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Boundary value problems for hybrid differential equations with fractional order

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Abstract

This note is motivated by some papers treating the fractional hybrid differential equations involving Riemann-Liouville differential operators of order $0 < \alpha < 1$. An existence theorem for this equation is proved under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities which are utilized to prove the existence of extremal solutions are also established. Necessary tools are considered and the comparison principle is proved, which will be useful for further study of qualitative behavior of solutions.

Keywords: hybrid differential equation; Caputo fractional derivative; maximal and minimal solutions

1 Introduction

During the past decades, fractional differential equations have attracted many authors (see [1–8]). The differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature (for instance, to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive study during recent years.

The quadratic perturbations of nonlinear differential equations have attracted much attention. We call them fractional hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [9–13].

Dhage and Lakshmikantham [12] discussed the following first order hybrid differential equation:

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) & \text{a.e. } t \in J = [0,T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and comparison results.

Zhao *et al.* [14] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$\begin{cases} D^q[\frac{x(t)}{f(t,x(t))}] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T], \\ x(0) = 0, \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. The authors of [14] established the existence theorem for fractional hybrid differential equations and some fundamental differential inequalities, they also established the existence of extremal solutions.

Benchohra *et al.* [15] discussed the following boundary value problems for differential equations with fractional order:

$$\begin{cases} {}^cD^\alpha y(t) = f(t,y(t)) \quad \text{for each } t \in J = [0,T], 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a, b, c are real constants with $a + b \neq 0$.

From the above works, we develop the theory of boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$. An existence theorem for boundary fractional hybrid differential equations is proved under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities which are utilized to prove the existence of extremal solutions are also established. Necessary tools are considered and the comparison principle is proved, which will be useful for further study of qualitative behavior of solutions.

2 Boundary value problems for hybrid differential equations with fractional order

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $X = C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0,T]$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\},$$

and let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) the map $x \mapsto g(t,x)$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By $L^1(J;\mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)|\, ds.$$

Definition 2.1 [16] The fractional integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds,$$

where Γ is the gamma function.

Definition 2.2 [16] For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$\left({}^{\mathrm{c}}D_{a^+}^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

In this paper we consider the boundary value problems for hybrid differential equations with fractional order (BVPHDEF for short) involving Caputo differential operators of order $0 < \alpha < 1$,

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

By a solution of BVPHDEF (1) we mean a function $x \in C(J,\mathbb{R})$ such that

(i) the function $t \mapsto \frac{x}{f(t,x)}$ is continuous for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (1).

The theory of strict and nonstrict differential inequalities related to the ODEs and hybrid differential equations is available in the literature (see [2, 12]). It is known that differential inequalities are useful for proving the existence of extremal solutions of the ODEs and hybrid differential equations defined on J.

3 Existence result

In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order (1) on the closed and bounded interval $J = [0,T]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it.

We define the multiplication in X by

$$(xy)(t) = x(t)y(t) \quad \text{for } x, y \in X.$$

Clearly, $X = C(J;\mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it.

Theorem 3.1 [11] *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X, and let $A : X \to X$ and $B : X \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is completely continuous,*

(c) $x = AxBy \Longrightarrow x \in S$ *for all* $y \in S$, *and*
(d) $M\psi(r) < r$, *where* $M = \|B(S)\| = \sup \|B(x)\| : x \in S$;
then the operator equation $AxBx = x$ *has a solution in* S.

We make the following assumptions:

(H_0) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.
(H_1) There exists a constant $L > 0$ such that

$$\bigl|f(t,x) - f(t,y)\bigr| \leq L|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.
(H_2) There exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\bigl|g(t,x)\bigr| \leq h(t) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$.

Lemma 3.1 *Assume that hypothesis* (H_0) *holds and* a, b, c *are real constants with* $a + b \neq 0$. *Then, for any* $h \in L^1(J;\mathbb{R})$, *the function* $x \in C(J;\mathbb{R})$ *is a solution of the BVPHDEF*

$$\begin{cases} D^\alpha\bigl(\frac{x(t)}{f(t,x(t))}\bigr) = h(t) \quad \textit{a.e. } t \in J = [0,T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases} \tag{2}$$

if and only if x satisfies the hybrid integral equation

$$\begin{aligned} x(t) = {}& \bigl[f\bigl(t,x(t)\bigr)\bigr]\biggl(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds \\ & - \frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - c\biggr)\biggr). \end{aligned} \tag{3}$$

Proof Assume that x is a solution of problem (3). By definition, $\frac{x(t)}{f(t,x(t))}$ is continuous. Applying the Caputo fractional operator of the order α, we obtain the first equation in (2). Again, substituting $t = 0$ and $t = T$ in (3) we have

$$\begin{aligned} \frac{x(0)}{f(0,x(0))} &= \frac{-1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - c\biggr), \\ \frac{x(T)}{f(T,x(T))} &= \biggl(\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - \frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - c\biggr)\biggr), \end{aligned}$$

then

$$\begin{aligned} a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = {}& \frac{-ab}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds + \frac{ac}{a+b} \\ & + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds \\ & - \frac{b^2}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds + \frac{bc}{a+b}, \end{aligned}$$

this implies that

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c.$$

Conversely, $D^\alpha(\frac{x(t)}{f(t,x(t))}) = h(t)$, so we get $\frac{x(t)}{f(t,x(t))} = \frac{x(0)}{f(0,x(0))} + I^\alpha h(t)$.

Then

$$b\frac{x(T)}{f(T,x(T))} = b\frac{x(0)}{f(0,x(0))} + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds,$$

thus

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = (a+b)\frac{x(0)}{f(0,x(0))} + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds$$

implies that

$$\frac{x(0)}{f(0,x(0))} = \frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds\right).$$

Consequently,

$$\begin{aligned} x(t) = {}& \bigl[f\bigl(t,x(t)\bigr)\bigr]\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds \right.\\ & \left. - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - c\right)\right). \end{aligned}$$ □

Theorem 3.2 *Assume that hypotheses* (H_0)-(H_2) *hold and a, b, c are real constants with* $a+b \neq 0$. *Further, if*

$$L\left(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}\left(1+\frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) < 1, \tag{4}$$

then the hybrid fractional-order differential equation (1) *has a solution defined on J.*

Proof We define a subset S of X by

$$S = \bigl\{x \in X / \|x\| \leq N\bigr\},$$

where $N = \frac{F_0(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}(1+\frac{|b|}{|a+b|})+\frac{|c|}{|a+b|})}{1-L(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}(1+\frac{|b|}{|a+b|})+\frac{|c|}{|a+b|})}$ and $F_0 = \sup_{t\in J}|f(t,0)|$.

It is clear that S satisfies the hypothesis of Theorem 3.1. By an application of Lemma 3.1, equation (1) is equivalent to the nonlinear hybrid integral equation

$$\begin{aligned} x(t) = {}& \bigl[f\bigl(t,x(t)\bigr)\bigr]\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds \right.\\ & \left. - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds - c\right)\right), \quad t \in J. \end{aligned} \tag{5}$$

Define two operators $A : X \to X$ and $B : S \to X$ by

$$Ax(t) = f\big(t, x(t)\big), \quad t \in J \tag{6}$$

and

$$\begin{aligned} Bx(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s, x(s)\big)\, ds \\ & - \frac{1}{a+b} \left(\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g\big(s, x(s)\big)\, ds - c \right). \end{aligned} \tag{7}$$

Then the hybrid integral equation (5) is transformed into the operator equation as

$$x(t) = Ax(t)Bx(t), \quad t \in J. \tag{8}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 3.1.

Claim 1. Let $x, y \in X$, then by hypothesis (H_1),

$$\big|Ax(t) - Ay(t)\big| = \big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\big| \le L\big|x(t) - y(t)\big| \le \|x - y\|$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \le L\|x - y\|$$

for all $x, y \in X$.

Claim 2. We show that B is continuous in S.

Let (x_n) be a sequence in S converging to a point $x \in S$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s, x_n(s)\big)\, ds = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\to\infty} g\big(s, x_n(s)\big)\, ds$$

and

$$\lim_{n\to\infty} \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g\big(s, x_n(s)\big)\, ds = \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \lim_{n\to\infty} g\big(s, x_n(s)\big)\, ds.$$

Then

$$\begin{aligned} \lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s, x_n(s)\big)\, ds \right. \\ &\quad \left. - \frac{1}{a+b} \left(\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g\big(s, x_n(s)\big)\, ds - c \right) \right] \\ &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s, x_n(s)\big)\, ds \\ &\quad - \lim_{n\to\infty} \frac{1}{a+b} \left(\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g\big(s, x(s)\big)\, ds - c \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s,x(s)\big)\,ds \\
&\quad - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g\big(s,x(s)\big)\,ds - c\right) \\
&= Bx(t)
\end{aligned}
$$

for all $t \in J$. This shows that B is a continuous operator on S.

Claim 3. B is a compact operator on S.

First, we show that $B(S)$ is a uniformly bounded set in X.

Let $x \in S$. Then, by hypothesis (H_2), for all $t \in J$,

$$
\begin{aligned}
\big|Bx(t)\big| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \big|(t-s)^{\alpha-1} g\big(s,x(s)\big)\big|\,ds \\
&\quad + \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)} \int_0^T \big|(T-s)^{\alpha-1} g\big(s,x(s)\big)\big|\,ds + |c|\right) \\
&\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \big|h(s)\big|\,ds + \frac{bT^{\alpha-1}}{|a+b|\Gamma(\alpha)} \int_0^T \big|h(s)\big|\,ds + \frac{|c|}{|a+b|} \\
&\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}.
\end{aligned}
$$

Thus $\|Bx\| \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|h\|_{L^1}(1 + |\frac{b}{a+b}|) + \frac{|c|}{|a+b|}$ for all $x \in S$.

This shows that B is uniformly bounded on S.

Next, we show that $B(S)$ is an equicontinuous set on X.

We set $p(t) = \int_0^t h(s)\,ds$.

Let $t_1, t_2 \in J$, then for any $x \in S$,

$$
\begin{aligned}
\big|Bx(t_1) - Bx(t_2)\big| &= \left|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} g\big(s,x(s)\big)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g\big(s,x(s)\big)\,ds\right| \\
&\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left|\int_{t_1}^{t_2} \big|g\big(s,x(s)\big)\big|\,ds\right| \\
&\leq \frac{T^{\alpha-1}}{\Gamma(\alpha)} \big|p(t_1) - p(t_2)\big|.
\end{aligned}
$$

Since p is continuous on compact J, it is uniformly continuous. Hence

$$
\forall \varepsilon > 0, \exists \eta > 0: \quad |t_1 - t_2| < \eta \quad \Longrightarrow \quad \big|Bx(t_1) - Bx(t_2)\big| < \varepsilon
$$

for all $t_1, t_2 \in J$ and for all $x \in X$.

This shows that $B(S)$ is an equicontinuous set in X.

Then, by the Arzelá-Ascoli theorem, B is a continuous and compact operator on S.

Claim 4. The hypothesis (c) of Theorem 3.1 is satisfied.

Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then

$$
\begin{aligned}
\big|x(t)\big| &= \big|Ax(t)\big|\big|By(t)\big| \\
&\leq \big|f\big(t,x(t)\big)\big| \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\big(s,x(s)\big)\,ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g\bigl(s,x(s)\bigr)\,ds - c\right)\Bigg| \\
&\leq \bigl[\bigl|f\bigl(t,x(t)\bigr) - f(t,0)\bigr| + \bigl|f(t,0)\bigr|\bigr] \\
&\quad \times \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\int_0^T h(s)\,ds + \frac{|b|T^{\alpha-1}}{|a+b|\Gamma(\alpha)}\int_0^T h(s)\,ds + \frac{|c|}{|a+b|}\right) \\
&\leq \bigl(L\bigl|x(t)\bigr| + F_0\bigr)\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right),
\end{aligned}
$$

and so

$$
\begin{aligned}
&\bigl|x(t)\bigr| - L\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right)\bigl|x(t)\bigr| \\
&\quad \leq F_0\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right),
\end{aligned}
$$

which implies

$$
\bigl|x(t)\bigr| \leq \frac{F_0(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}(1 + \frac{|b|}{|a+b|}) + \frac{|c|}{|a+b|})}{1 - L(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}(1 + \frac{|b|}{|a+b|}) + \frac{|c|}{|a+b|})}.
$$

Taking supremum over t,

$$
\|x\| \leq \frac{F_0(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}(1 + \frac{|b|}{|a+b|}) + \frac{|c|}{|a+b|})}{1 - L(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}(1 + \frac{|b|}{|a+b|}) + \frac{|c|}{|a+b|})} = N.
$$

Then $x \in S$ and the hypothesis (c) of Theorem 3.1 is satisfied.

Finally, we have

$$
M = \bigl\|B(S)\bigr\| = \sup\bigl\{\|Bx\| : x \in S\bigr\} \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}\left(1 + \left|\frac{b}{a+b}\right|\right) + \frac{|c|}{|a+b|},
$$

and so

$$
\alpha M \leq \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\|h\|_{L^1}\left(1 + \left|\frac{b}{a+b}\right|\right) + \frac{|c|}{|a+b|}\right) < 1.
$$

Thus, all the conditions of Theorem 3.1 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, BVPHDEF (1) has a solution defined on J. This completes the proof. □

4 Fractional hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for BVPHDEF (1).

We begin with the definition of the class $C_p([0,T],\mathbb{R})$.

Definition 4.1 $m \in C_p([0,T],\mathbb{R})$ means that $m \in C([0,T],\mathbb{R})$ and $t^p m(t) \in C([0,T],\mathbb{R})$.

Lemma 4.1 [17] *Let $m \in C_p([0,T],\mathbb{R})$. Suppose that for any $t_1 \in (0,+\infty)$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$.*

Then it follows that

$$D^q m(t_1) \geq 0.$$

Theorem 4.1 *Assume that hypothesis* (H_0) *holds. Suppose that there exist functions* $y, z \in C_p([0, T], \mathbb{R})$ *such that*

$$D^\alpha\left(\frac{y(t)}{f(t, y(t))}\right) \leq g\bigl(t, y(t)\bigr) \quad \text{a.e. } t \in J \tag{9}$$

and

$$D^\alpha\left(\frac{z(t)}{f(t, z(t))}\right) \geq g\bigl(t, z(t)\bigr) \quad \text{a.e. } t \in J, \tag{10}$$

$0 < t \leq T$, *with one of the inequalities being strict. Then*

$$y^0 < z^0,$$

where $y^0 = t^{1-\alpha}y(t)|_{t=0}$ *and* $z^0 = t^{1-\alpha}z(t)|_{t=0}$ *implies*

$$y(t) < z(t)$$

for all $t \in J$.

Proof Suppose that inequality (10) holds. Assume that the claim is false. Then, since $y^0 < z^0$ and $t^{1-\alpha}y(t)$ and $t^{1-\alpha}z(t)$ are continuous functions, there exists t_1 such that $0 < t_1 \leq T$ with $y(t_1) = z(t_1)$ and $y(t) < z(t)$, $0 \leq t < t_1$.

Define

$$Y(t) = \frac{y(t)}{f(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t)}{f(t, z(t))}.$$

Then we have $Y(t_1) = Z(t_1)$, and by virtue of hypothesis (H_0), we get $Y(t) < Z(t)$ for all $0 \leq t < t_1$.

Setting $m(t) = Y(t) - Z(t)$, $0 \leq t \leq t_1$, we find that $m(t) < 0$, $0 \leq t < t_1$ and $m(t_1) = 0$ with $m \in C_p([0, T], \mathbb{R})$. Then, by Lemma 4.1, we have $D^q m(t_1) \geq 0$. By (9) and (10), we obtain

$$g\bigl(t_1, y(t_1)\bigr) \geq D^q Y(t_1) \geq D^q Z(t_1) > g\bigl(t_1, z(t_1)\bigr).$$

This is a contradiction with $y(t_1) = z(t_1)$. Thus the conclusion of the theorem holds and the proof is complete. □

Theorem 4.2 *Assume that hypothesis* (H_0) *holds and* a, b, c *are real constants with* $a + b \neq 0$. *Suppose that there exist functions* $y, z \in C_p([0, T], \mathbb{R})$ *such that*

$$D^\alpha\left(\frac{y(t)}{f(t, y(t))}\right) \leq g\bigl(t, y(t)\bigr) \quad \text{a.e. } t \in J \tag{11}$$

and

$$D^\alpha\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t)) \quad a.e.\ t \in J, \tag{12}$$

one of the inequalities being strict, and if $a > 0$, $b < 0$ *and* $y(T) < z(T)$, *then*

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))} \tag{13}$$

implies

$$y(t) < z(t) \tag{14}$$

for all $t \in J$.

Proof We have $a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))}$.

This implies $a(\frac{y(0)}{f(0,y(0))} - \frac{z(0)}{f(0,z(0))}) < b(\frac{z(T)}{f(T,z(T))} - \frac{y(T)}{f(T,y(T))})$.

Since $b < 0$ and $y(T) < z(T)$ by hypothesis (H_0), then $\frac{z(T)}{f(T,z(T))} - \frac{y(T)}{f(T,y(T))} > 0$.

This shows that $\frac{y(0)}{f(0,y(0))} - \frac{z(0)}{f(0,z(0))} < 0$ since $a > 0$, and by hypothesis (H_0) we have $y(0) < z(0)$.

Hence the application of Theorem 4.1 yields that $y(t) < z(t)$. □

Theorem 4.3 *Assume that the conditions of Theorem* 4.2 *hold with inequalities* (9) *and* (10). *Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad a.e.\ t \in J \tag{15}$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \geq x_2$. *Then*

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))}$$

implies, provided $M \leq \Gamma(1+\alpha)$,

$$y(t) < z(t)$$

for all $t \in J$.

Proof We set $\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))} = \frac{z(t)}{f(t,z(t))} + \varepsilon(1+t^\alpha)$ for small $\varepsilon > 0$ and let $Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}$ and $Z(t) = \frac{z(t)}{f(t,z(t))}$ for $t \in J$.

So that we have

$$Z_\varepsilon(t) > Z(t) \quad \Longrightarrow \quad z_\varepsilon(t) > z(t).$$

Since $g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^\alpha}(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)})$ and $D^\alpha(\frac{z(t)}{f(t,z(t))}) \geq g(t,z(t))$ for all $t \in J$, one has

$$\begin{aligned} D^\alpha Z_\varepsilon(t) &= D^\alpha Z(t) + \varepsilon D^\alpha t^\alpha \\ &\geq g(t,z(t)) + \varepsilon\Gamma(\alpha+1) \end{aligned}$$

$$\begin{aligned}
&\geq g\big(t, z_\varepsilon(t)\big) - \frac{M}{1+t^\alpha}(Z_\varepsilon - Z) + \varepsilon\Gamma(1+\alpha)\\
&\geq g\big(t, z_\varepsilon(t)\big) + \varepsilon\big(\Gamma(1+\alpha) - M\big)\\
&> g\big(t, z_\varepsilon(t)\big)
\end{aligned}$$

provided $M \leq \Gamma(1+\alpha)$.

Also, we have $z_\varepsilon(0) > z(0) \geq y(0)$. Hence, the application of Theorem 4.1 yields that $y(t) < z_\varepsilon(t)$ for all $t \in J$.

By the arbitrariness of $\varepsilon > 0$, taking the limits as $\varepsilon \to 0$, we have $y(t) \leq z(t)$ for all $t \in J$. This completes the proof. □

Remark 4.1 Let $f(t,x) = 1$ and $g(t,x) = x$. We can easily verify that f and g satisfy condition (15).

5 Existence of maximal and minimal solutions

In this section, we shall prove the existence of maximal and minimal solutions for BVPHDEF (1) on $J = [0, T]$. We need the following definition in what follows.

Definition 5.1 A solution r of BVPHDEF (1) is said to be maximal if for any other solution x to BVPHDEF (1) one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a solution ρ of BVPHDEF (1) is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any solution of BVPHDEF (1) on J.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrarily small real number $\varepsilon > 0$, consider the following boundary value problem of BVPHDEF of order $0 < \alpha < 1$:

$$\begin{cases} D^\alpha(\frac{x(t)}{f(t,x(t))}) = g(t, x(t)) + \varepsilon \quad \text{a.e. } t \in J = [0, T],\\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c + \varepsilon, \end{cases} \tag{16}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

An existence theorem for BVPHDEF (16) can be stated as follows.

Theorem 5.1 *Assume that hypotheses* (H_0)-(H_2) *hold and a, b, c are real constants with* $a + b \neq 0$. *Suppose that inequality* (4) *holds. Then, for every small number* $\varepsilon > 0$, *BVPHDEF* (16) *has a solution defined on J.*

Proof By hypothesis, since

$$L\left(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) < 1,$$

there exists $\varepsilon_0 > 0$ such that

$$L\left(\frac{T^{\alpha-1}\|h\|_{L^1} + \varepsilon\frac{T}{\alpha}}{\Gamma(\alpha)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c| + \varepsilon}{|a+b|}\right) < 1$$

for all $0 < \varepsilon \leq \varepsilon_0$. Now the rest of the proof is similar to Theorem 3.2. □

Our main existence theorem for maximal solution for BVPHDEF (1) is the following.

Theorem 5.2 *Assume that hypotheses* (H_0)-(H_2) *hold with the conditions of Theorem* 4.2 *and* a, b, c *are real constants with* $a + b \neq 0$. *Furthermore, if condition* (4) *holds, then BVPHDEF* (1) *has a maximal solution defined on* J.

Proof Let $\{\varepsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, where ε_0 is a positive real number satisfying the inequality

$$L\left(\frac{T^{\alpha-1}\|h\|_{L^1} + \varepsilon_0 \frac{T}{\alpha}}{\Gamma(\alpha)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c| + \varepsilon_0}{|a+b|}\right) < 1.$$

The number ε_0 exists in view of inequality (4). By Theorem 5.1, there exists a solution $r(t, \varepsilon_n)$ defined on J of the BVPHDEF

$$\begin{cases} D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \varepsilon_n & \text{a.e. } t \in J, \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c + \varepsilon_n. \end{cases} \tag{17}$$

Then any solution u of BVPHDEF (1) satisfies

$$D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) \leq g\big(t, u(t)\big),$$

and any solution of auxiliary problem (17) satisfies

$$D^\alpha\left(\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right) = g\big(t, r(t,\varepsilon_n)\big) + \varepsilon_n > g\big(t, r(t,\varepsilon_n)\big),$$

where $a\frac{u(0)}{f(0,u(0))} + b\frac{u(T)}{f(T,u(T))} = c \leq c + \varepsilon_n = a\frac{r(0,\varepsilon_n)}{f(0,r(0,\varepsilon_n))} + b\frac{r(T,\varepsilon_n)}{f(T,r(T,\varepsilon_n))}$. By Theorem 4.2, we infer that

$$u(t) \leq r(t, \varepsilon_n) \tag{18}$$

for all $t \in J$ and $n \in IN$.

Since

$$\begin{aligned} c + \varepsilon_2 &= a\frac{r(0,\varepsilon_2)}{f(0,r(0,\varepsilon_2))} + b\frac{r(T,\varepsilon_2)}{f(T,r(T,\varepsilon_2))} \\ &\leq a\frac{r(0,\varepsilon_1)}{f(0,r(0,\varepsilon_1))} + b\frac{r(T,\varepsilon_1)}{f(T,r(T,\varepsilon_1))} = c + \varepsilon_1, \end{aligned}$$

then by Theorem 4.2, we infer that $r(t,\varepsilon_2) \leq r(t,\varepsilon_1)$. Therefore, $r(t,\varepsilon_n)$ is a decreasing sequence of positive real numbers, and the limit

$$r(t) = \lim_{n\to\infty} r(t, \varepsilon_n) \tag{19}$$

exists. We show that the convergence in (19) is uniform on J. To finish, it is enough to prove that the sequence $r(t,\varepsilon_n)$ is equicontinuous in $C(J,R)$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ be arbitrary. Then

$$\begin{aligned}
\left|r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\right| = \Bigg| & \left[f\left(t_1,r(t_1,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right.\\
&\left.-\frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds-c-\varepsilon_n\right)\right)\\
&-\left[f\left(t_2,r(t_2,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right.\\
&\left.-\frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds-c-\varepsilon_n\right)\right)\Bigg|.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\right| &= \Bigg|\left[f\left(t_1,r(t_1,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)\right)ds\\
&\quad-\left[f\left(t_2,r(t_2,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right)\\
&\quad-\left(f\left(t_1,r(t_1,\varepsilon_n)\right)-f\left(t_2,r(t_2,\varepsilon_n)\right)\right)\\
&\quad\times\frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds-c-\varepsilon_n\right)\Bigg|\\
&\leq\Bigg|\left[f\left(t_1,r(t_1,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right)\\
&\quad-\left[f\left(t_2,r(t_2,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right)\Bigg|\\
&\quad+\Bigg|\left(f\left(t_1,r(t_1,\varepsilon_n)\right)-f\left(t_2,r(t_2,\varepsilon_n)\right)\right)\\
&\quad\times\frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds-c-\varepsilon_n\right)\Bigg|\\
&\quad+\Bigg|\left[f\left(t_2,r(t_2,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right)\\
&\quad-\left[f\left(t_2,r(t_2,\varepsilon_n)\right)\right]\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\left(g\left(s,r(s,\varepsilon_n)\right)+\varepsilon_n\right)ds\right)\Bigg|\\
&\leq\left|f\left(t_1,r(t_1,\varepsilon_n)\right)-f\left(t_2,r(t_2,\varepsilon_n)\right)\right|\\
&\quad\times\left[\frac{(\|h\|_{L_1}+\varepsilon_n)T^\alpha}{\Gamma(\alpha+1)}+\frac{|b|(\|h\|_{L_1}+\varepsilon_n)T^\alpha}{|a+b|\Gamma(\alpha+1)}+\frac{|c|+\varepsilon_n}{|a+b|}\right]\\
&\quad+F\frac{(\|h\|_{L_1}+\varepsilon_n)}{\Gamma(\alpha+1)}\left[\left|t_2^\alpha-t_1^\alpha-(t_2-t_1)^\alpha\right|+(t_2-t_1)^\alpha\right],
\end{aligned}$$

where $F=\sup_{(t,x)\in J\times[-N,N]}|f(t,x)|$.

Since f is continuous on a compact set $J\times[-N,N]$, it is uniformly continuous there. Hence,

$$\left|f\left(t_1,r(t_1,\varepsilon_n)\right)-f\left(t_2,r(t_2,\varepsilon_n)\right)\right|\to 0\quad\text{as } t_1\to t_2$$

uniformly for all $n \in N$. Therefore, from the above inequality, it follows that

$$\left|r(t_1,\varepsilon_n) - r(t_2,\varepsilon_n)\right| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly for all $n \in N$. Therefore,

$$r(t,\varepsilon_n) \to r(t) \quad \text{as } n \to \infty \text{ for all } t \in J.$$

Next, we show that the function $r(t)$ is a solution of BVPHDEF (1) defined on J. Now, since $r(t,\varepsilon_n)$ is a solution of BVPHDEF (17), we have

$$\begin{aligned} r(t,\varepsilon_n) = {} & \left[f\bigl(t, r(t,\varepsilon_n)\bigr)\right]\Biggl(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl(g\bigl(s, r(s,\varepsilon_n)\bigr) + \varepsilon_n\bigr)\,ds \\ & - \frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl(g\bigl(s, r(s,\varepsilon_n)\bigr) + \varepsilon_n\bigr)\,ds - c - \varepsilon_n\biggr)\Biggr) \end{aligned}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above equation yields

$$\begin{aligned} r(t) = {} & \left[f\bigl(t, r(t)\bigr)\right]\Biggl(\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t-s)^{\alpha-1} g\bigl(s, r(s)\bigr)\,ds \\ & - \frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g\bigl(s, r(s)\bigr)\,ds - c\biggr)\Biggr) \end{aligned}$$

for all $t \in J$. Thus, the function r is a solution of BVPHDEF (1) on J. Finally, from inequality (18) it follows that $u(t) \le r(t)$ for all $t \in J$. Hence, BVPHDEF (1) has a maximal solution on J. This completes the proof. □

6 Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to BVPHDEF (1). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to BVPHDEF (1) on $J = [0, T]$.

Theorem 6.1 *Assume that hypotheses* (H_0)-(H_2) *and condition* (4) *hold and* a, b, c *are real constants with* $a + b \neq 0$. *Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad \text{a.e. } t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \ge x_2$, *where* $M \le \Gamma(1+\alpha)$. *Furthermore, if there exists a function* $u \in C(J,\mathbb{R})$ *such that*

$$\begin{cases} D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) \le g(t,u(t)) & \text{a.e. } t \in J, \\ a\frac{u(0)}{f(0,u(0))} + b\frac{u(T)}{f(T,u(T))} \le c, \end{cases} \tag{20}$$

then

$$u(t) \le r(t) \tag{21}$$

for all $t \in J$, *where* r *is a maximal solution of BVPHDEF* (1) *on* J.

Proof Let $\varepsilon > 0$ be arbitrarily small. By Theorem 5.2, $r(t,\varepsilon)$ is a maximal solution of BVPHDEF (16) so that the limit

$$r(t) = \lim_{\varepsilon \to 0} r(t,\varepsilon) \tag{22}$$

is uniform on J and the function r is a maximal solution of BVPHDEF (1) on J. Hence, we obtain

$$\begin{cases} D^\alpha(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}) = g(t,r(t,\varepsilon)) + \varepsilon \quad \text{a.e. } t \in J, \\ a\frac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))} + b\frac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))} = c + \varepsilon. \end{cases} \tag{23}$$

From the above inequality it follows that

$$\begin{cases} D^\alpha(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}) > g(t,r(t,\varepsilon)) \quad \text{a.e. } t \in J, \\ a\frac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))} + b\frac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))} = c + \varepsilon. \end{cases} \tag{24}$$

Now we apply Theorem 4.3 to inequalities (20) and (24) and conclude that $u(t) < r(t,\varepsilon)$ for all $t \in J$. This, in view of limit (22), further implies that inequality (21) holds on J. This completes the proof. □

Theorem 6.2 *Assume that hypotheses* (H_0)-(H_2) *and condition* (4) *hold and a, b, c are real constants with $a + b \neq 0$. Suppose that there exists a real number $M > 0$ such that*

$$g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad \textit{a.e. } t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geq x_2$, where $M \leq \Gamma(1+\alpha)$. Furthermore, if there exists a function $v \in C(J,\mathbb{R})$ such that

$$\begin{cases} D^\alpha(\frac{v(t)}{f(t,v(t))}) \geq g(t,v(t)) \quad \textit{a.e. } t \in J, \\ a\frac{v(0)}{f(0,v(0))} + b\frac{v(T)}{f(T,v(T))} > c, \end{cases}$$

then

$$\rho(t) \leq v(t)$$

for all $t \in J$, where ρ is a minimal solution of BVPHDEF (1) *on J.*

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for BVPHDEF (1) on J. A result in this direction is as follows.

Theorem 6.3 *Assume that hypotheses* (H_0)-(H_2) *and condition* (4) *hold and a, b, c are real constants with $a + b \neq 0$. Suppose that there exists a real number $M > 0$ such that*

$$g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad \textit{a.e. } t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \geq x_2$, *where* $M \leq \Gamma(1+\alpha)$. *If an identically zero function is the only solution of the differential equation*

$$D^\alpha m(t) = \frac{M}{1+t^\alpha} m(t) \quad a.e.\ t \in J, \qquad am(0) + bm(T) = 0, \tag{25}$$

then BVPHDEF (1) *has a unique solution on* J.

Proof By Theorem 3.2, BVPHDEF (1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of BVPHDEF (1) existing on J with $u_1 > u_2$. Define a function $m : J \to \mathbb{R}$ by

$$m(t) = \frac{u_1(t)}{f(t, u_1(t))} - \frac{u_2(t)}{f(t, u_2(t))}.$$

In view of hypothesis (H_0), we conclude that $m(t) > 0$. Then we have

$$\begin{aligned} D^\alpha m(t) &= D^\alpha \left[\frac{u_1(t)}{f(t, u_1(t))} \right] - D^\alpha \left[\frac{u_2(t)}{f(t, u_2(t))} \right] \\ &= g\bigl(t, u_1(t)\bigr) - g\bigl(t, u_2(t)\bigr) \\ &\leq \frac{M}{1+t^\alpha} \left(\frac{u_1}{f(t, u_1(t))} - \frac{u_2(t)}{f(t, u_2(t))} \right) \\ &= \frac{M}{1+t^\alpha} m(t) \end{aligned}$$

for almost everywhere $t \in J$, and since $m(0) = \frac{u_1(0)}{f(0,u_1(0))} - \frac{u_2(0)}{f(0,u_2(0))}$ and $m(T) = \frac{u_1(T)}{f(T,u_1(T))} - \frac{u_2(T)}{f(T,u_2(T))}$ and $a\frac{u_1(0)}{f(0,u_1(0))} + b\frac{u_1(T)}{f(T,u_1(T))} = a\frac{u_2(0)}{f(0,u_2(0))} + b\frac{u_2(T)}{f(T,u_2(T))}$, we have $am(0) + bm(T) = 0$.

Now, we apply Theorem 6.1 with $f(t,x) = 1$ and $c = 0$ to get that $m(t) \leq 0$ for all $t \in J$, where an identically zero function is the only solution of the differential equation (25) $m(t) \leq 0$ is a contradiction with $m(t) > 0$. Then we can get $u_1 = u_2$. This completes the proof. □

7 Existence of extremal solutions in vector segment

Sometimes it is desirable to have knowledge of the existence of extremal positive solutions for BVPHDEF (1) on J. In this section, we shall prove the existence of maximal and minimal positive solutions for BVPHDEF (1) between the given upper and lower solutions on $J = [0, T]$. We use a hybrid fixed point theorem of Dhage [10] in ordered Banach spaces for establishing our results. We need the following preliminaries in what follows. A nonempty closed set K in a Banach algebra X is called a cone with vertex 0 if

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geq 0$,
(iii) $(-K) \cap K = 0$, where 0 is the zero element of X,
(iv) a cone K is called positive if $K \circ K \subseteq K$, where $\circ$ is a multiplication composition in X.

We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is said to be normal if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Heikkila and Lakshmikantham [18].

Lemma 7.1 [10] *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $a, b \in X$, the order interval $[a,b]$ is a set in X given by

$$[a,b] = \{x \in X : a \leq x \leq b\}.$$

Definition 7.1 A mapping $Q : [a,b] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Qx \leq Qy$ for all $x, y \in [a,b]$.

We use the following fixed point theorems of Dhage [19] for proving the existence of extremal solutions for problem (1) under certain monotonicity conditions.

Lemma 7.2 [19] *Let K be a cone in a Banach algebra X and let $a, b \in X$ be such that $a \leq b$. Suppose that $A, B : [a,b] \to K$ are two nondecreasing operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is complete,*
(c) *$AxBx \in [a,b]$ for each $x \in [a,b]$.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has the least and the greatest positive solution in $[a,b]$, whenever $\alpha M < 1$, where $M = \|B([a,b])\| = \sup\{\|Bx\| : x \in [a,b]\}$.

We equip the space $C(J,R)$ with the order relation $\leq$ with the help of cone K defined by

$$K = \{x \in C(J,R) : x(t) \geq 0, \forall t \in J\}. \tag{26}$$

It is well known that the cone K is positive and normal in $C(J,R)$. We need the following definitions in what follows.

Definition 7.2 A function $a \in C(J,R)$ is called a lower solution of BVPHDEF (1) defined on J if it satisfies (11). Similarly, a function $a \in C(J,R)$ is called an upper solution of BVPHDEF (1) defined on J if it satisfies (12). A solution to BVPHDEF (1) is a lower as well as an upper solution for BVPHDEF (1) defined on J and *vice versa*.

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R} \to \mathbb{R}^+ - 0$, $g : J \times \mathbb{R} \to \mathbb{R}^+$.
(B_1) BVPHDEF (1) has a lower solution a and an upper solution b defined on J with $a \leq b$.
(B_2) The function $x \to \frac{x}{f(t,x)}$ is increasing in the interval $[\min_{t\in J} a(t), \max_{t\in J} b(t)]$ almost everywhere for $t \in J$.
(B_3) The functions $f(t,x)$ and $g(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.
(B_4) There exists a function $k \in L^1(J,\mathbb{R})$ such that $g(t,b(t)) \leq k(t)$.

We remark that hypothesis (B_4) holds in particular if f is continuous and g is L^1-Carathéodory on $J \times \mathbb{R}$.

Theorem 7.1 *Suppose that assumptions* (H_1) *and* (B_0)-(B_4) *hold and* a, b, c *are real constants with* $a+b\neq 0$. *Furthermore, if*

$$L\biggl(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}\biggl(1+\frac{|b|}{|a+b|}\biggr)+\frac{|c|}{|a+b|}\biggr)<1 \quad \text{and} \quad \frac{b}{a+b}\leq 0, \tag{27}$$

then BVPHDEF (1) *has a minimal and a maximal positive solution defined on* J.

Proof Now, BVPHDEF (1) is equivalent to integral equation (5) defined on J. Let $X = C(J,R)$. Define two operators A and B on X by (6) and (7), respectively. Then the integral equation (5) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in the Banach algebra X. Notice that hypothesis (B_0) implies $A,B:[a,b]\to K$. Since the cone K in X is normal, $[a,b]$ is a norm-bounded set in X. Now it is shown, as in the proof of Theorem 3.2, that A is a Lipschitzian with the Lipschitz constant L and B is a completely continuous operator on $[a,b]$. Again, hypothesis (B_3) implies that A and B are nondecreasing on $[a,b]$. To see this, let $x,y\in[a,b]$ be such that $x\leq y$. Then, by hypothesis (B_3),

$$Ax(t)=f\bigl(t,x(t)\bigr)\leq f\bigl(t,y(t)\bigr)=Ay(t)$$

for all $t\in J$. Similarly, we have

$$\begin{aligned}
Bx(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds \\
&\quad -\frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds-c\biggr) \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,y(s)\bigr)\,ds \\
&\quad -\frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,y(s)\bigr)\,ds-c\biggr) \\
&= By(t)
\end{aligned}$$

for all $t\in J$. So A and B are nondecreasing operators on $[a,b]$. Lemma 7.1 and hypothesis (B_3) together imply that

$$\begin{aligned}
a(t) &\leq f\bigl(t,a(t)\bigr)\biggl[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds \\
&\quad -\frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds-c\biggr)\biggr] \\
&\leq f\bigl(t,x(t)\bigr)\biggl[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds \\
&\quad -\frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds-c\biggr)\biggr] \\
&\leq f\bigl(t,b(t)\bigr)\biggl[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds \\
&\quad -\frac{1}{a+b}\biggl(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g\bigl(s,x(s)\bigr)\,ds-c\biggr)\biggr] \\
&\leq b(t)
\end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. As a result $a(t) \leq Ax(t)Bx(t) \leq b(t)$ for all $t \in J$ and $x \in [a,b]$. Hence, $AxBx \in [a,b]$ for all $x \in [a,b]$. Again,

$$M = \|B([a,b])\| = \sup\{\|Bx\| : x \in [a,b]\} \leq L\left(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right),$$

and so $\alpha M \leq L(\frac{T^{\alpha-1}\|h\|_{L^1}}{\Gamma(\alpha)}(1 + \frac{|b|}{|a+b|}) + \frac{|c|}{|a+b|}) < 1$. Now, we apply Lemma 7.2 to the operator equation $AxBx = x$ to yield that BVPHDEF (1) has a minimal and a maximal positive solution in $[a,b]$ defined on J. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read, and approved the final manuscript.

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