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Existence results for multi-term fractional differential inclusions

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Abstract
In this paper, we study the existence of solutions for a new class of boundary value problems for nonlinear multi-term fractional differential inclusions. Our main result relies on the multi-valued form of Krasnoselskii's fixed point theorem. An illustrative example is also presented.

MSC: 34A08; 34A60; 34B10; 34B15

Keywords: fractional differential inclusion; boundary value problem; existence; fixed point theorem

1 Introduction and preliminaries

In this paper we study the existence of solutions for the following multi-term fractional differential inclusions:

$$\begin{aligned} {}^cD^\alpha u(t) \in{}& F\bigl(t,u(t),u'(t),u''(t),{}^cD^{q_1}u(t),\dots,{}^cD^{q_k}u(t)\bigr) \\ &+ G\bigl(t,u(t),u'(t),u''(t),{}^cD^{q_1}u(t),\dots,{}^cD^{q_k}u(t)\bigr) \end{aligned} \tag{1.1}$$

supplemented with boundary conditions

$$u(0)=0, \qquad u'(0)=-u(1)-u'(1), \qquad u''(0)=-u''(1)-{}^cD^pu(1), \tag{1.2}$$

where ${}^cD^\alpha$, ${}^cD^{q_i}$ denote the Caputo fractional derivatives, $2<\alpha\le 3$, $1<q_i\le 2$, $i=1,2,\dots,k$, $t\in J:=[0,1]$, $1<p\le 2$, $k\ge 1$, and $F,G:J\times\mathbb{R}^{k+3}\to\mathcal{P}(\mathbb{R})$ are multifunctions.

Many of published papers about fractional differential equations and inclusions apply the fixed point theory for proving the existence results. For instance, one can find a lot of papers in this field (see [1–25] and the references therein).

Let $\alpha>0$, $n-1<\alpha<n$, $n=[\alpha]+1$, and $u\in C([a,b],\mathbb{R})$. The Caputo derivative of fractional of order α for the function u is defined by ${}^cD^\alpha u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)\,d\tau$ (see for more details [11, 23, 25–27]). Also, the Riemann-Liouville fractional order integral of the function u is defined by $I^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{u(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau$ $(t>0)$ whenever the integral exists [11, 23, 25–27]. In [28], it has been proved that the general solution of the fractional differential equation ${}^cD^\alpha u(t)=0$ is given by $u(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$, where $c_0,\dots,c_{n-1}$ are real constants and $n=[\alpha]+1$. Also, for each $T>0$ and $u\in C([0,T])$ we have

$$I^{\alpha\,c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_0, \ldots, c_{n-1}$ are real constants and $n = [\alpha] + 1$ [28].

Now, we review some definitions and notations as regards multifunctions [29, 30].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{\mathrm{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{\mathrm{b}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{\mathrm{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{\mathrm{cp,cv}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$, $\mathcal{P}_{\mathrm{b,cl,cv}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded, closed, and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \bigcup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$ (*i.e.*, $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_{\mathrm{b}}(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, *i.e.*, $u_n \to u_*$, $y_n \to y_*$, $y_n \in G(u_n)$ imply $y_* \in G(u_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multi-valued operator G will be denoted by $\operatorname{Fix} G$. A multi-valued map $G : J \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

Consider the Pompeiu-Hausdorff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. A multi-valued operator $N : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called contraction if there exists $\gamma \in (0,1)$ such that $H_d(N(x),N(y)) \le \gamma d(x,y)$ for each $x, y \in X$.

We say that $F : J \times \mathbb{R}^{k+3} \to \mathcal{P}(\mathbb{R})$ is a Carathéodory multifunction if $t \mapsto F(t, u_1, \ldots, u_{k+3})$ is measurable for all $u_i \in \mathbb{R}$ and $(u_1, \ldots, u_{k+3}) \mapsto F(t, u_1, \ldots, u_{k+3})$ is upper semi-continuous for almost all $t \in J$ [29, 31]. Also, a Carathéodory multifunction $F : J \times \mathbb{R}^{k+3} \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\big\|F(t, u_1, \ldots, u_{k+3})\big\| = \sup_{t\in J}\big\{|s| : s \in F(t, u_1, \ldots, u_{k+3})\big\} \le \phi_\rho(t)$$

for all $|u_1|, \ldots, |u_{k+3}| \le \rho$ and for almost all $t \in J$ [29, 31].

Define the set of selections of F and G at $u \in C(J, \mathbb{R})$ by

$$S_{F,u} := \big\{v \in L^1(J,\mathbb{R}) : v(t) \in F\big(t, u(t), u'(t), u''(t), {}^cD^{q_1}u(t), \ldots, {}^cD^{q_k}u(t)\big)\big\}$$

and

$$S_{G,u} := \big\{v_1 \in L^1(J,\mathbb{R}) : v_1(t) \in G\big(t, u(t), u'(t), u''(t), {}^cD^{q_1}u(t), \ldots, {}^cD^{q_k}u(t)\big)\big\}$$

for almost all $t \in J$. If F is an arbitrary multifunction, then it has been proved that $S_F(u) \neq \emptyset$ for all $u \in C(J, X)$ if $\dim X < \infty$ [32].

The graph of a function F is the set $\mathrm{Gr}(F) = \{(x,y) \in X \times Y : y \in F(x)\}$ [29]. The graph $\mathrm{Gr}(F)$ of $F : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is said to be a closed subset of $X \times Y$, if for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, when $n \to \infty$, $u_n \to u_0$, $y_n \to y_0$, and $y_n \in F(u_n)$, then $y_0 \in F(u_0)$ [29].

We will use the following lemmas and theorem in our main result.

Lemma 1.1 ([29], Proposition 1.2) *If $F : X \to \mathcal{P}_{\mathrm{cl}}(Y)$ is u.s.c., then $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$. Conversely, if F is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.2 ([32]) *Let X be a separable Banach space. Let $F : [0,1] \times X^{k+3} \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be an L^1-Carathéodory function. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{\mathrm{cp,cv}}\big(C(J,X)\big), \qquad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator.

Theorem 1.3 ([33], Krasnoselskii's fixed point theorem) *Let X be a Banach space, $Y \in \mathcal{P}_{\mathrm{b,cl,cv}}(X)$ and $A, B : Y \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ two multi-valued operators. If the following conditions are satisfied*:

(i) *$Ay + By \subset Y$ for all $y \in Y$*;
(ii) *A is a contraction*;
(iii) *B is u.s.c. and compact,*

then there exists $y \in Y$ such that $y \in Ay + By$.

2 Main results

Now, we are ready to prove our main result. Let $X = \{u : u, u', u'', {}^cD^{q_i}u \in C(J,\mathbb{R}), i = 1,2,\ldots,k\}$. Then $(X, \|\cdot\|)$ endowed with the norm

$$\|u\| = \sup_{t\in J}\big|u(t)\big| + \sup_{t\in J}\big|u'(t)\big| + \sup_{t\in J}\big|u''(t)\big| + \sup_{t\in J}\big|{}^cD^{q_i}u(t)\big| \quad (i = 1,\ldots,k)$$

is a Banach space [34].

We need the following auxiliary lemma. See also [35, 36].

Lemma 2.1 *Let $y \in C(J,\mathbb{R})$ and $u \in C^2([0,1],\mathbb{R})$ is a solution to the fractional boundary value problem*

$$\begin{cases} {}^cD^\alpha u(t) = y(t), \\ u(0) = 0, \qquad u'(0) = -u(1) - u'(1), \qquad u''(0) = -u''(1) - {}^cD^p u(1), \end{cases} \tag{2.1}$$

then

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\,ds \\ & + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} y(s)\,ds + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds, \end{aligned} \tag{2.2}$$

and vice versa, where $\Delta = \frac{\Gamma(3-p)}{4\Gamma(3-p)+2} \neq 0$.

Proof It is well known that the solution of equation ${}^cD^\alpha u(t) = y(t)$ can be written as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + c_0 + c_1 t + c_2 t^2, \tag{2.3}$$

where $c_0, c_1, c_2 \in \mathbb{R}$. Then we get

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\,ds + c_1 + 2c_2 t,$$
$$u''(t) = \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} y(s)\,ds + 2c_2$$

and

$${}^cD^p u(t) = \int_0^t \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds + c_2 \frac{2t^{2-p}}{\Gamma(3-p)} \quad (1 < p \le 2).$$

By using the boundary value conditions, we obtain $c_0 = 0$ and

$$\begin{aligned} c_1 = {} & -\frac{1}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \frac{1}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\,ds \\ & + \Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} y(s)\,ds + \Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds \end{aligned}$$

and

$$c_2 = -\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} y(s)\,ds - \Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds.$$

Substituting the values of c_0, c_1, and c_2 in (2.3) we get (2.2).

Conversely, applying the operator ${}^cD^\alpha$ on (2.2) and taking into account (2.1), it follows that ${}^cD^\alpha u(t) = y(t)$. From (2.2) it is easily to verify that the boundary conditions $u(0) = 0$, $u'(0) = -u(1) - u'(1)$, $u''(0) = -u''(1) - {}^cD^p u(1)$ are satisfied. This establishes the equivalence between (2.1) and (2.2). The proof is completed. □

Definition 2.2 A function $u \in C^2([0,1], \mathbb{R})$ is called a solution for the problem (1.1)-(1.2) if it satisfies the boundary value conditions $u(0) = 0$, $u'(0) = -u(1) - u'(1)$, and $u''(0) = -u''(1) - {}^cD^p u(1)$, there exist functions $v, v_1 \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, u(t), u'(t), u''(t), {}^cD^{q_1} u(t), \dots, {}^cD^{q_k} u(t))$, $v_1(t) \in G(t, u(t), u'(t), u''(t), {}^cD^{q_1} u(t), \dots, {}^cD^{q_k} u(t))$ for almost all $t \in J$ and

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds \\ & - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)\,ds + (t-t^2)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v(s)\,ds \\ & + (t-t^2)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v(s)\,ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds \end{aligned}$$

$$- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_1(s)\,ds + \bigl(t-t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v_1(s)\,ds$$
$$+ \bigl(t-t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v_1(s)\,ds. \tag{2.4}$$

Remark 2.3 For the sake of brevity, we set

$$\Lambda_1 = \frac{4}{3\Gamma(\alpha+1)} + \frac{1}{3\Gamma(\alpha)} + \frac{\Delta}{4\Gamma(\alpha-1)} + \frac{\Delta}{4\Gamma(\alpha-p+1)}, \tag{2.5}$$
$$\Lambda_2 = \frac{4}{3\Gamma(\alpha)} + \frac{1}{3\Gamma(\alpha+1)} + \frac{\Delta}{\Gamma(\alpha-1)} + \frac{\Delta}{\Gamma(\alpha-p+1)}, \tag{2.6}$$
$$\Lambda_3 = \frac{1+2\Delta}{\Gamma(\alpha-1)} + \frac{2\Delta}{\Gamma(\alpha-p+1)}, \tag{2.7}$$

and, for each $i = 1,\dots,k$,

$$\Lambda_4^i = \frac{1}{\Gamma(\alpha-q_i+1)} + \frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-1)} + \frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-p+1)}. \tag{2.8}$$

Also in the following we use the notation $\|x\|_\infty = \sup\{|x(t)| : t \in J\}$.

Theorem 2.4 *Suppose that*:

(H_1) $F : J \times \mathbb{R}^{k+3} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a multifunction and* $G : J \times \mathbb{R}^{k+3} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory multifunction*;

(H_2) *there exist continuous functions* $p, m : J \to (0,\infty)$ *such that* $t \mapsto F(t, w_1, w_2, w_3, z_1, \dots, z_k)$ *is measurable and*

$$\bigl\|F(t, w_1, w_2, w_3, z_1, \dots, z_k)\bigr\| \le m(t), \qquad \bigl\|G(t, w_1, w_2, w_3, z_1, \dots, z_k)\bigr\| \le p(t);$$

(H_3) *there exists a continuous function* $h : J \to (0,\infty)$ *such that*

$$H_d\bigl(F(t, w_1, w_2, w_3, z_1, \dots, z_k), F\bigl(t, w_1', w_2', w_3', z_1', \dots, z_k'\bigr)\bigr)$$
$$\le h(t)\left[\sum_{i=1}^{3}\bigl|w_i - w_i'\bigr| + \sum_{i=1}^{k}\bigl|z_i - z_i'\bigr|\right]$$

for all $t \in J$ *and for each* $w_1, w_2, w_3, z_1, \dots, z_k, w_1', w_2', w_3', z_1', \dots, z_k' \in \mathbb{R}$.

If

$$L := \|h\|_\infty\bigl(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4^i\bigr) < 1$$

for $i = 1, 2, \dots, k$, *where the* Λ_j ($j = 1, \dots, 4$) *are defined in* (2.5)-(2.8), *then the inclusion problem* (1.1)-(1.2) *has at least one solution.*

Proof We define the subset Y of X by $Y = \{u \in X : \|u\| \le M\}$, where

$$M = \bigl(\|p\|_\infty + \|m\|_\infty\bigr)\bigl(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4^i\bigr) \quad (i = 1, \dots, k).$$

It is clear that Y is closed, bounded, and convex subset of Banach space X. We define the multi-valued operators $A, B : Y \to \mathcal{P}(X)$ such that for some $v \in S_{F,u}$,

$$\begin{aligned} A(u) = \bigg\{ u \in X : u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)\,ds + \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v(s)\,ds \\ &+ \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v(s)\,ds \bigg\}, \end{aligned}$$

and for some $v_1 \in S_{G,u}$,

$$\begin{aligned} B(u) = \bigg\{ u \in X : u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_1(s)\,ds + \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v_1(s)\,ds \\ &+ \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v_1(s)\,ds \bigg\}. \end{aligned}$$

In this way, the fractional differential inclusion (1.1)-(1.2) is equivalent to the inclusion problem $u \in Au + Bu$. We show that the multi-valued operators A and B satisfy the conditions of Theorem 1.3 on Y.

First, we show that the operators A and B define the multi-valued operators $A, B : Y \to \mathcal{P}_{\mathrm{cp,cv}}(X)$. First we prove that A is compact-valued on Y. Note that the operator A is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1(J, \mathbb{R})$ into X, defined by

$$\begin{aligned} \mathcal{L}(v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)\,ds + \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v(s)\,ds \\ &+ \big(t-t^2\big)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v(s)\,ds. \end{aligned}$$

Suppose that $u \in Y$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,u}$. Then, by definition of $S_{F,u}$, we have $v_n(t) \in F(t, u(t), u'(t), u''(t), {}^cD^{q_1}u(t), \ldots, {}^cD^{q_k}u(t))$ for almost all $t \in J$. Since $F(t, u(t), u'(t), u''(t), {}^cD^{q_1}u(t), \ldots, {}^cD^{q_k}u(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,u}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on J.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned} &\big|\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)\big| \\ &\quad \le \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] v_n(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} v_n(s)\,ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{|t_2-t_1|}{3\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\big|v_n(s)\big|\,ds+\frac{|t_2-t_1|}{3\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\big|v_n(s)\big|\,ds\\
&+\frac{\Delta|[(t_2-t_1)-(t_2^2-t_1^2)]|}{\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\big|v_n(s)\big|\,ds\\
&+\frac{\Delta|[(t_2-t_1)-(t_2^2-t_1^2)]|}{\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}\big|v_n(s)\big|\,ds\\
\le{}&\|m\|_\infty\left\{\frac{|t_2^\alpha-t_1^\alpha|}{\Gamma(\alpha+1)}+\frac{|t_2-t_1|}{3\Gamma(\alpha+1)}+\frac{|t_2-t_1|}{3\Gamma(\alpha)}+\frac{\Delta|[(t_2-t_1)-(t_2^2-t_1^2)]|}{\Gamma(\alpha-1)}\right.\\
&\left.+\frac{\Delta|[(t_2-t_1)-(t_2^2-t_1^2)]|}{\Gamma(\alpha-p+1)}\right\}.
\end{aligned}
$$

Continuing this process, we have

$$
\big|\big(\mathcal{L}'(v_n)(t_2)\big)-\big(\mathcal{L}'(v_n)(t_1)\big)\big|\le\|m\|_\infty\left\{\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\Gamma(\alpha)}+\frac{\Delta|t_2-t_1|}{\Gamma(\alpha-1)}+\frac{\Delta|t_2-t_1|}{\Gamma(\alpha-p+1)}\right\}
$$

and

$$
\begin{aligned}
\big|\big(\mathcal{L}''(v_n)(t_2)\big)-\big(\mathcal{L}''(v_n)(t_1)\big)\big|&\le\left|\frac{1}{\Gamma(\alpha-2)}\int_0^{t_1}\big[(t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}\big]v_n(s)\,ds\right.\\
&\quad\left.+\frac{1}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-3}v_n(s)\,ds\right|\\
&\le\|m\|_\infty\frac{|t_2^{\alpha-2}-t_1^{\alpha-2}|}{\Gamma(\alpha-1)},
\end{aligned}
$$

and, finally, for every $i=1,\dots,k$,

$$
\begin{aligned}
&\big|\big({}^cD^{q_i}\mathcal{L}(v_n)(t_2)\big)-\big({}^cD^{q_i}\mathcal{L}(v_n)(t_1)\big)\big|\\
&\quad\le\|m\|_\infty\left\{\frac{|t_2^{\alpha-q_i}-t_1^{\alpha-q_i}|}{\Gamma(\alpha-q_i+1)}+\frac{2\Delta|t_2^{2-q_i}-t_1^{2-q_i}|}{\Gamma(3-q_i)\Gamma(\alpha-1)}+\frac{2\Delta|t_2^{2-q_i}-t_1^{2-q_i}|}{\Gamma(3-q_i)\Gamma(\alpha-p+1)}\right\}.
\end{aligned}
$$

We see that the right-hand sides of the above inequalities tend to zero as $t_2\to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equi-continuous and by using the Arzelá-Ascoli theorem, we see that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n)\to\mathcal{L}(v)$. Note that $\mathcal{L}(v)\in\mathcal{L}(S_{F,u})$. Hence, $A(u)=\mathcal{L}(S_{F,u})$ is compact for all $u\in Y$. So $A(u)$ is compact.

Now, we show that $A(u)$ is convex for all $u\in X$. Let $z_1,z_2\in A(u)$. We select $f_1,f_2\in S_{F,u}$ such that

$$
\begin{aligned}
z_i(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f_i(s)\,ds-\frac{t}{3}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f_i(s)\,ds\\
&-\frac{t}{3}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_i(s)\,ds+\big(t-t^2\big)\Delta\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}f_i(s)\,ds\\
&+\big(t-t^2\big)\Delta\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}f_i(s)\,ds,\quad i=1,2
\end{aligned}
$$

for almost all $t \in J$. Let $0 \leq \lambda \leq 1$. Then we have

$$\begin{aligned}\big[\lambda z_1 + (1-\lambda)z_2\big](t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big[\lambda f_1(s) + (1-\lambda)f_2(s)\big]\,ds \\ &\quad - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big[\lambda f_1(s) + (1-\lambda)f_2(s)\big]\,ds \\ &\quad - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big[\lambda f_1(s) + (1-\lambda)f_2(s)\big]\,ds \\ &\quad + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\big[\lambda f_1(s) + (1-\lambda)f_2(s)\big]\,ds \\ &\quad + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\big[\lambda f_1(s) + (1-\lambda)f_2(s)\big]\,ds.\end{aligned}$$

Since F has convex values, $S_{F,u}$ is convex and $\lambda f_1(s) + (1-\lambda)f_2(s) \in S_{F,u}$. Thus

$$\lambda z_1 + (1-\lambda)z_2 \in A(u).$$

Consequently, A is convex-valued. Similarly, B is compact and convex-valued.

Here, we show that $A(u) + B(u) \subset Y$ for all $u \in Y$. Suppose that $u \in Y$ and $z_1 \in A(u)$, $z_2 \in B(u)$ are arbitrary elements. Choose $v_1 \in S_{F,u}$ and $v_2 \in S_{G,u}$ such that

$$\begin{aligned}z_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds \\ &\quad - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(s)\,ds + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}v_1(s)\,ds \\ &\quad + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}v_1(s)\,ds\end{aligned}$$

and

$$\begin{aligned}z_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_2(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_2(s)\,ds \\ &\quad - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}v_2(s)\,ds + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}v_2(s)\,ds \\ &\quad + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}v_2(s)\,ds\end{aligned}$$

for almost all $t \in J$. Hence, we get

$$\begin{aligned}\big|z_1(t) + z_2(t)\big| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(\big|v_1(s)\big| + \big|v_2(s)\big|\big)\,ds \\ &\quad + \frac{|t|}{3\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\big(\big|v_1(s)\big| + \big|v_2(s)\big|\big)\,ds \\ &\quad + \frac{|t|}{3\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\big(\big|v_1(s)\big| + \big|v_2(s)\big|\big)\,ds \\ &\quad + \frac{\Delta|t-t^2|}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\big(\big|v_1(s)\big| + \big|v_2(s)\big|\big)\,ds\end{aligned}$$

$$+\frac{\Delta|t-t^2|}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}\big(|v_1(s)|+|v_2(s)|\big)\,ds$$
$$\leq \big(\|p\|_\infty+\|m\|_\infty\big)\left\{\frac{4}{3\Gamma(\alpha+1)}+\frac{1}{3\Gamma(\alpha)}+\frac{\Delta}{4\Gamma(\alpha-1)}+\frac{\Delta}{4\Gamma(\alpha-p+1)}\right\}.$$

Hence, $\sup_{t\in J}|z_1(t)+z_2(t)|\leq(\|p\|_\infty+\|m\|_\infty)\Lambda_1$. Also we have

$$\big|z_1'(t)+z_2'(t)\big|\leq\big(\|p\|_\infty+\|m\|_\infty\big)\left\{\frac{4}{3\Gamma(\alpha)}+\frac{1}{3\Gamma(\alpha+1)}+\frac{\Delta}{\Gamma(\alpha-1)}+\frac{\Delta}{\Gamma(\alpha-p+1)}\right\},$$

which implies that $\sup_{t\in J}|z_1'(t)+z_2'(t)|\leq(\|p\|_\infty+\|m\|_\infty)\Lambda_2$ and

$$\big|z_1''(t)+z_2''(t)\big|\leq\big(\|p\|_\infty+\|m\|_\infty\big)\left\{\frac{1+2\Delta}{\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(\alpha-p+1)}\right\}$$

from which $\sup_{t\in J}|z_1''(t)+z_2''(t)|\leq(\|p\|_\infty+\|m\|_\infty)\Lambda_3$. Finally, for all $i=1,\ldots,k$, we have

$$\big|{}^cD^{q_i}z_1(t)+{}^cD^{q_i}z_2(t)\big|$$
$$\leq\big(\|p\|_\infty+\|m\|_\infty\big)\left\{\frac{1}{\Gamma(\alpha-q_i+1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-p+1)}\right\},$$

and so $\sup_{t\in J}|{}^cD^{q_i}z_1(t)+{}^cD^{q_i}z_2(t)|\leq(\|p\|_\infty+\|m\|_\infty)\Lambda_4^i$, $i=1,2,\ldots,k$. Hence, it follows that

$$\|z_1+z_2\|\leq\big(\|p\|_\infty+\|m\|_\infty\big)\big(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4^i\big)=M,\quad i=1,2,\ldots,k.$$

Now, we show that the operator B is compact on Y. To do this, it is enough to prove that $B(Y)$ is uniformly bounded and equi-continuous in X. Let $z\in B(Y)$ be arbitrary. For some $u\in Y$, choose $v_1\in S_{G,u}$ such that

$$\begin{aligned}z(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds-\frac{t}{3}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds\\&-\frac{t}{3}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(s)\,ds+\big(t-t^2\big)\Delta\int_0^1\frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}v_1(s)\,ds\\&+\big(t-t^2\big)\Delta\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}v_1(s)\,ds,\quad t\in J.\end{aligned}\tag{2.9}$$

Hence,

$$\big|z(t)\big|\leq\|p\|_\infty\left\{\frac{4}{3\Gamma(\alpha+1)}+\frac{1}{3\Gamma(\alpha)}+\frac{\Delta}{4\Gamma(\alpha-1)}+\frac{\Delta}{4\Gamma(\alpha-p+1)}\right\},$$
$$\big|z'(t)\big|\leq\|p\|_\infty\left\{\frac{4}{3\Gamma(\alpha)}+\frac{1}{3\Gamma(\alpha+1)}+\frac{\Delta}{\Gamma(\alpha-1)}+\frac{\Delta}{\Gamma(\alpha-p+1)}\right\},$$
$$\big|z''(t)\big|\leq\|p\|_\infty\left\{\frac{1+2\Delta}{\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(\alpha-p+1)}\right\},$$
$$\big|{}^cD^{q_i}z(t)\big|\leq\|p\|_\infty\left\{\frac{1}{\Gamma(\alpha-q_i+1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-p+1)}\right\}$$

for $i=1,\ldots,k$. Hence, $\|z\|\leq\|p\|_\infty(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4^i)$, $i=1,\ldots,k$.

Now, we show that B maps Y to equi-continuous subsets of X. Let $t_1, t_2 \in J$ with $t_1 < t_2$, $u \in Y$, and $z \in B(u)$. Choose $v_1 \in S_{G,u}$ such that $z(t)$ is given by (2.9). Then we have

$$\begin{aligned} \bigl|z(t_2) - z(t_1)\bigr| &\le \|p\|_\infty \biggl\{ \frac{|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)} + \frac{|t_2 - t_1|}{3\Gamma(\alpha+1)} + \frac{|t_2 - t_1|}{3\Gamma(\alpha)} + \frac{\Delta|[(t_2 - t_1) - (t_2^2 - t_1^2)]|}{\Gamma(\alpha-1)} \\ &\quad + \frac{\Delta|[(t_2 - t_1) - (t_2^2 - t_1^2)]|}{\Gamma(\alpha-p+1)} \biggr\}, \\ \bigl|z'(t_2) - z'(t_1)\bigr| &\le \|p\|_\infty \biggl\{ \frac{|t_2^{\alpha-1} - t_1^{\alpha-1}|}{\Gamma(\alpha)} + \frac{2\Delta|t_2 - t_1|}{\Gamma(\alpha-1)} + \frac{2\Delta|t_2 - t_1|}{\Gamma(\alpha-p+1)} \biggr\}, \\ \bigl|z''(t_2) - z''(t_1)\bigr| &\le \|p\|_\infty \frac{|t_2^{\alpha-2} - t_1^{\alpha-2}|}{\Gamma(\alpha-1)} \end{aligned}$$

and

$$\begin{aligned} &\bigl|{}^cD^{q_i} z(t_2) - {}^cD^{q_i} z(t_1)\bigr| \\ &\quad \le \|p\|_\infty \biggl\{ \frac{|t_2^{\alpha-q_i} - t_1^{\alpha-q_i}|}{\Gamma(\alpha-q_i+1)} + \frac{2\Delta|t_2^{2-q_i} - t_1^{2-q_i}|}{\Gamma(3-q_i)\Gamma(\alpha-1)} + \frac{2\Delta|t_2^{2-q_i} - t_1^{2-q_i}|}{\Gamma(3-q_i)\Gamma(\alpha-p+1)} \biggr\} \end{aligned}$$

for each $i = 1, \ldots, k$. It is seen that the right-hand sides of the above inequalities tend to zero as $t_2 \to t_1$. Hence, by using the Arzelá-Ascoli theorem, B is compact.

Next, we prove that B has a closed graph. Let $u_n \in Y$ and $z_n \in B(u_n)$ for all n such that $u_n \to u_0$ and $z_n \to z_0$. We show that $z_0 \in B(u_0)$. Associated with $z_n \in B(u_n)$ for each $n \in \mathbb{N}$, there exists $v_n \in S_{G,u_n}$ such that

$$\begin{aligned} z_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\,ds - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)\,ds \\ &\quad - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_n(s)\,ds + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v_n(s)\,ds \\ &\quad + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v_n(s)\,ds \end{aligned}$$

for all $t \in J$. It suffices to show that there exists $v_0 \in S_{G,u_0}$ such that, for each $t \in J$,

$$\begin{aligned} z_0(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_0(s)\,ds - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_0(s)\,ds \\ &\quad - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_0(s)\,ds + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v_0(s)\,ds \\ &\quad + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v_0(s)\,ds. \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(J, \mathbb{R}) \to X$ by

$$\begin{aligned} \Theta(v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds \\ &\quad - \frac{t}{3} \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v(s)\,ds + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v(s)\,ds \\ &\quad + \bigl(t - t^2\bigr)\Delta \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v(s)\,ds. \end{aligned}$$

Notice that

$$\begin{aligned}\|z_n(t)-z_0(t)\| = \bigg\| &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(v_n(s)-v_0(s)\big)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big(v_n(s)-v_0(s)\big)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big(v_n(s)-v_0(s)\big)\,ds \\ &+ \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}\big(v_n(s)-v_0(s)\big)\,ds \\ &+ \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\big(v_n(s)-v_0(s)\big)\,ds\bigg\| \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

By using Lemma 1.2, $\Theta\circ S_G$ is a closed graph operator. Since $z_n(t)\in\Theta(S_{G,u_n})$ for all n, and $u_n\to u_0$, there is $v_0\in S_{G,u_0}$ such that

$$\begin{aligned}z_0(t) = {}&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_0(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_0(s)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}v_0(s)\,ds + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}v_0(s)\,ds \\ &+ \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}v_0(s)\,ds.\end{aligned}$$

Hence, $z_0\in B(u_0)$. So, it follows that B has a closed graph and this implies that the operator B is upper semi-continuous.

Finally, we show that A is a contraction multifunction. Let $u,w\in X$ and $z_1\in A(w)$ is given. Then we can select $v_1\in S_{F,w}$ such that

$$\begin{aligned}z_1(t) = {}&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v_1(s)\,ds \\ &- \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}v_1(s)\,ds + \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}v_1(s)\,ds \\ &+ \big(t-t^2\big)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}v_1(s)\,ds\end{aligned}$$

for all $t\in J$. Since

$$\begin{aligned}&H_d\big(F\big(t,u(t),u'(t),u''(t),{}^cD^{q_1}u(t),\dots,{}^cD^{q_k}u(t)\big) \\ &\quad - F\big(t,w(t),w'(t),w''(t),{}^cD^{q_1}w(t),\dots,{}^cD^{q_k}w(t)\big)\big) \\ &\quad\le h(t)\Bigg[\big|u(t)-w(t)\big| + \big|u'(t)-w'(t)\big| + \big|u''(t)-w''(t)\big| + \sum_{i=1}^k\big|{}^cD^{q_i}u(t)-{}^cD^{q_i}w(t)\big|\Bigg]\end{aligned}$$

for almost all $t\in J$, there exists $y\in F(t,u(t),u'(t),u''(t),{}^cD^{q_1}u(t),\dots,{}^cD^{q_k}u(t))$ such that

$$\begin{aligned}\big|v_1(t)-y\big| \le m(t)\Bigg[&\big|u(t)-w(t)\big| + \big|u'(t)-w'(t)\big| + \big|u''(t)-w''(t)\big| \\ &+ \sum_{i=1}^k\big|{}^cD^{q_i}u(t)-{}^cD^{q_i}w(t)\big|\Bigg]\end{aligned}$$

for almost all $t \in J$. Consider the multifunction $U : J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{s \in \mathbb{R} : |v_1(t) - s| \le m(t)g(t) \text{ for almost all } t \in J\},$$

where

$$g(t) = \left[|u(t) - w(t)| + |u'(t) - w'(t)| + |u''(t) - w''(t)| + \sum_{i=1}^{k} |{}^cD^{q_i}u(t) - {}^cD^{q_i}w(t)| \right].$$

Since v_1 and $\varphi = mg$ are measurable, $U(\cdot) \cap F(\cdot, u(\cdot), u'(\cdot), u''(\cdot), {}^cD^{q_1}u(\cdot), \ldots, {}^cD^{q_k}u(\cdot))$ is a measurable multifunction. Thus, we can choose

$$v_2(t) \in F(t, u(t), u'(t), u''(t), {}^cD^{q_1}u(t), \ldots, {}^cD^{q_k}u(t))$$

such that

$$|v_1(t) - v_2(t)| \le m(t)\left[|u(t) - w(t)| + |u'(t) - w'(t)| + |u''(t) - w''(t)| + \sum_{i=1}^{k} |{}^cD^{q_i}u(t) - {}^cD^{q_i}w(t)| \right]$$

and

$$\begin{aligned} z_2(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)\,ds - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)\,ds \\ & - \frac{t}{3}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} v_2(s)\,ds + (t-t^2)\Delta\int_0^1 \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} v_2(s)\,ds \\ & + (t-t^2)\Delta\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} v_2(s)\,ds \end{aligned}$$

for all $t \in J$. Now, we have

$$\begin{aligned} |z_1(t) - z_2(t)| \le {}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_1(s) - v_2(s)|\,ds \\ & + \frac{|t|}{3\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|v_1(s) - v_2(s)|\,ds \\ & + \frac{|t|}{3\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}|v_1(s) - v_2(s)|\,ds \\ & + \frac{\Delta|t-t^2|}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}|v_1(s) - v_2(s)|\,ds \\ & + \frac{\Delta|t-t^2|}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}|v_1(s) - v_2(s)|\,ds \\ \le {}& \|h\|_\infty\left\{ \frac{4}{3\Gamma(\alpha+1)} + \frac{1}{3\Gamma(\alpha)} + \frac{\Delta}{4\Gamma(\alpha-1)} + \frac{\Delta}{4\Gamma(\alpha-p+1)} \right\}\|u - w\|. \end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|z_1'(t)-z_2'(t)\right| \le \|h\|_\infty \left\{\frac{4}{3\Gamma(\alpha)}+\frac{1}{3\Gamma(\alpha+1)}+\frac{\Delta}{\Gamma(\alpha-1)}+\frac{\Delta}{\Gamma(\alpha-p+1)}\right\}\|u-w\|, \\
&\left|z_1''(t)-z_2''(t)\right| \le \|h\|_\infty \left\{\frac{1+2\Delta}{\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(\alpha-p+1)}\right\}\|u-w\|, \\
&\left|{}^cD^{q_i}z_1(t)-{}^cD^{q_i}z_2(t)\right| \\
&\quad \le \|h\|_\infty \left\{\frac{1}{\Gamma(\alpha-q_i+1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-1)}+\frac{2\Delta}{\Gamma(3-q_i)\Gamma(\alpha-p+1)}\right\}\|u-w\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sup_{t\in J}\left|z_1(t)-z_2(t)\right| \le \|h\|_\infty \Lambda_1\|u-w\|, \\
&\sup_{t\in J}\left|z_1'(t)-z_2'(t)\right| \le \|h\|_\infty \Lambda_2\|u-w\|, \\
&\sup_{t\in J}\left|z_1''(t)-z_2''(t)\right| \le \|h\|_\infty \Lambda_3\|u-w\|, \\
&\sup_{t\in J}\left|{}^cD^{q_i}z_1(t)-{}^cD^{q_i}z_2(t)\right| \le \|h\|_\infty \Lambda_4^i\|u-w\|
\end{aligned}$$

for each $1\le i\le k$. So

$$\|z_1-z_2\| \le \|h\|_\infty\left(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4^i\right)\|u-w\|, \quad i=1,2,\dots,k.$$

This implies that $H_d(A(u),A(w)) \le L\|u-w\|$. Thus A and B satisfy all the conditions of Theorem 1.3 and so the inclusion $u\in A(u)+B(u)$ has a solution in Y. Therefore the inclusion problem (1.1)-(1.2) has a solution in Y and the proof is completed. □

Finally, we give an example to illustrate the validity of our main result.

Example 2.5 Consider the following fractional differential inclusion:

$$\begin{aligned}
{}^cD^{\frac{5}{2}}u(t) \in{}& \left[0, \frac{t|u(t)|^3}{100(1+|u(t)|^3)}+\frac{t|2\sin(u'(t))|}{200(|\sin(u'(t))|+1)}+\frac{0.01t|u''(t)|}{|u''(t)|+1}\right. \\
&\left.+\frac{t|\cos({}^cD^{\frac{3}{2}}u(t))|}{100(1+|\cos({}^cD^{\frac{3}{2}}u(t))|)}+\frac{t^2|\sin\frac{\pi}{2}t||{}^cD^{\frac{3}{2}}u(t)|^2}{100t(|{}^cD^{\frac{3}{2}}u(t)|^2+1)}\right] \\
&+\left[0, \frac{e^{-t}|u(t)|}{(1+e^t)(1+|u(t)|)}+\frac{|\cos\pi t||u'(t)|e^{-t}}{(1+e^t)(1+|u'(t)|)}+\frac{e^{-t}|u''(t)|^2}{(1+|u''(t)|^2)(1+e^t)}\right. \\
&\left.+\frac{e^{-2t}|\sin({}^cD^{\frac{3}{2}}u(t))|}{(1+|\sin({}^cD^{\frac{3}{2}}u(t))|)e^t(1+e^t)}+\frac{e^{-3t}|{}^cD^{\frac{3}{2}}u(t)|^3}{(e^{2t}+e^{3t})(1+|{}^cD^{\frac{3}{2}}u(t)|^3)}\right],
\end{aligned} \tag{2.10}$$

with the following boundary conditions:

$$u(0)=0, \qquad u'(0)=-u(1)-u'(1), \qquad u''(0)=-u''(1)-{}^cD^{\frac{3}{2}}u(1), \tag{2.11}$$

where $t\in[0,1]$. In the above inclusion problem, we have $\alpha=5/2$, $p=3/2$, $k=2$, and $q_1=q_2=3/2$. Also, we have $\Delta=0.1597$.

Now, we define $F : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(t,x,y,z,v,w) = \left[0, \frac{t|x|^3}{100(1+|x|^3)} + \frac{t|2\sin y|}{200(|\sin y|+1)} + \frac{0.01t|z|}{|z|+1} + \frac{t|\cos v|}{100(1+|\cos v|)} + \frac{t^2|\sin\frac{\pi}{2}t|w^2}{100t(w^2+1)}\right],$$

and $G : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$G(t,x,y,z,v,w) = \left[0, \frac{e^{-t}|x|}{(1+e^t)(1+|x|)} + \frac{|\cos\pi t||y|e^{-t}}{(1+e^t)(1+|y|)} + \frac{e^{-t}z^2}{(1+z^2)(1+e^t)} + \frac{e^{-2t}|\sin v|}{(1+|\sin v|)e^t(1+e^t)} + \frac{e^{-3t}|w|^3}{(e^{2t}+e^{3t})(1+|w|^3)}\right].$$

Then there exist continuous functions $m, p : [0,1] \to (0,\infty)$ given by

$$m(t) = 5 + \frac{t}{100}, \qquad p(t) = \frac{e^{-t}}{1+e^t}.$$

On the other hand, we can easily check that, for every $x_i, y_i, z_i, v_i, w_i \in \mathbb{R}$ $(i = 1,2)$,

$$H_d\big(F(t,x_1,y_1,z_1,v_1,w_1) - F(t,x_2,y_2,z_2,v_2,w_2)\big) \leq h(t)\big(|x_1-x_2| + |y_1-y_2| + |z_1-z_2| + |v_1-v_2| + |w_1-w_2|\big),$$

where $h : [0,1] \to (0,\infty)$ is defined by $h(t) = \frac{t}{100}$. It can easily be found that $\Lambda_1 = 0.7369$, $\Lambda_2 = 1.4434$, $\Lambda_3 = 1.8102$, $\Lambda_4^1 = 1.7687$, and $\Lambda_4^2 = 1.7687$. Since $\|h\|_\infty = \frac{1}{100}$, we have $L := \|h\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4^1 + \Lambda_4^2) = 0.01 \times 7.5279 = 0.075279 < 1$. Consequently all assumptions and conditions of Theorem 2.4 are satisfied. Hence, Theorem 2.4 implies that the fractional differential inclusion problem (2.10)-(2.11) has a solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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