_q}{[2]_{q^d}} \sum_{l=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dx)\, d\mu_{-q^d}(x), \tag{2.24}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

By (2.24), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (x_1 \mid \lambda)_n \, d\mu_{-q}(x_1) \\
&\quad = \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} (a + dx_1 \mid \lambda)_n \, d\mu_{-q^d}(x_1) \\
&\quad = \frac{[2]_q}{[2]_{q^d}} d^n \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} \left(\frac{a}{d} + x_1 \,\Big|\, \frac{\lambda}{d} \right)_n d\mu_{-q^d}(x_1) \\
&\quad = d^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} (-q)^a \mathcal{E}_{n,q^d}\left(\frac{a}{d} \,\Big|\, \frac{\lambda}{d} \right),
\end{aligned} \tag{2.25}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and $n \geq 0$.

Therefore, by (2.25), we obtain the following theorem.

Theorem 2.6 *For* $n \geq 0$, $d \in \mathbb{N}$ *with* $d \equiv 1 \pmod 2$, *we have*

$$\mathcal{E}_{n,q}(\lambda) = d^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} (-q)^a \mathcal{E}_{n,q^d}\left(\frac{a}{d} \,\Big|\, \frac{\lambda}{d} \right).$$

Moreover,

$$\mathcal{E}_{n,q}(x \mid \lambda) = d^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} (-q)^a \mathcal{E}_{n,q^d}\left(\frac{a+x}{d} \,\Big|\, \frac{\lambda}{d} \right).$$

Now, we consider the degenerate q-Euler polynomials of the second kind as follows:

$$\widehat{\mathcal{E}}_{n,q}(x \mid \lambda) = \int_{\mathbb{Z}_p} \bigl(-(x_1 + x) \mid \lambda \bigr)_n \, d\mu_{-q}(x_1) \quad (n \geq 0). \tag{2.26}$$

From (2.26), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \hat{\mathcal{E}}_{n,q}(x \mid \lambda) \frac{t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \binom{-\frac{x_1+x}{\lambda}}{n} d\mu_{-q}(x_1) t^n
\end{aligned}$$

$$= (1+\lambda t)^{-x/\lambda}\int_{\mathbb{Z}_p}(1+\lambda t)^{-x_1/\lambda}\,d\mu_{-q}(x_1)$$

$$= \frac{[2]_q}{(1+\lambda t)^{1/\lambda}+q}(1+\lambda t)^{(1-x)/\lambda}. \tag{2.27}$$

When $x=0$, $\hat{\mathcal{E}}_{n,q}(\lambda)=\hat{\mathcal{E}}_{n,q}(0\mid\lambda)$ are called the degenerate q-Euler numbers of the second kind.

By (2.26), we get

$$\begin{aligned}
&\hat{\mathcal{E}}_{n,q}(x\mid\lambda)\\
&\quad=\lambda^n\int_{\mathbb{Z}_p}\left(-\frac{x_1+x}{\lambda}\right)_n d\mu_{-q}(x)\\
&\quad=\lambda^n\sum_{l=0}^{n}S_1(n,l)\frac{(-1)^l}{\lambda^l}\int_{\mathbb{Z}_p}(x_1+x)^l\,d\mu_{-q}(x)\\
&\quad=\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}(-1)^l E_{l,q}(x).
\end{aligned} \tag{2.28}$$

Thus, from (2.28), we have

$$\begin{aligned}
&(-1)^n\hat{\mathcal{E}}_{n,q}(x\mid\lambda)\\
&\quad=\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)\lambda^{n-l}E_{l,q}(x)\\
&\quad=\sum_{l=0}^{n}\left|S_1(n,l)\right|\lambda^{n-l}E_{l,q}(x).
\end{aligned} \tag{2.29}$$

We observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}E_{n,q^{-1}}(1-x)\frac{t^n}{n!}\\
&\quad=\frac{1+q^{-1}}{q^{-1}e^t+1}e^{(1-x)t}=\frac{1+q}{qe^{-t}+1}e^{-xt}\\
&\quad=\frac{[2]_q}{qe^{-t}+1}e^{-xt}=\sum_{n=0}^{\infty}(-1)^nE_{n,q}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.30}$$

From (2.30), we have

$$E_{n,q^{-1}}(1-x)=(-1)^nE_{n,q}(x)\quad(n\geq 0). \tag{2.31}$$

By replacing t by $\frac{e^{\lambda t}-1}{\lambda}$ in (2.27), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\hat{\mathcal{E}}_{n,q}(x\mid\lambda)\frac{1}{n!}\frac{1}{\lambda^n}\left(e^{\lambda t}-1\right)^n\\
&\quad=\frac{1+q}{e^t+q}e^{(1-x)t}
\end{aligned}$$

$$= \frac{[2]_{q^{-1}}}{q^{-1}e^t + 1} e^{(1-x)t}$$
$$= \sum_{n=0}^{\infty} E_{n,q^{-1}}(1-x) \frac{t^n}{n!}. \tag{2.32}$$

On the other hand, we have

$$\sum_{m=0}^{\infty} \hat{\mathcal{E}}_{m,q}(x \mid \lambda) \frac{1}{m!} \frac{1}{\lambda^m} \left(e^{\lambda t} - 1\right)^m$$
$$= \sum_{m=0}^{\infty} \hat{\mathcal{E}}_{m,q}(x \mid \lambda) \frac{1}{\lambda^m} \sum_{n=m}^{\infty} S_2(n,m) \frac{\lambda^n t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \hat{\mathcal{E}}_{m,q}(x \mid \lambda) S_2(m,n) \lambda^{n-m} \right) \frac{t^n}{n!}. \tag{2.33}$$

From (2.32) and (2.33), we note that

$$(-1)^n E_{n,q^{-1}}(x) = \sum_{m=0}^{n} \hat{\mathcal{E}}_{m,q}(x \mid \lambda) S_2(n,m) \lambda^{n-m}. \tag{2.34}$$

Therefore, by (2.29) and (2.34), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$(-1)^n \hat{\mathcal{E}}_{n,q}(x \mid \lambda) = \sum_{l=0}^{n} \left| S_1(n,l) \right| \lambda^{n-l} E_{l,q}(x)$$

and

$$(-1)^n E_{n,q^{-1}}(x) = \sum_{l=0}^{n} S_2(n,l) \lambda^{n-l} \hat{\mathcal{E}}_{l,q}(x \mid \lambda).$$

It is easy to show that

$$\binom{x+y}{n} = \sum_{l=0}^{n} \binom{x}{l} \binom{y}{n-l} \quad (n \geq 0). \tag{2.35}$$

From (2.35), we have

$$\frac{(-1)^n \mathcal{E}_{n,q}(\lambda)}{n!}$$
$$= \frac{(-1)^n}{n!} \int_{\mathbb{Z}_p} (x_1 \mid \lambda)_n \, d\mu_{-q}(x_1)$$
$$= \lambda^n \int_{\mathbb{Z}_p} \binom{-\frac{x_1}{\lambda} + n - 1}{n} d\mu_{-q}(x_1)$$
$$= \lambda^n \sum_{l=0}^{n} \binom{n-1}{n-l} \int_{\mathbb{Z}_p} \binom{-\frac{x_1}{\lambda}}{l} d\mu_{-q}(x_1)$$

$$= \lambda^n \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{1}{\lambda^l l!} \int_{\mathbb{Z}_p} (-x_1 \mid \lambda)_l \, d\mu_{-q}(x_1)$$

$$= \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{1}{l!} \hat{\mathcal{E}}_{l,q}(\lambda) \tag{2.36}$$

and

$$\frac{(-1)^n}{n!} \hat{\mathcal{E}}_{n,q}(\lambda) = \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} \frac{1}{l!} \mathcal{E}_{l,q}(\lambda). \tag{2.37}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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