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A space-time spectral method for the time fractional diffusion optimal control problems

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Abstract

In this paper, we study the Galerkin spectral approximation to an unconstrained convex distributed optimal control problem governed by the time fractional diffusion equation. We construct a suitable weak formulation, study its well-posedness, and design a Galerkin spectral method for its numerical solution. The contribution of the paper is twofold: *a priori* error estimate for the spectral approximation is derived; a conjugate gradient optimization algorithm is designed to efficiently solve the discrete optimization problem. In addition, some numerical experiments are carried out to confirm the efficiency of the proposed method. The obtained numerical results show that the convergence is exponential for smooth exact solutions.

Keywords: fractional optimal control problem; time fractional diffusion equation; spectral method; *a priori* error

1 Introduction

Optimal control problems (OCPs) can be found in many scientific and engineering applications, and it has become a very active and successful research area in recent years. Considerable work has been done in the area of OCPs governed by integral order differential equations, the literature on this field is huge, and it is impossible to give even a very brief review here. Recently, fractional differential equations (FDEs) have gained considerable importance due to their application in various sciences, such as control theory [1, 2], viscoelastic materials [3, 4], anomalous diffusion [5–7], advection and dispersion of solutes in porous or fractured media [8], *etc.* [9–12]. Therefore, the optimal control problem for the fractional-order system initiated a new research direction and has received increasing attention.

A general formulation and a solution scheme for the fractional optimal control problem (FOCP) were first proposed in [13], where the fractional variational principle and the Lagrange multiplier technique were used. Following this idea, Frederico and Torres [14, 15] formulated a Noether-type theorem in the general context and studied fractional conservation laws. Mophou [16] applied the classical control theory to a fractional diffusion equation, involving a Riemann-Liouville fractional time derivative. Dorville *et al.* [17] later extended the results of [16] to a boundary fractional optimal control. Also we refer the interested reader in FOCP to [18–24] for some recent work on the subject.

Recently, considerable efforts have been made in developing spectral methods for solving FDEs. For instance, a Legendre spectral approximation was proposed in [25–27] to solve the fractional diffusion equations. Bhrawy *et al.* [28] applied the shifted Legendre spectral collocation method to obtain the numerical solution of the space-time fractional Burger's equation. A spectral collocation scheme based upon the generalized Laguerre polynomials was investigated in [29] to obtain a numerical solution of the fractional pantograph equation with variable coefficients on a semi-infinite domain. With the help of operational matrices of fractional derivatives for orthogonal polynomials, the Jacobi tau spectral method is also utilized in [30] to solve multi-term space-time fractional partial differential equations. On the other hand, there exist also limited but very promising efforts in developing spectral methods for solving FOCPs. In [31], a numerical direct method based on the Legendre orthonormal basis and operational matrix of Riemann-Liouville fractional integration were introduced to solve a general class of FOCP, and the convergence of the proposed method was also extensively discussed. Ye and Xu [32] proposed a Galerkin spectral method to solve the linear-quadratic FOCP associated to the time fractional diffusion equation with control constraints, and detailed error analysis was carried out. Doha *et al.* [33] derived an efficient numerical scheme based on the shifted orthonormal Jacobi polynomials to solve a general form of the FOCPs.

The main aim of this work is to derive *a priori* error estimates for spectral approximation to an unconstrained FOCP with general convex cost functional, and propose an efficient algorithm to solve the discrete control problem. As compared to the linear-quadratic FOCP considered by Ye and Xu [32], the presence of the general cost functional here leads to many additional difficulties, one of which is that the derivation of the optimality condition.

The rest of the paper is organized as follows. In the next section we formulate the optimal control problem under consideration and derive the optimality conditions. In Section 3, the space-time spectral discretization is presented. Thereafter, the main result on the error analysis for the considered optimal control problem is given in Section 4. In this section, error estimates for the error in the control, state, and adjoint variables are analyzed. In Section 5, we describe the overall algorithm and present some numerical examples to illustrate our results. Some concluding remarks are given at the end of this article.

2 Fractional optimal control problem and optimization

Let $\Lambda = (-1,1)$, $I = (0,T)$, and $\Omega = I \times \Lambda$. We consider the following optimal control problem for the state variable u and the control variable q:

$$\min_q \{g(u) + h(q)\}, \tag{2.1}$$

where g and h are given convex functionals, u is governed by the time fractional diffusion equation as follows:

$$\begin{aligned}
&{}_0\partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = f(x,t) + q(x,t), \quad \forall (x,t) \in \Omega,\\
&u(-1,t) = u(1,t) = 0, \quad \forall t \in I,\\
&u(x,0) = u_0(x), \quad \forall x \in \Lambda,
\end{aligned} \tag{2.2}$$

with ${}_0\partial_t^\alpha$ denoting the left-sided Caputo fractional derivative of order $\alpha \in (0,1)$, following [12], defined as

$$ {}_0\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \partial_\tau u(x,\tau) \frac{\mathrm{d}\tau}{(t-\tau)^\alpha}. \tag{2.3} $$

In order to define well the FOCP, we first introduce some notations that will be used to construct the weak problem of the time fractional diffusion equation (2.2). We use the symbol $\mathcal{O}$ to denote a domain which may stand for Λ, I or Ω. Let $L^2(\mathcal{O})$ be the space of measurable functions whose square is Lebesgue integrable in $\mathcal{O}$. The inner product and norm of $L^2(\mathcal{O})$ are defined by

$$ (u,v)_{\mathcal{O}} := \int_{\mathcal{O}} uv \,\mathrm{d}\mathcal{O}, \qquad \|u\|_{0,\mathcal{O}} := (u,u)_{\mathcal{O}}^{\frac{1}{2}}, \quad \forall u,v \in L^2(\mathcal{O}). $$

For a nonnegative real number s, we also use $H^s(\mathcal{O})$ and $H_0^s(\mathcal{O})$ to denote the usual Sobolev spaces, whose norms are denoted by $\|\cdot\|_{s,\mathcal{O}}$; see, *e.g.* [34].

Particularly, we will need to recall the definitions of some fractional Sobolev spaces introduced in [25]. For a bounded domain I, the space

$$ {}^lH^s(I) := \{v; \|v\|_{{}^lH^s(I)} < \infty\} $$

is endowed with the norm

$$ \|v\|_{{}^lH^s(I)} := \left(\|v\|_{0,I}^2 + |v|_{{}^lH^s(I)}^2\right)^{\frac{1}{2}}, \qquad |v|_{{}^lH^s(I)} := \left\|{}_0^R\partial_t^s v\right\|_{0,I}, $$

and the space

$$ {}^rH^s(I) := \{v; \|v\|_{{}^rH^s(I)} < \infty\} $$

is endowed with the norm

$$ \|v\|_{{}^rH^s(I)} := \left(\|v\|_{0,I}^2 + |v|_{{}^rH^s(I)}^2\right)^{\frac{1}{2}}, \qquad |v|_{{}^rH^s(I)} := \left\|{}_t^R\partial_T^s v\right\|_{0,I}, $$

where ${}_0^R\partial_t^s v$ and ${}_t^R\partial_T^s v$, respectively, denote the left and right Riemann-Liouville fractional derivative, whose definitions will be given later. It has been showed in [25] that the spaces ${}^lH^s(I)$, ${}^rH^s(I)$ and the usual Sobolev space $H^s(I)$ are equivalent for $s \neq n - \frac{1}{2}$.

For the Sobolev space X with norm $\|\cdot\|_X$, let

$$ H^s(I;X) := \{v \mid \|v(\cdot,t)\|_X \in H^s(I)\}, \quad s \geq 0, $$

endowed with the norm

$$ \|v\|_{H^s(I;X)} := \left\| \|v(\cdot,t)\|_X \right\|_{s,I} = \left\| \|v(\cdot,t)\|_X \right\|_{0,I} + \left\| {}_0^R\partial_t^\alpha \left(\|v(\cdot,t)\|_X\right) \right\|_{0,I}. $$

When X stands for $H^\mu(\Lambda)$ or $H_0^\mu(\Lambda)$, $\mu \geq 0$, the norm of the space $H^s(I;X)$ will be denoted by $\|\cdot\|_{\mu,s,\Omega}$. Hereafter, in the cases where no confusion would arise, the domain symbols I, Λ, and Ω may be dropped from the notations.

We also need some definitions regarding fractional derivatives allowing us to formulate the FOCP. The right Caputo fractional derivative [12] is given by

$$ {}_t\partial_T^\alpha u(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{u'(\tau)}{(\tau - t)^\alpha} \,\mathrm{d}\tau, \quad 0 < \alpha < 1. \tag{2.4} $$

The left Riemann-Liouville fractional derivative is defined as

$$ {}_0^R\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{u(\tau)}{(t-\tau)^\alpha} \,\mathrm{d}\tau, \quad 0 < \alpha < 1, \tag{2.5} $$

and the right Riemann-Liouville fractional derivative is given by

$$ {}_t^R\partial_T^\alpha u(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^T \frac{u(\tau)}{(\tau - t)^\alpha} \,\mathrm{d}\tau, \quad 0 < \alpha < 1. \tag{2.6} $$

The definitions of the Riemann-Liouville and the Caputo fractional derivative are linked by the following relationship:

$$ {}_0^R\partial_t^\alpha v(t) = \frac{v(0)t^{-\alpha}}{\Gamma(1-\alpha)} + {}_0\partial_t^\alpha v(t), \tag{2.7} $$

$$ {}_t^R\partial_T^\alpha v(t) = \frac{v(T)(T-t)^{-\alpha}}{\Gamma(1-\alpha)} + {}_t\partial_T^\alpha v(t). \tag{2.8} $$

We employ the space introduced in [25]

$$ B^s(\Omega) = H^s\bigl(I, L^2(\Lambda)\bigr) \cap L^2\bigl(I, H_0^1(\Lambda)\bigr) $$

equipped with the norm

$$ \|v\|_{B^s(\Omega)} = \bigl(\|v\|_{H^s(I,L^2(\Lambda))}^2 + \|v\|_{L^2(I,H_0^1(\Lambda))}^2\bigr)^{\frac{1}{2}}. $$

In this setting, the weak formulation of the state equation (2.2) reads [25]: given $q, f \in L^2(\Omega)$, find $u \in B^{\frac{\alpha}{2}}(\Omega)$, such that

$$ \mathcal{A}(u,v) = (f + q, v)_\Omega + \left(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)}, v\right)_\Omega, \quad \forall v \in B^{\frac{\alpha}{2}}(\Omega), \tag{2.9} $$

where the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$ \mathcal{A}(u,v) := \bigl({}_0^R\partial_t^{\frac{\alpha}{2}} u, {}_t^R\partial_T^{\frac{\alpha}{2}} v\bigr)_\Omega + (\partial_x u, \partial_x v)_\Omega. $$

It has been proved [25] that the problem (2.9) is well-posed.

We now define the cost functional as follows:

$$ \mathcal{J}(q,u) := g(u) + h(q), \tag{2.10} $$

then the optimal control problem (2.1)-(2.2) reads: find $(q^*, u(q^*)) \in L^2(\Omega) \times B^{\frac{\alpha}{2}}(\Omega)$, such that

$$ \mathcal{J}\bigl(q^*, u\bigl(q^*\bigr)\bigr) = \min_{(q,u)\in L^2(\Omega)\times B^{\frac{\alpha}{2}}(\Omega)} \mathcal{J}(q,u) \quad \text{subject to (2.9)}. \tag{2.11} $$

We further assume that $h(q) \to +\infty$ as $\|q\|_{0,\Omega} \to +\infty$, and the functional $g(\cdot)$ is bounded below, then the optimal control problem (2.11) admits a unique solution $(q^*, u(q^*)) \in L^2(\Omega) \times B^{\frac{\alpha}{2}}(\Omega)$.

The well-posedness of the state problem ensures the existence of a control-to-state mapping $q \mapsto u = u(q)$ defined through (2.9). By means of this mapping we introduce the reduced cost functional $J(q) := \mathcal{J}(q, u(q))$, $q \in L^2(\Omega)$. Then the optimal control problem (2.11) is equivalent to: find $q^* \in L^2(\Omega)$, such that

$$J(q^*) = \min_{q \in L^2(\Omega)} J(q). \tag{2.12}$$

The first order necessary optimality condition for (2.12) takes the form

$$J'(q^*)(\delta q) = 0, \quad \forall \delta q \in L^2(\Omega), \tag{2.13}$$

where $J'(q^*)(\delta q)$ is usually called the gradient of $J(q)$, which is defined through the Gâteaux differential of $J(q)$ at q^* along the direction δq.

Lemma 2.1 *We have*

$$J'(q)(\delta q) = (h'(q) + z(q), \delta q)_\Omega, \quad \forall \delta q \in L^2(\Omega), \tag{2.14}$$

where $z(q) = z$ *is the solution of the following adjoint state equation*:

$$\begin{aligned} &{}_t\partial_T^\alpha z(x,t) - \partial_x^2 z(x,t) = g'(u), \quad \forall (x,t) \in \Omega, \\ &z(-1,t) = z(1,t) = 0, \quad \forall t \in I, \\ &z(x,T) = 0, \quad \forall x \in \Lambda. \end{aligned} \tag{2.15}$$

Proof We first obtain by using the chain rule

$$\begin{aligned} J'(q)(\delta q) &= (g(u(q)))'(\delta q) + h'(q)(\delta q) \\ &= \int_\Omega g'(u(q))u'(q)(\delta q)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega h'(q)\delta q\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.16}$$

We now compute $u'(q)(\delta q)$. For simplicity, let δu denote the derivative of $u = u(q)$ in the direction δq, that is,

$$\delta u(x,t) := u'(q)(\delta q) = \lim_{\varepsilon \to 0} \frac{u(q + \varepsilon \delta q) - u(q)}{\varepsilon}.$$

Then it is readily seen that δu is the solution of the following problem:

$$\begin{cases} {}_0\partial_t^\alpha \delta u - \partial_x^2 \delta u = \delta q, \quad \forall (x,t) \in \Omega, \\ \delta u(-1,t) = \delta u(1,t) = 0, \quad \forall t \in I, \\ \delta u(x,0) = 0, \quad \forall x \in \Lambda. \end{cases} \tag{2.17}$$

To prove (2.14), we multiply each side of the first equation in (2.15) by δu, then integrate the resulted equation on the domain Ω to find

$$\int_\Omega g'(u)\delta u \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \big({}_t\partial_T^\alpha z - \partial_x^2 z\big)\delta u \,\mathrm{d}x\,\mathrm{d}t. \tag{2.18}$$

On one side, taking into account the boundary conditions in (2.17) and (2.15), we have

$$\int_\Omega \partial_x^2 z\delta u \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega z\partial_x^2 \delta u \,\mathrm{d}x\,\mathrm{d}t. \tag{2.19}$$

On the other side, by means of (2.7), (2.8), the terminal condition in (2.15), the initial condition in (2.17), and the fractional integration by parts demonstrated in [25], we have

$$\begin{aligned}
\int_\Omega {}_t\partial_T^\alpha z\delta u \,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \left({}_t^R\partial_T^\alpha z - \frac{z(x,T)(T-t)^{-\alpha}}{\Gamma(1-\alpha)}\right)\delta u \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega {}_t^R\partial_T^\alpha z\delta u \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega z{}_0^R\partial_t^\alpha \delta u \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega z{}_0\partial_t^\alpha \delta u \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \frac{z\delta u(x,0)}{\Gamma(1-\alpha)t^\alpha}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega z{}_0\partial_t^\alpha \delta u \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.20}$$

Finally, combining (2.17), (2.18), (2.19), and (2.20), we obtain

$$\begin{aligned}
\int_\Omega g'(u)\delta u \,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \big({}_0\partial_t^\alpha \delta u - \partial_x^2 \delta u\big)z \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega \delta q z \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.21}$$

This, together with (2.16), leads to (2.14). □

The weak form of (2.15) reads: find $z \in B^{\frac{\alpha}{2}}(\Omega)$, such that

$$\mathcal{A}(\varphi, z) = \big(g'(u), \varphi\big)_\Omega, \quad \forall \varphi \in B^{\frac{\alpha}{2}}(\Omega). \tag{2.22}$$

Following the same idea as for the problem (2.9), it can be proved that (2.22) admits a unique solution $z \in B^{\frac{\alpha}{2}}(\Omega)$ for any given $u \in B^{\frac{\alpha}{2}}(\Omega)$.

In what follows we will need the mapping $q \to u(q) \to z(q)$, where for any given q, $u(q)$ is defined by (2.9), and once $u(q)$ is known $z(q)$ is defined by (2.22).

3 Space-time spectral discretization

In this section we investigate a space-time spectral approximation to the optimization problem (2.12).

We first define the polynomial space

$$P_M^0(\Lambda) := P_M(\Lambda) \cap H_0^1(\Lambda), \qquad S_L := P_M^0(\Lambda) \otimes P_N(I),$$

where P_M denotes the space of all polynomials of degree less than or equal to M, L stands for the parameter pair (M, N).

We then define the discrete cost functional, which is an approximation to the reduced cost functional J, as follows:

$$J_L(q_L) := g(u_L) + h(q_L), \quad \forall q_L \in P_M(\Lambda) \otimes P_N(I), \tag{3.1}$$

where $u_L = u_L(q_L) \in S_L$ is the solution of the following problem:

$$\mathcal{A}(u_L, v_L) = (f + q_L, v_L)_\Omega + \left(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)}, v_L \right)_\Omega, \quad \forall v_L \in S_L. \tag{3.2}$$

We propose the following space-time spectral approximation to the optimization problem (2.12): find $q_L^* \in P_M(\Lambda) \otimes P_N(I)$ such that

$$J_L(q_L^*) = \min_{q_L \in P_M(\Lambda)\otimes P_N(I)} J_L(q_L). \tag{3.3}$$

It can be proved that the discrete optimization problem (3.3) admits a unique solution $q_L^* \in P_M(\Lambda) \otimes P_N(I)$, which fulfills the first order optimality condition:

$$J_L'(q_L^*)(\delta q) = 0, \quad \forall \delta q \in P_M(\Lambda) \otimes P_N(I), \tag{3.4}$$

where

$$J_L'(q_L)(\delta q) = \left(h'(q_L) + z_L, \delta q\right)_\Omega \tag{3.5}$$

with $z_L = z_L(q_L) \in S_L$, the solution of the discrete adjoint state equation:

$$\mathcal{A}(\varphi_L, z_L) = \left(g'(u_L), \varphi_L\right)_\Omega, \quad \forall \varphi_L \in S_L. \tag{3.6}$$

4 *A priori* error estimation

We now carry out an error analysis for the spectral approximation (3.3). To simplify the notations, we let c be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leq cB$.

We now introduce the auxiliary problem:

$$\mathcal{A}\left(u_L(q), v_L\right) = (f + q, v_L)_\Omega + \left(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)}, v_L \right)_\Omega, \quad \forall v_L \in S_L, \tag{4.1}$$

$$\mathcal{A}\left(\varphi_L, z_L(q)\right) = \left(g'\left(u_L(q)\right), \varphi_L\right)_\Omega, \quad \forall \varphi_L \in S_L, \tag{4.2}$$

where $q \in L^2(\Omega)$ and $u_L(q), z_L(q) \in S_L$. Let

$$J_L(q) = g\left(u_L(q)\right) + h(q), \quad \forall q \in L^2(\Omega), \tag{4.3}$$

then it can be verified by a direct calculation that

$$J_L'(q)(\delta q) = \left(h'(q) + z_L(q), \delta q\right)_\Omega, \quad \delta q \in L^2(\Omega). \tag{4.4}$$

Following [25], the error between the solution of (2.9) and the solution of (4.1) can be estimated as follows.

Lemma 4.1 *For any $q \in L^2(\Omega)$, let $u(q)$ be the solution of* (2.9), *$u_L(q)$ be the solution of* (4.1). *Suppose $u \in H^{\frac{\alpha}{2}}(I;H^\mu(\Lambda)) \cap H^\nu(I;H_0^1(\Lambda))$, $0 < \alpha < 1$, $\nu > 1$, $\mu \geq 1$, then we have*

$$\begin{aligned}\left\| u(q) - u_L(q)\right\|_{B^{\frac{\alpha}{2}}(\Omega)} &\lesssim N^{\frac{\alpha}{2}-\nu}\|u\|_{0,\nu} + N^{-\nu}\|u\|_{1,\nu} + N^{\frac{\alpha}{2}-\nu}M^{-\mu}\|u\|_{\mu,\nu} \\ &\quad + M^{-\mu}\|u\|_{\mu,\frac{\alpha}{2}} + M^{1-\mu}\|u\|_{\mu,0}. \end{aligned} \tag{4.5}$$

We are now in a position to analyze the approximation error of the proposed space-time spectral method. The proof of the main result will be accomplished with a series of lemmas which we present below.

Lemma 4.2 *If $g(\cdot)$ is convex and $h(\cdot)$ is uniformly convex, that is, there exists a constant c such that*

$$\left(h'(p) - h'(q), p - q\right)_\Omega \geq c\|p-q\|_{0,\Omega}^2, \quad \forall p,q \in L^2(\Omega).$$

Then, for all $p,q \in L^2(\Omega)$, we have

$$J_L'(p)(p-q) - J_L'(q)(p-q) \geq c\|p-q\|_{0,\Omega}^2. \tag{4.6}$$

Proof Note that

$$\begin{aligned} J_L'(p)(p-q) - J_L'(q)(p-q) &= \left(z_L(p) + h'(p), p-q\right)_\Omega - \left(z_L(q) + h'(q), p-q\right)_\Omega \\ &= \left(z_L(p) - z_L(q), p-q\right)_\Omega + \left(h'(p) - h'(q), p-q\right)_\Omega. \end{aligned} \tag{4.7}$$

Moreover, it follows from (4.1) and (4.2) that

$$\begin{aligned} \left(z_L(p) - z_L(q), p-q\right)_\Omega &= \mathcal{A}\left(u_L(p) - u_L(q), z_L(p) - z_L(q)\right) \\ &= \left(g'\left(u_L(p)\right) - g'\left(u_L(q)\right), u_L(p) - u_L(q)\right)_\Omega. \end{aligned} \tag{4.8}$$

Noting that $g(\cdot)$ is convex, and $h(\cdot)$ is uniformly convex, (4.7) and (4.8) imply that

$$J_L'(p)(p-q) - J_L'(q)(p-q) \geq \left(h'(p) - h'(q), p-q\right)_\Omega \geq c\|p-q\|_{0,\Omega}^2.$$

This proves (4.6). □

Lemma 4.3 *Let q^* be the solution of the continuous optimization problem* (2.12), *q_L^* be the solution of the discrete optimization problem* (3.3). *Assume that $h(\cdot)$ and $g(\cdot)$ are Lipschitz continuous with Lipschitz constants L_1 and L_2, respectively. Moreover, suppose $q^* \in L^2(I;H^\mu(\Lambda)) \cap H^\nu(I;L^2(\Lambda))$, $\nu > 1$, $\mu \geq 1$, then we have*

$$\left\|q^* - q_L^*\right\|_{0,\Omega} \lesssim N^{-\nu}\left\|q^*\right\|_{0,\nu} + M^{-\mu}\left\|q^*\right\|_{\mu,0} + \left\|z_L\left(q^*\right) - z\left(q^*\right)\right\|_{0,\Omega}. \tag{4.9}$$

Proof To obtain the asserted result, we split the error to be estimated in the following way:

$$\left\|q^* - q_L^*\right\|_{0,\Omega} \leq \left\|q^* - p_L\right\|_{0,\Omega} + \left\|p_L - q_L^*\right\|_{0,\Omega}, \quad \forall p_L \in P_M(\Lambda) \otimes P_N(I). \tag{4.10}$$

First it follows from Lemma 4.2 that

$$c\|p_L - q_L^*\|_{0,\Omega}^2 \le J_L'(p_L)(p_L - q_L^*) - J_L'(q_L^*)(p_L - q_L^*), \quad \forall p_L \in P_M(\Lambda) \otimes P_N(I). \tag{4.11}$$

In virtue of (2.13) and (3.4), we have

$$J_L'(q_L^*)(p_L - q_L^*) = J'(q^*)(p_L - q_L^*) = 0, \quad \forall p_L \in P_M(\Lambda) \otimes P_N(I).$$

Combining these equalities with (2.14), (3.5), and (4.4), we obtain

$$\begin{aligned}
&c\|p_L - q_L^*\|_{0,\Omega}^2 \\
&\quad \le J_L'(p_L)(p_L - q_L^*) - J'(q^*)(p_L - q_L^*) \\
&\quad = J_L'(p_L)(p_L - q_L^*) - J_L'(q^*)(p_L - q_L^*) + J_L'(q^*)(p_L - q_L^*) - J'(q^*)(p_L - q_L^*) \\
&\quad = \big(h'(p_L) - h'(q^*), p_L - q_L^*\big)_\Omega + \big(z_L(p_L) - z_L(q^*), p_L - q_L^*\big)_\Omega \\
&\qquad + \big(z_L(q^*) - z(q^*), p_L - q_L^*\big)_\Omega \\
&\quad \le \|h'(p_L) - h'(q^*)\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega} + \|z_L(p_L) - z_L(q^*)\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega} \\
&\qquad + \|z_L(q^*) - z(q^*)\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega} \\
&\quad \le L_1\|p_L - q^*\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega} + \|z_L(p_L) - z_L(q^*)\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega} \\
&\qquad + \|z_L(q^*) - z(q^*)\|_{0,\Omega}\|p_L - q_L^*\|_{0,\Omega}.
\end{aligned}$$

By simplifying both sides, we obtain

$$c\|p_L - q_L^*\|_{0,\Omega} \le L_1\|p_L - q^*\|_{0,\Omega} + \|z_L(p_L) - z_L(q^*)\|_{0,\Omega} + \|z_L(q^*) - z(q^*)\|_{0,\Omega}. \tag{4.12}$$

Note that $z_L(p_L) - z_L(q^*)$ solves

$$\mathcal{A}\big(\varphi_L, z_L(p_L) - z_L(q^*)\big) = \big(g'(u_L(p_L)) - g'(u_L(q^*)), \varphi_L\big)_\Omega, \quad \forall \varphi_L \in S_L, \tag{4.13}$$

and $u_L(p_L) - u_L(q^*)$ satisfies

$$\mathcal{A}\big(u_L(p_L) - u_L(q^*), v_L\big) = \big(p_L - q^*, v_L\big)_\Omega, \quad \forall v_L \in S_L. \tag{4.14}$$

On the other hand, the bilinear form $\mathcal{A}(\cdot,\cdot)$ satisfies the following continuity and coercivity [25]:

$$\mathcal{A}(u,v) \lesssim \|u\|_{B^{\frac{\alpha}{2}}(\Omega)}\|v\|_{B^{\frac{\alpha}{2}}(\Omega)}, \qquad \mathcal{A}(v,v) \gtrsim \|v\|_{B^{\frac{\alpha}{2}}(\Omega)}^2, \quad \forall u, v \in B^{\frac{\alpha}{2}}(\Omega).$$

Thus, taking $v_L = u_L(p_L) - u_L(q^*)$ in (4.14) gives

$$\|u_L(p_L) - u_L(q^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \|p_L - q^*\|_{0,\Omega}. \tag{4.15}$$

Similarly, taking $\varphi_L = z_L(p_L) - z_L(q^*)$ in (4.13) gives

$$\|z_L(p_L) - z_L(q^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \|g'(u_L(p_L)) - g'(u_L(q^*))\|_{0,\Omega} \lesssim \|u_L(p_L) - u_L(q^*)\|_{B^{\frac{\alpha}{2}}(\Omega)}.$$

Putting (4.15) into the above inequality, we obtain

$$\|z_L(p_L) - z_L(q^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \|p_L - q^*\|_{0,\Omega}. \tag{4.16}$$

Then combining (4.12) and (4.16), we get

$$c\|p_L - q_L^*\|_{0,\Omega} \le (C + L_1)\|p_L - q^*\|_{0,\Omega} + \|z_L(q^*) - z(q^*)\|_{0,\Omega}. \tag{4.17}$$

Plugging (4.17) into (4.10) yields

$$\|q^* - q_L^*\|_{0,\Omega} \lesssim \|q^* - p_L\|_{0,\Omega} + \|z_L(q^*) - z(q^*)\|_{0,\Omega}. \tag{4.18}$$

Since the above estimate is true for all $p_L \in P_M(\Lambda) \otimes P_N(I)$, we take $p_L = \Pi_M \Pi_N q^*$ in (4.18), with Π_M and Π_N standing for the standard L^2-orthogonal projectors, respectively, defined in Λ and I, to obtain

$$\|q^* - q_L^*\|_{0,\Omega} \lesssim N^{-\nu}\|q^*\|_{0,\nu} + M^{-\mu}\|q^*\|_{\mu,0} + \|z_L(q^*) - z(q^*)\|_{0,\Omega}. \qquad \square$$

Lemma 4.4 *Let $z = z(q) \in B^{\frac{\alpha}{2}}(\Omega)$ be the solution of the continuous adjoint state problem* (2.22), *$z_L(q)$ be the solution of its approximation problem* (4.2). *Assume that $g(\cdot)$ is Lipschitz continuous, then we have*

$$\|z(q) - z_L(q)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \|u(q) - u_L(q)\|_{0,\Omega} + \inf_{\forall \varphi_L \in S_L} \|z - \varphi_L\|_{B^{\frac{\alpha}{2}}(\Omega)}, \tag{4.19}$$

where $u(q)$ and $u_L(q)$ are, respectively, the solutions of (2.9) *and* (4.1).

Proof The proof goes along the same lines as Lemma 4.5 in [35]. $\square$

Using the above lemmas and following the same lines as the proof of Theorem 4.1 in [35], we obtain the main result concerning the approximation errors.

Theorem 4.1 *Suppose q^* and q_L^* are, respectively, the solutions of the continuous optimization problem* (2.12) *and its discrete counterpart* (3.3), *$u(q^*)$ and $u_L(q_L^*)$ are the state solutions of* (2.9) *and* (3.2) *associated to q^* and q_L^*, respectively, $z(q^*)$ and $z_L(q_L^*)$ are the associated solutions of* (2.22) *and* (3.6), *respectively. Suppose, moreover, $h(\cdot)$ and $g(\cdot)$ are Lipschitz continuous. If $q^* \in L^2(I;H^\mu(\Lambda)) \cap H^\nu(I;L^2(\Lambda))$ and $u(q^*), z(q^*) \in H^{\frac{\alpha}{2}}(I;H^\mu(\Lambda)) \cap H^\nu(I;H_0^1(\Lambda))$, $0 < \alpha < 1$, $\nu > 1$, $\mu \ge 1$, then the following estimate holds*:

$$\begin{aligned}
&\|q^* - q_L^*\|_{0,\Omega} + \|u(q^*) - u_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} + \|z(q^*) - z_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \\
&\quad \lesssim N^{-\nu}\|q^*\|_{0,\nu} + M^{-\mu}\|q^*\|_{\mu,0} + N^{\frac{\alpha}{2}-\nu}\big(\|u(q^*)\|_{0,\nu} + \|z(q^*)\|_{0,\nu}\big) \\
&\qquad + N^{-\nu}\big(\|u(q^*)\|_{1,\nu} + \|z(q^*)\|_{1,\nu}\big) + N^{\frac{\alpha}{2}-\nu}M^{-\mu}\big(\|u(q^*)\|_{\mu,\nu} + \|z(q^*)\|_{\mu,\nu}\big) \\
&\qquad + M^{-\mu}\big(\|u(q^*)\|_{\mu,\frac{\alpha}{2}} + \|z(q^*)\|_{\mu,\frac{\alpha}{2}}\big) + M^{1-\mu}\big(\|u(q^*)\|_{\mu,0} + \|z(q^*)\|_{\mu,0}\big).
\end{aligned} \tag{4.20}$$

5 Conjugate gradient optimization algorithm and numerical results

We carry out in this section a series of numerical experiments to numerically verify the *a priori* error estimates we obtained in the previous sections. We will focus on the linear-quadratic optimal control problem. Precisely, we consider the quadratic cost functional $J(q)$ with

$$g(u) = \frac{1}{2}\int_\Omega (u-\bar{u})^2\,\mathrm{d}x\,\mathrm{d}t, \qquad h(q) = \frac{1}{2}\int_\Omega q^2\,\mathrm{d}x\,\mathrm{d}t,$$

where $\bar{u}$ is a given observation data. Then we have

$$J'(q)(\delta q) = \bigl(q + z(q), \delta q\bigr)_\Omega, \quad \forall \delta q \in L^2(\Omega).$$

5.1 Conjugate gradient optimization algorithm

In the following, we propose a conjugate gradient algorithm for the associated linear-quadratic optimization problem. The details are described below.

Given an initial control $q_L^{(0)}$, the corresponding state $u_L(q_L^{(0)})$ is given by the solution of the state equation in (3.2). To apply the stopping criterion $\|J_L'(q_L^{(0)})\| \le \varepsilon$, with ε being a pre-defined tolerance, we need information on the adjoint state $z_L(q_L^{(0)})$, which is obtained from the adjoint state equation (3.6) for given $u_L(q_L^{(0)})$ and $q_L^{(0)}$. Then the descent direction, that is, the gradient of the objective functional at $q_L^{(0)}$ is calculated through

$$d_L^{(0)} := J_L'\bigl(q_L^{(0)}\bigr) = z_L\bigl(q_L^{(0)}\bigr) + h'\bigl(q_L^{(0)}\bigr) = z_L\bigl(q_L^{(0)}\bigr) + q_L^{(0)}.$$

We simultaneously let the first conjugate direction be the gradient direction, namely

$$s_L^{(0)} = d_L^{(0)}.$$

Then, assuming known $q_L^{(k)}$, $d_L^{(k)}$, and $s_L^{(k)}$ at the current (kth) iteration, we update $q_L^{(k)}$ via

$$q_L^{(k+1)} = q_L^{(k)} - \rho_k s_L^{(k)},$$

where ρ_k is the iteration step size, determined in such a way that

$$J_L\bigl(q_L^{(k)} - \rho_k d_L^{(k)}\bigr) = \min_{\rho>0} J_L\bigl(q_L^{(k)} - \rho d_L^{(k)}\bigr).$$

Due to $(d_L^{(k+1)}, s_L^{(k)})_\Omega = 0$ and

$$d_L^{(k+1)} = z_L\bigl(q_L^{(k+1)}\bigr) + q_L^{(k+1)} = z_L\bigl(q_L^{(k+1)}\bigr) + \bigl(q_L^{(k)} - \rho_k s_L^{(k)}\bigr),$$

ρ_k is characterized as

$$\bigl(z_L\bigl(q_L^{(k+1)}\bigr) + \bigl(q_L^{(k)} - \rho_k s_L^{(k)}\bigr), s_L^{(k)}\bigr)_\Omega = 0, \tag{5.1}$$

where $z_L(q_L^{(k+1)})$ is the solution of

$$\mathcal{A}\bigl(\varphi_L, z_L\bigl(q_L^{(k+1)}\bigr)\bigr) = \bigl(u_L^{(k+1)} - \bar{u}, \varphi_L\bigr)_\Omega, \quad \forall \varphi_L \in S_L, \tag{5.2}$$

with $u_L^{(k+1)} \in S_L$ given by

$$\mathcal{A}\bigl(u_L^{(k+1)}, v_L\bigr) = \bigl(f + q_L^{(k)} - \rho_k s_L^{(k)}, v_L\bigr)_\Omega + \left(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)}, v_L\right)_\Omega, \quad \forall v_L \in S_L. \tag{5.3}$$

The optimal iteration step size ρ_k can be efficiently calculated through solving (5.1). Indeed we first notice there exists an explicit expression of $z_L(q_L^{(k+1)})$ on ρ_k. Let $\tilde{u}_L^{(k)}$ and $\tilde{z}_L^{(k)}$ denote, respectively, the solutions of

$$\mathcal{A}\bigl(\tilde{u}_L^{(k)}, v_L\bigr) = \bigl(s_L^{(k)}, v_L\bigr)_\Omega, \quad \forall v_L \in S_L, \tag{5.4}$$

$$\mathcal{A}\bigl(\varphi_L, \tilde{z}_L^{(k)}\bigr) = \bigl(\tilde{u}_L^{(k)}, \varphi_L\bigr)_\Omega, \quad \forall \varphi_L \in S_L, \tag{5.5}$$

$u_L(q_L^{(k)})$ and $z_L(q_L^{(k)})$ are, respectively, the solutions of

$$\mathcal{A}\bigl(u_L\bigl(q_L^{(k)}\bigr), v_L\bigr) = \bigl(f + q_L^{(k)}, v_L\bigr)_\Omega + \left(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)}, v_L\right)_\Omega, \quad \forall v_L \in S_L, \tag{5.6}$$

$$\mathcal{A}\bigl(\varphi_L, z_L\bigl(q_L^{(k)}\bigr)\bigr) = \bigl(u_L\bigl(q_L^{(k)}\bigr) - \bar{u}, \varphi_L\bigr)_\Omega, \quad \forall \varphi_L \in S_L, \tag{5.7}$$

then it can be checked that $z_L(q_L^{(k)}) - \rho_k \tilde{z}_L^{(k)}$ solves (5.2)-(5.3), that is,

$$z_L\bigl(q_L^{(k+1)}\bigr) = z_L\bigl(q_L^{(k)}\bigr) - \rho_k \tilde{z}_L^{(k)}.$$

Putting this expression into (5.1) gives

$$\bigl(z_L\bigl(q_L^{(k)}\bigr) - \rho_k \tilde{z}_L^{(k)} + \bigl(q_L^{(k)} - \rho_k s_L^{(k)}\bigr), s_L^{(k)}\bigr)_\Omega = 0.$$

Obviously, $d_L^{(k)} = z_L(q_L^{(k)}) + q_L^{(k)}$ holds. Let $\tilde{d}_L^{(k)} = \tilde{z}_L^{(k)} + s_L^{(k)}$, then we obtain

$$\rho_k = \frac{(d_L^{(k)}, s_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, s_L^{(k)})_\Omega}. \tag{5.8}$$

In addition, it is easy to prove that

$$d_L^{(k+1)} = d_L^{(k)} - \rho_k \tilde{d}_L^{(k)}$$

holds. Let

$$\beta_k = \frac{\|d_L^{(k+1)}\|_{0,\Omega}^2}{\|d_L^{(k)}\|_{0,\Omega}^2}$$

be the conjugate coefficient, we update the conjugate direction via

$$s_L^{(k+1)} = d_L^{(k+1)} + \beta_k s_L^{(k)}.$$

Using the result

$$\bigl(d_L^{(k)}, s_L^{(k-1)}\bigr)_\Omega = 0,$$

we improve the optimum iterative step ρ_k as follows:

$$\rho_k = \frac{(d_L^{(k)}, s_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, s_L^{(k)})_\Omega} = \frac{(d_L^{(k)}, d_L^{(k)} + \beta_{k-1} s_L^{(k-1)})_\Omega}{(\tilde{d}_L^{(k)}, s_L^{(k)})_\Omega} = \frac{(d_L^{(k)}, d_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, s_L^{(k)})_\Omega}. \tag{5.9}$$

The overall process is summarized below.

Conjugate gradient optimization algorithm Choose an initial control $q_L^{(0)}$. Set $k = 0$.

(a) Solve problems (5.6)-(5.7), let $d_L^{(k)} = z_L(q_L^{(k)}) + q_L^{(k)}$, $s_L^{(k)} = g_L^{(k)}$.

(b) Solve problems (5.4)-(5.5), and set $\tilde{d}_L^{(k)} = \tilde{z}_L^{(k)} + s_L^{(k)}$, $\rho_k = \frac{(d_L^{(k)}, d_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, s_L^{(k)})_\Omega}$.

(c) Update $q_L^{(k+1)} = q_L^{(k)} - \rho_k s_L^{(k)}$, $d_L^{(k+1)} = d_L^{(k)} - \rho_k \tilde{d}_L^{(k)}$.

(d) If $\|d_L^{(k+1)}\|_{0,\Omega} \le$ tolerance, then take $q_L^* = q_L^{(k+1)}$, and solve problems (3.2) and (3.6) to get $u_L(q_L^*)$ and $z_L(q_L^*)$; else, let $\beta_k = \frac{\|d_L^{(k+1)}\|_{0,\Omega}^2}{\|d_L^{(k)}\|_{0,\Omega}^2}$, $s_L^{(k+1)} = d_L^{(k+1)} + \beta_k s_L^{(k)}$. Set $k = k + 1$, repeat (a)-(d).

5.2 Numerical results

We are now in a position to carry out some numerical experiments and present some results to validate the obtained error estimates. In all the calculations, we take $T = 1$.

Example 5.1 We now let the observation data $\bar{u}(x,t) = \sin \pi t \sin \pi x$, and consider problem (2.1)-(2.2) with exact analytical solution:

$$u(q^*) = \sin \pi t \sin \pi x, \qquad z(q^*) = 0, \quad q^* = 0.$$

For this choice of data, that is, the exact solution $u(q^*)$ serves as the observation data, problem (2.1)-(2.2) is indeed an inverse problem about unknown parameter in the right-hand side and the corresponding objective function is expected to attain its minimum 0.

In this example, we fix the initial guess at $q^{(0)} = 1$. We should mention here that any initial guess is possible. We will show in our next example that the presence of perturbation or noise in the control has limited influences on the optimization algorithm.

We first check the convergence behavior of numerical solutions with respect to the polynomial degrees M. In Figure 1, we plot the errors as functions of the polynomial degrees M with $\alpha = 0.3$, $N = 20$. Also, in Table 1, we list the maximum absolute errors of q, u, and

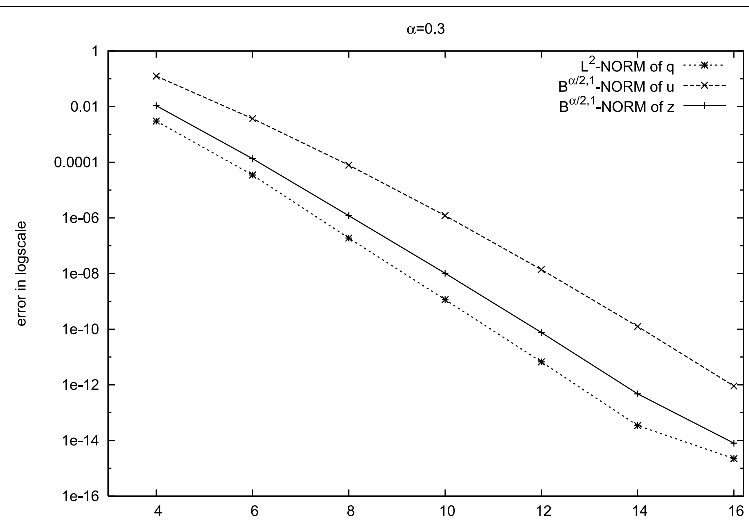

Figure 1 Errors of *u*, *z*, *q* *versus* *M* with *N* = 20, *α* = 0.3.

Table 1 Maximum absolute errors for *q*, *u*, and *J* at $N = 20$, $\alpha = 0.3$, and various choices of *M* for Example 5.1

M	4	6	8	10	12	14	16
q	4.08E–03	5.24E–05	2.93E–07	1.58E–09	9.97E–12	5.17E–14	2.30E–14
u	4.75E–02	7.96E–04	1.17E–05	1.39E–07	1.28E–09	9.82E–12	3.43E–13
J	2.86E–05	5.77E–09	8.23E–13	8.83E–17	6.38E–21	3.09E–25	1.12E–29

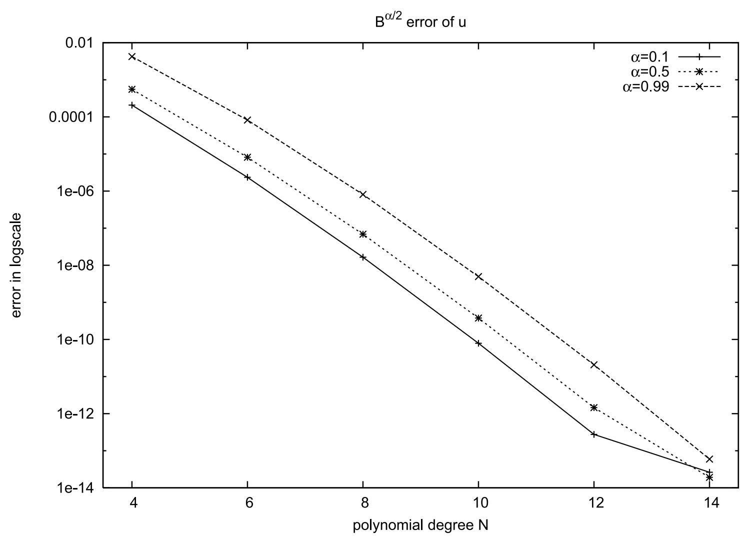

Figure 2 $B^{\frac{\alpha}{2}}$ **Errors of *u versus N* with $M = 20$, $\alpha = 0.1, 0.5, 0.99$.**

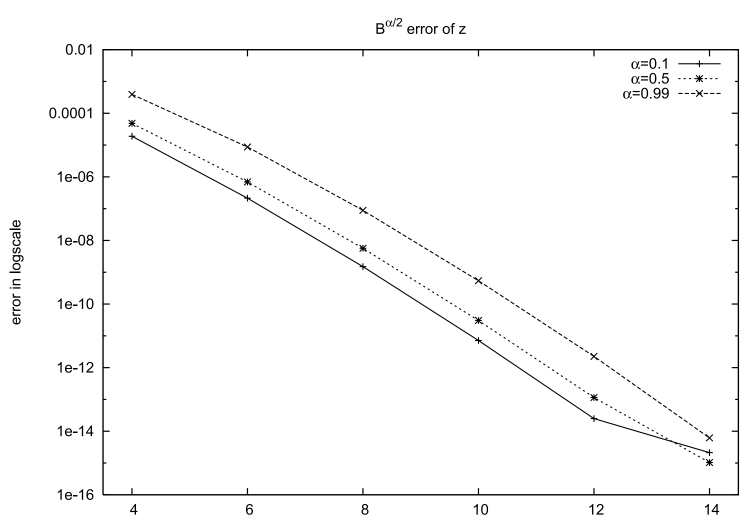

Figure 3 $B^{\frac{\alpha}{2}}$ **Errors of *z versus N* with $M = 20$, $\alpha = 0.1, 0.5, 0.99$.**

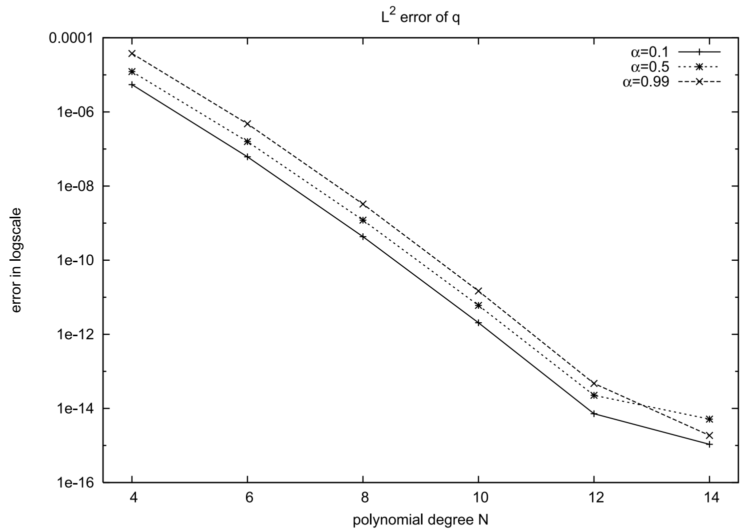

Figure 4 L^2 **Errors of** q ***versus*** N **with** $M = 20$, $\alpha = 0.1, 0.5, 0.99$.

Table 2 Maximum absolute errors for q, u, **and** J **at** $M = 20$ **and various choices of** α **and** N **for Example 5.1**

Errors	α	$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 12$	$N = 14$
q	0.1	7.11E–06	8.06E–08	6.19E–10	3.27E–12	1.79E–14	7.75E–15
	0.5	1.68E–05	2.14E–07	1.61E–09	8.24E–12	2.94E–14	4.91E–16
	0.99	6.96E–05	1.07E–06	8.98E–09	4.70E–11	1.63E–13	2.89E–16
u	0.1	7.74E–05	9.14E–07	7.45E–09	3.98E–11	2.14E–13	8.67E–14
	0.5	1.82E–04	2.53E–06	1.97E–08	1.00E–10	3.66E–13	1.14E–14
	0.99	6.69E–04	9.16E–06	7.07E–08	3.82E–10	1.75E–12	5.11E–15
J	0.1	8.39E–11	8.02E–15	3.24E–19	6.35E–24	6.76E–29	1.98E–31
	0.5	4.64E–10	6.15E–13	3.01E–18	6.74E–23	7.83E–28	3.16E–32
	0.99	5.00E–09	7.90E–13	3.88E–17	8.27E–22	9.04E–27	6.78E–32

J at $\alpha = 0.3$, $N = 20$, and various choices of M. As expected, the errors show an exponential decay, since in this semi-log representation one observes that the error variations are essentially linear versus the degrees of polynomial.

We then investigate the temporal errors, which is more interesting to us because of the fractional derivative in time. In Figures 2 to 4, we plot the errors as functions of N with $M = 20$ for three values $\alpha = 0.1, 0.5, 0.99$. The straight line of the error curves indicates that the convergence in time is also exponential. The maximum absolute errors of q, u, and J at $M = 20$ are also listed in Table 2.

Example 5.2 We choose another exact analytical solutions as

$$u(q^*) = \sin \pi x e^t, \qquad z(q^*) = \sin \pi x(1-t)e^{2t}, \quad q^* = -z(q^*).$$

Unlike the previous example which uses the exact solution $u(q^*)$ as the observation data, the observation data $\bar{u}(x,t)$ here is calculated through (2.22) using $u(q^*)$ and $z(q^*)$.

First, as mentioned in the previous example, we investigate the impact of the initial guess on the convergence of the conjugate gradient algorithm. We start by consider-

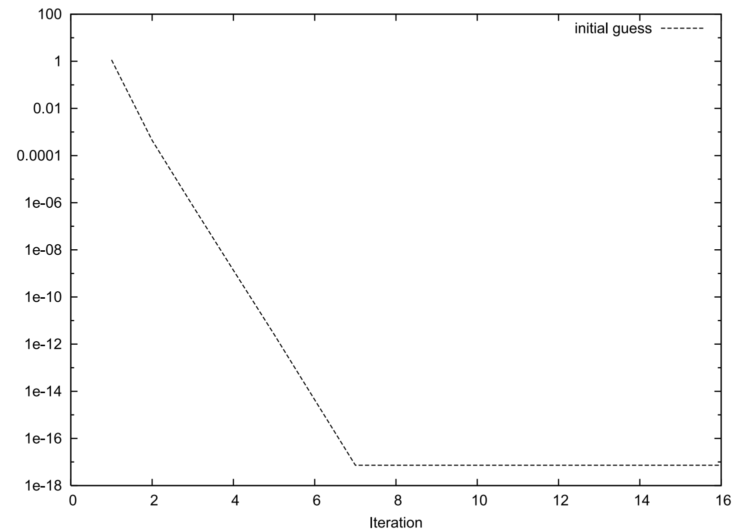

Figure 5 Convergence history of the gradient of the cost function for $q^{(0)} = 20q^*$.

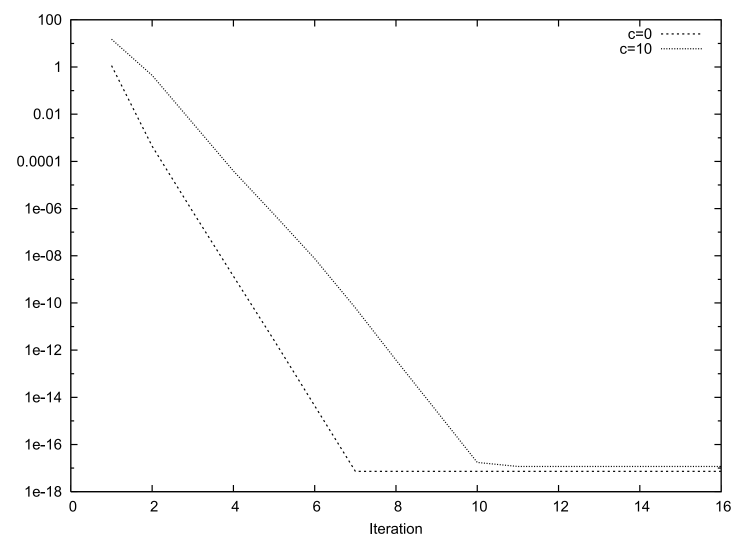

Figure 6 Convergence history of the gradient of the cost function for $q^{(0)} = c$.

ing $q^{(0)} = 20q^*$. This represents a strong perturbation in the initial guess. We now fix $M = N = 18$, $\alpha = 0.6$. In Figure 5, we plot the convergence history of the gradient of the objective function as a function of the iteration number with $M = N = 18$, $\alpha = 0.6$. We see that the iterative method converges within seven iterations. We then take $q^{(0)}$ to be constant c with $c = 0$ or 10, which has nothing to compare with the exact control q^*. We repeat the same computation as in Figure 5. The result is given in Figure 6. These results seem to tell that the type and amplitude of the perturbation have no significant effects on the convergence of the optimization algorithm, since in any case the iterative algorithm converges with the same rate. In the following, the initial guess is set to 0.

We now investigate the errors of the numerical solution with respect to the temporal and spatial approximations. In Figure 7 and Figures 8-10 we report the errors in logarithmic

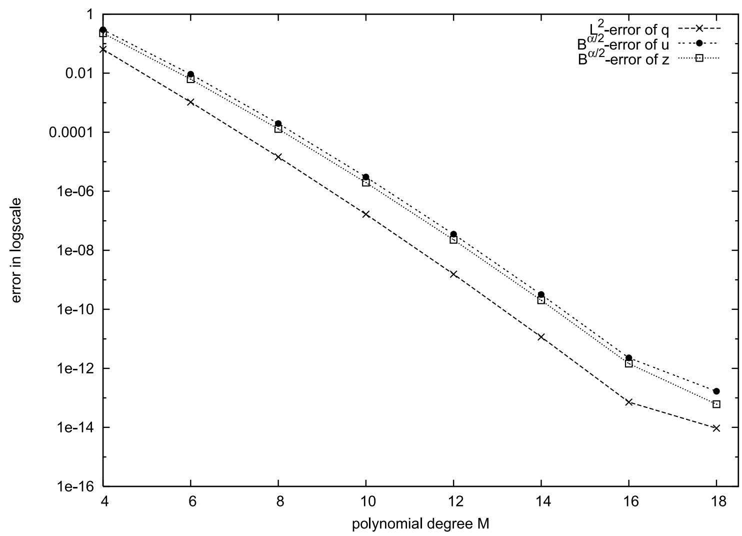

Figure 7 Errors of u, z, q *versus* M with $\alpha = 0.6$, $N = 20$.

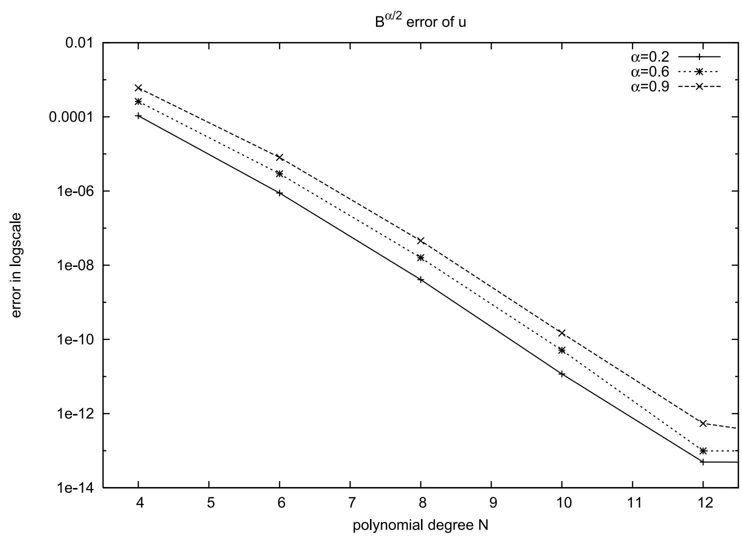

Figure 8 $B^{\frac{\alpha}{2}}$ Errors of u *versus* N with $M = 20$, $\alpha = 0.2, 0.6, 0.9$.

scale as a function of the polynomial degrees M and N, respectively. Also, in Tables 3 and 4, we list the maximum absolute errors of q, u, and J. Clearly, all the errors show an exponential decay.

6 Concluding remarks

In the present work, we have shown an efficient optimization algorithm for the space-time fractional equation optimal control problem based on the spectral approximation. *A priori* error estimates are derived. Some numerical experiments have been carried out to confirm the theoretical results.

There are many important issues that still need to be addressed. Firstly, the same formulation and solution scheme can be used with minor changes for the problem defined in

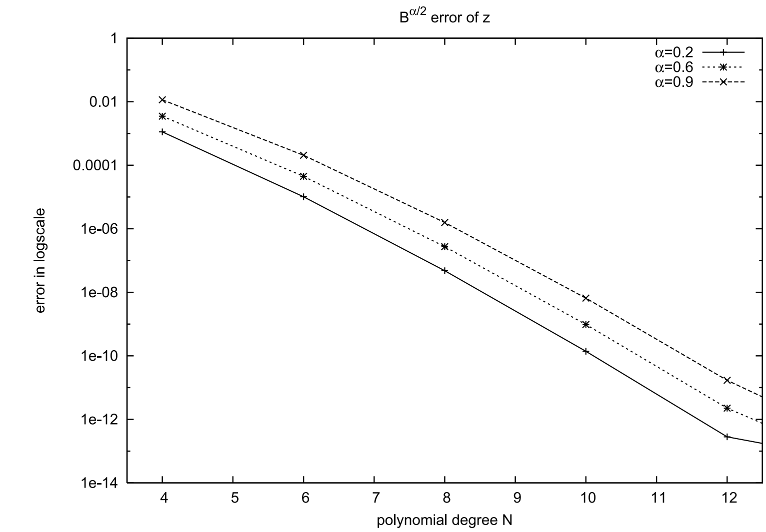

Figure 9 $B^{\frac{\alpha}{2}}$ **Errors of** z ***versus*** N **with** $M = 20$, $\alpha = 0.2, 0.6, 0.9$.

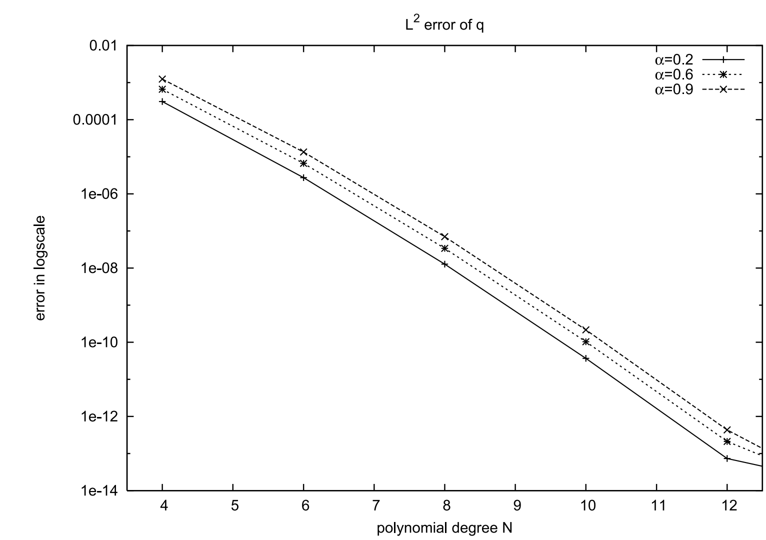

Figure 10 L^2 **Errors of** q ***versus*** N **with** $M = 20$, $\alpha = 0.2, 0.6, 0.9$.

Table 3 Maximum absolute errors for q, u, **and** J **at** $N = 20$, $\alpha = 0.6$ **and various choices of** M **for Example 5.2**

M	4	6	8	10	12	14	16
q	7.24E–02	1.18E–03	1.62E–05	1.92E–07	1.75E–09	1.34E–11	1.57E–13
u	1.30E–01	2.17E–03	3.19E–05	3.78E–07	3.48E–09	2.66E–11	2.22E–13
J	9.47E–02	1.07E–03	2.72E–06	2.30E–09	1.08E–12	6.66E–16	8.88E–16

Table 4 Maximum absolute errors for *q*, *u*, and *J* at $M = 20$ and various choices of α and *N* for Example 5.2

Errors	α	$N = 4$	$N = 6$	$N = 8$	$N = 10$	$N = 12$
q	0.2	9.18E–02	1.01E–05	5.45E–08	1.76E–10	3.72E–13
	0.6	2.06E–03	2.55E–05	1.50E–07	5.13E–10	1.15E–12
	0.9	3.50E–03	4.68E–05	2.85E–07	9.94E–10	2.23E–12
u	0.2	8.55E–05	8.52E–07	4.46E–09	1.40E–11	9.37E–14
	0.6	1.32E–04	1.23E–06	5.65E–09	1.55E–11	2.35E–14
	0.9	1.01E–04	1.24E–06	7.83E–09	2.66E–11	6.33E–14
J	0.2	3.17E–05	2.37E–08	3.38E–11	4.55E–14	1.33E–15
	0.6	5.52E–05	3.85E–08	2.22E–11	2.08E–13	2.44E–15
	0.9	8.53E–05	8.61E–08	2.57E–11	9.77E–14	2.22E–16

terms of Riemann-Liouville derivatives. Secondly, studies for more complicated control problems and constraint sets are needed. Thirdly, although our analysis and algorithm are designed for the optimization of the distributed control problem, we hope that they are generalizable for the minimization problems of other parameters, such as boundary conditions and so on.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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