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A fourth order accurate approximation of the first and pure second derivatives of the Laplace equation on a rectangle

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Abstract

In this paper, we discuss an approximation of the first and pure second order derivatives for the solution of the Dirichlet problem on a rectangular domain. The boundary values on the sides of the rectangle are supposed to have the sixth derivatives satisfying the Hölder condition. On the vertices, besides the continuity condition, the compatibility conditions, which result from the Laplace equation for the second and fourth derivatives of the boundary values, given on the adjacent sides, are also satisfied. Under these conditions a uniform approximation of order $O(h^4)$ (h is the grid size) is obtained for the solution of the Dirichlet problem on a square grid, its first and pure second derivatives, by a simple difference scheme. Numerical experiments are illustrated to support the analysis made.

Keywords: finite difference method; approximation of derivatives; uniform error; Laplace equation

1 Introduction

Since the operation of differentiation is ill conditioned, to find a highly accurate approximation for the derivatives of the solution of a differential equation becomes problematic, especially when the smoothness is restricted.

In [1], it was proved that the higher order difference derivatives uniformly converge to the corresponding derivatives of the solution of the Laplace equation in any strictly interior subdomain, with the same order of h as which the difference solution converges on the given domain. In [2], by using the difference solution of the Dirichlet problem for the Laplace equation on a rectangle, the uniform convergence of its first and pure second divided difference over the whole grid domain to the corresponding derivatives of the exact solution with the rate $O(h^2)$ is proved. In [3], the difference schemes on a rectangular parallelepiped were constructed, where solutions approximate the Dirichlet problem for the Laplace equation and its first and second derivatives. Under the assumptions that the boundary functions belong to $C^{4,\lambda}$, $0 < \lambda < 1$, on the faces, are continuous on the edges, and their second-order derivatives satisfy the compatibility condition, the solution to their difference schemes converges uniformly on the grid with the rate of $O(h^2)$.

In this paper, we consider the Dirichlet problem for the Laplace equation on a rectangle, when the boundary values belong to $C^{6,\lambda}$, $0 < \lambda < 1$, on the sides of the rectangle and as a whole are continuous on the vertices. Also the 2τ, $\tau = 1, 2$, order derivatives satisfy the compatibility conditions on the vertices which result from the Laplace equation. Under these conditions, we construct the difference problems, the solutions of which converge to the first and pure second derivatives of the exact solution with the order $O(h^4)$. Finally, numerical experiments are given in the last part of the paper to support the theoretical results.

2 The Dirichlet problem on rectangular domains

Let $\Pi = \{(x,y) : 0 < x < a, 0 < y < b\}$ be a rectangle, a/b be rational, γ_j (γ_j'), $j = 1,2,3,4$, be the sides, including (excluding) the ends, enumerated counterclockwise starting from the left side ($\gamma_0 \equiv \gamma_4$, $\gamma_5 \equiv \gamma_1$), and let $\gamma = \bigcup_{j=1}^{4} \gamma_j$ be the boundary of Π. Denote by s the arclength, measured along γ, and by s_j the value of s at the beginning of γ_j. We say that $f \in C^{k,\lambda}(D)$, if f has kth derivatives on D satisfying a Hölder condition with exponent $\lambda \in (0,1)$.

We consider the boundary value problem

$$\Delta u = 0 \quad \text{on } \Pi, \qquad u = \varphi_j(s) \quad \text{on } \gamma_j, j = 1,2,3,4, \tag{1}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j are given functions of s. Assume that

$$\varphi_j \in C^{6,\lambda}(\gamma_j), \quad 0 < \lambda < 1, j = 1,2,3,4, \tag{2}$$

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \quad q = 0,1,2. \tag{3}$$

Lemma 2.1 *The solution u of problem* (1) *is from* $C^{5,\lambda}(\overline{\Pi})$.

The proof of Lemma 2.1 follows from Theorem 3.1 in [4].

Lemma 2.2 *The inequality is true*

$$\max_{0 \le p \le 3} \sup_{(x,y)\in\Pi} \left| \frac{\partial^6 u}{\partial x^{2p}\, \partial y^{6-2p}} \right| < \infty, \tag{4}$$

where u is the solution of problem (1).

Proof From Lemma 2.1 it follows that the functions $\frac{\partial^4 u}{\partial x^4}$ and $\frac{\partial^4 u}{\partial y^4}$ are continuous on $\overline{\Pi}$. We put $w = \frac{\partial^4 u}{\partial x^4}$. The function w is harmonic in Π and is the solution of the problem

$$\Delta w = 0 \quad \text{on } \Pi, \qquad w = \Phi_j \quad \text{on } \gamma_j, j = 1,2,3,4,$$

where

$$\Phi_\tau = \frac{\partial^4 \varphi_\tau}{\partial y^4}, \quad \tau = 1,3,$$

$$\Phi_\nu = \frac{\partial^4 \varphi_\nu}{\partial x^4}, \quad \nu = 2,4.$$

From the conditions (2) and (3) it follows that

$$\Phi_j \in C^{2,\lambda}(\gamma_j), \quad 0 < \lambda < 1, \qquad \Phi_j(s_j) = \Phi_{j-1}(s_j), \quad j = 1,2,3,4.$$

Hence, on the basis of Theorem 6.1 in [5], we have

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^2 w}{\partial x^2}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^6}\right| < \infty, \tag{5}$$

$$\sup_{(x,y)\in\Pi}\left|\frac{\partial^2 w}{\partial y^2}\right| = \sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^4\,\partial y^2}\right| < \infty. \tag{6}$$

Similarly, it is proved that

$$\sup_{(x,y)\in\Pi}\left\{\left|\frac{\partial^6 u}{\partial y^6}\right|,\left|\frac{\partial^6 u}{\partial y^4\,\partial x^2}\right|\right\} < \infty. \tag{7}$$

From (5)-(7), estimation (4) follows. □

Lemma 2.3 *Let $\rho(x,y)$ be the distance from a current point of the open rectangle Π to its boundary and let $\partial/\partial l \equiv \alpha\partial/\partial x + \beta\partial/\partial y$, $\alpha^2+\beta^2=1$. Then the next inequality holds*:

$$\left|\frac{\partial^8 u}{\partial l^8}\right| \le c\rho^{-2}, \tag{8}$$

where c is a constant independent of the direction of the derivative $\partial/\partial l$, u is a solution of problem (1).

Proof According to Lemma 2.2, we have

$$\max_{0\le p\le 3}\sup_{(x,y)\in\Pi}\left|\frac{\partial^6 u}{\partial x^{2p}\,\partial y^{6-2p}}\right| \le c < \infty.$$

Since any eighth order derivative can be obtained by two times differentiating some of the derivatives $\partial^6/\partial x^{2p}\,\partial y^{6-2p}$, $0\le p\le 3$, on the basis of estimations (29) and (30) from [6], we obtain

$$\max_{\nu+\mu=8}\left|\frac{\partial^8 u}{\partial x^\nu\,\partial y^\mu}\right| \le c_1\rho^{-2}(x,y) < \infty. \tag{9}$$

From (9), inequality (8) follows. □

Let $h>0$, and $a/h\ge 6$, $b/h\ge 6$ be integers. We assign Π^h, a square net on Π, with step h, obtained by the lines $x,y=0,h,2h,\ldots$. Let γ_j^h be a set of nodes on the interior of γ_j, and let

$$\gamma^h=\bigcup_{j=1}^4\gamma_j^h,\qquad \dot\gamma_j=\gamma_{j-1}\cap\gamma_j,\qquad \overline{\gamma}^h=\bigcup_{j=1}^4(\gamma_j^h\cup\dot\gamma_j),\qquad \overline{\Pi}^h=\Pi^h\cup\overline{\gamma}^h.$$

Let the operator B be defined as follows:

$$\begin{aligned}Bu(x,y) &= \bigl(u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)\bigr)/5\\&\quad+\bigl(u(x+h,y+h)+u(x+h,y-h)\\&\quad+u(x-h,y+h)+u(x-h,y-h)\bigr)/20.\end{aligned} \tag{10}$$

We consider the classical 9-point finite difference approximation of problem (1):

$$u_h = Bu_h \quad \text{on } \Pi^h, \qquad u_h = \varphi_j \quad \text{on } \gamma_j^h \cup \dot{\gamma}_j, j = 1,2,3,4. \tag{11}$$

By the maximum principle, problem (11) has a unique solution.

In what follows, and for simplicity, we will denote by $c, c_1, c_2, \ldots$ constants which are independent of h and the nearest factor, and identical notation will be used for various constants.

Let Π^{1h} be the set of nodes of the grid Π^h that are at a distance h from γ, and let $\Pi^{2h} = \Pi^h \setminus \Pi^{1h}$.

Lemma 2.4 *The following inequality holds*:

$$\max_{(x,y)\in(\Pi^{1h}\cup\Pi^{2h})} |Bu - u| \leq ch^6, \tag{12}$$

where u is a solution of problem (1).

Proof Let (x_0, y_0) be a point of Π^{1h}, and let

$$R_0 = \{(x,y) : |x - x_0| < h, |y - y_0| < h\} \tag{13}$$

be an elementary square, some sides of which lie on the boundary of the rectangle Π. On the vertices of R_0 and on the mid-points of its sides lie the nodes of which the function values are used to evaluate $Bu(x_0, y_0)$.

We represent a solution of problem (1) in some neighborhood of $(x_0, y_0) \in \Pi^{1h}$ by Taylor's formula

$$u(x,y) = p_7(x,y) + r_8(x,y), \tag{14}$$

where $p_7(x,y)$ is the seventh order Taylor's polynomial, $r_8(x,y)$ is the remainder term. Taking into account that the function u is harmonic, by exhaustive calculations, we have

$$Bp_7(x_0, y_0) = u(x_0, y_0). \tag{15}$$

Now, we estimate r_8 at the nodes of the operator B. We take a node $(x_0 + h, y_0 + h)$ which is one of the eight nodes of B, and we consider the function

$$\widetilde{u}(s) = u\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \quad -\sqrt{2}h \leq s \leq \sqrt{2}h \tag{16}$$

of one variable s. By virtue of Lemma 2.3, we have

$$\left|\frac{d^8\widetilde{u}(s)}{ds^8}\right| \leq c(\sqrt{2}h - s)^{-2}, \quad 0 \leq s < \sqrt{2}h. \tag{17}$$

We represent function (16) around the point $s = 0$ by Taylor's formula

$$\widetilde{u}(s) = \widetilde{p}_7(s) + \widetilde{r}_8(s),$$

where $\widetilde{p}_7(s) \equiv p_7(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}})$ is the seventh order Taylor's polynomial of the variable s, and

$$\widetilde{r}_8(s) \equiv r_8\left(x_0 + \frac{s}{\sqrt{2}}, y_0 + \frac{s}{\sqrt{2}}\right), \quad 0 \le |s| < \sqrt{2}h, \tag{18}$$

is the remainder term.

On the basis of (17) and the integral form of the remainder term of Taylor's formula, we have

$$\left|\widetilde{r}_8(\sqrt{2}h - \varepsilon)\right| \le c\frac{1}{7!}\int_0^{\sqrt{2}h-\varepsilon} (\sqrt{2}h - \varepsilon - t)^7(\sqrt{2}h - t)^{-2}\,dt \le c_1h^6, \quad 0 < \varepsilon \le \frac{h}{\sqrt{2}}. \tag{19}$$

Taking into account the continuity of the function $\widetilde{r}_8(s)$ on $[-\sqrt{2}h, \sqrt{2}h]$, from (18) and (19), we obtain

$$\left|r_8(x_0 + h, y_0 + h)\right| \le c_1h^6, \tag{20}$$

where c_1 is a constant independent of the point taken, (x_0, y_0) on Π^{1h}.

Estimation (20) is obtained analogously for the remaining seven nodes of the operator B. Since the norm of the operator is equal to 1 in the uniform metric, by using (20), we have

$$\left|Br_8(x_0, y_0)\right| \le c_2h^6. \tag{21}$$

Hence, on the basis of (14), (15), (17), and linearity of the operator B, we obtain

$$\left|Bu(x_0, y_0) - u(x_0, y_0)\right| \le ch^6,$$

for any $(x_0, y_0) \in \Pi^{1h}$.

Now, let (x_0, y_0) be a point of Π^{2h}, and let in the Taylor formula (14) corresponding to this point, the remainder term $r_8(x, y)$ be represented in the Lagrange form. Then $Br_8(x_0, y_0)$ contains eighth order derivatives of the solution of problem (1) at some points of the open square R_0 defined by (13), when $(x_0, y_0) \in \Pi^{2h}$. The square R_0 lies at a distance from the boundary γ of the rectangle Π; it is not less than h. Therefore, on the basis of Lemma 2.3, we obtain

$$\left|Br_8(x_0, y_0)\right| \le c_3h^6, \tag{22}$$

where c_3 is a constant independent of the point $(x_0, y_0) \in \Pi^{2h}$. Again, on the basis of (14), (15), and (22) follows estimation (12) at any point $(x_0, y_0) \in \Pi^{2h}$. Lemma 2.4 is proved. □

We present two more lemmas. Consider the following systems:

$$q_h = Bq_h + g_h \quad \text{on } \Pi^h, \qquad q_h = 0 \quad \text{on } \gamma^h, \tag{23}$$

$$\overline{q}_h = B\overline{q}_h + \overline{g}_h \quad \text{on } \Pi^h, \qquad \overline{q}_h \ge 0 \quad \text{on } \gamma^h, \tag{24}$$

where g_h and $\overline{g}_h$ are given functions, and $|g_h| \le \overline{g}_h$ on Π^h.

Lemma 2.5 *The solutions q_h and $\overline{q}_h$ of systems* (23) *and* (24) *satisfy the inequality*

$$|q_h| \leq \overline{q}_h \quad on\ \overline{\Pi}^h.$$

The proof of Lemma 2.5 follows from the comparison theorem (see Chapter 4 in [7]).

Lemma 2.6 *For the solution of the problem*

$$q_h = Bq_h + h^6 \quad on\ \Pi^h, \qquad q_h = 0 \quad on\ \gamma^h, \tag{25}$$

the following inequality holds:

$$q_h \leq \frac{5}{3}\rho d h^4 \quad on\ \overline{\Pi}^h,$$

where $d = \max\{a,b\}$, $\rho = \rho(x,y)$ is the distance from the current point $(x,y) \in \overline{\Pi}^h$ to the boundary of the rectangle Π.

Proof We consider the functions

$$\overline{q}_h^{(1)}(x,y) = \frac{5}{3}h^4\left(ax - x^2\right) \geq 0, \qquad \overline{q}_h^{(2)}(x,y) = \frac{5}{3}h^4\left(by - y^2\right) \geq 0 \quad \text{on } \overline{\Pi},$$

which are solutions of the equation $\overline{q}_h = B\overline{q}_h + h^6$ on Π^h. By virtue of Lemma 2.5, we obtain

$$q_h \leq \min_{i=1,2} \overline{q}_h^{(i)}(x,y) \leq \frac{5}{3}\rho d h^4 \quad \text{on } \overline{\Pi}^h. \qquad \square$$

Theorem 2.7 *Assume that the boundary functions φ_j, $j = 1,2,3,4$ satisfy conditions* (2) *and* (3). *Then*

$$\max_{\overline{\Pi}^h} |u_h - u| \leq c\rho h^4, \tag{26}$$

where u_h is the solution of the finite difference problem (11), *and u is the exact solution of problem* (1).

Proof Let

$$\varepsilon_h = u_h - u \quad \text{on } \overline{\Pi}^h. \tag{27}$$

It is obvious that

$$\varepsilon_h = B\varepsilon_h + (Bu - u) \quad \text{on } \Pi^h, \qquad \varepsilon_h = 0 \quad \text{on } \gamma^h. \tag{28}$$

By virtue of estimation (12) for $(Bu - u)$ and by applying Lemma 2.5 to the problems (25) and (28), on the basis of Lemma 2.6 we obtain

$$\max_{\overline{\Pi}^h} |\varepsilon_h| \leq c\rho h^4. \tag{29}$$

From (27) and (29) follows the proof of Theorem 2.7. $\square$

3 Approximation of the first derivative

We denote $\Psi_j = \frac{\partial u}{\partial x}$ on γ_j, $j = 1,2,3,4$, and consider the boundary value problem:

$$\Delta v = 0 \quad \text{on } \Pi, \qquad v = \Psi_j \quad \text{on } \gamma_j, j = 1,2,3,4, \tag{30}$$

where u is a solution of the boundary value problem (1).

We put

$$\begin{aligned}\Psi_{1h}(u_h) &= \frac{1}{12h}\bigl(-25\varphi_1(y) + 48u_h(h,y) - 36u_h(2h,y) \\ &\quad + 16u_h(3h,y) - 3u_h(4h,y)\bigr) \quad \text{on } \gamma_1^h,\end{aligned} \tag{31}$$

$$\begin{aligned}\Psi_{3h}(u_h) &= \frac{1}{12h}\bigl(25\varphi_3(y) - 48u_h(a-h,y) + 36u_h(a-2h,y) \\ &\quad - 16u_h(a-3h,y) + 3u_h(a-4h,y)\bigr) \quad \text{on } \gamma_3^h,\end{aligned} \tag{32}$$

$$\Psi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x} \quad \text{on } \gamma_p^h, p = 2,4, \tag{33}$$

where u_h is the solution of the finite difference boundary value problem (11).

Lemma 3.1 *The following inequality is true*:

$$\bigl|\Psi_{kh}(u_h) - \Psi_{kh}(u)\bigr| \le c_3h^4, \quad k = 1,3, \tag{34}$$

where u_h is the solution of problem (11), *u is the solution of problem* (1).

Proof On the basis of (31), (32), and Theorem 2.7, we have

$$\begin{aligned}\bigl|\Psi_{kh}(u_h) - \Psi_{kh}(u)\bigr| &\le \frac{1}{12h}\bigl(48(ch)h^4 + 36(c2h)h^4 + 16(c3h)h^4 + 3(c4h)h^4\bigr) \\ &\le c_3h^4, \quad k = 1,3.\end{aligned}$$

□

Lemma 3.2 *The following inequality holds*

$$\max_{(x,y)\in\gamma_k^h} \bigl|\Psi_{kh}(u_h) - \Psi_k\bigr| \le c_4h^4, \quad k = 1,3. \tag{35}$$

Proof From Lemma 2.1 it follows that $u \in C^{5,0}(\overline{\Pi})$. Then, at the end points $(0,\nu h) \in \gamma_1^h$ and $(a,\nu h) \in \gamma_3^h$ of each line segment $\{(x,y) : 0 \le x \le a, 0 < y = \nu h < b\}$, (31) and (32) give the fourth order approximation of $\frac{\partial u}{\partial x}$, respectively. From the truncation error formulas (see [8]) it follows that

$$\max_{(x,y)\in\gamma_k^h} \bigl|\Psi_{kh}(u) - \Psi_k\bigr| \le \frac{h^4}{5} \max_{(x,y)\in\overline{\Pi}} \left|\frac{\partial^5 u}{\partial x^5}\right| \le c_5h^4, \quad k = 1,3. \tag{36}$$

On the basis of Lemma 3.1 and estimation (33) follows (35).

We consider the finite difference boundary value problem

$$v_h = Bv_h \quad \text{on } \Pi^h, \qquad v_h = \Psi_{jh} \quad \text{on } \gamma_j^h, j = 1,2,3,4, \tag{37}$$

where Ψ_{jh}, $j = 1,2,3,4$, are defined by (31)-(33). □

Theorem 3.3 *The estimation is true*

$$\max_{(x,y)\in\overline{\Pi}^h}\left|\nu_h-\frac{\partial u}{\partial x}\right|\le ch^4, \tag{38}$$

where u is the solution of problem (1), ν_h *is the solution of the finite difference problem* (37).

Proof Let

$$\epsilon_h=\nu_h-\nu \quad \text{on } \overline{\Pi}^h, \tag{39}$$

where $\nu=\frac{\partial u}{\partial x}$. From (37) and (39), we have

$$\begin{aligned} &\epsilon_h=B\epsilon_h+(B\nu-\nu) \quad \text{on } \Pi^h, \qquad \epsilon_h=\Psi_{kh}(u_h)-\nu \quad \text{on } \gamma_k^h, k=1,3,\\ &\epsilon_h=0 \quad \text{on } \gamma_p^h, p=2,4. \end{aligned} \tag{40}$$

We represent

$$\epsilon_h=\epsilon_h^1+\epsilon_h^2, \tag{41}$$

where

$$\epsilon_h^1=B\epsilon_h^1 \quad \text{on } \Pi^h, \tag{42}$$

$$\epsilon_h^1=\Psi_{kh}(u_h)-\nu \quad \text{on } \gamma_k^h, k=1,3, \qquad \epsilon_h^1=0 \quad \text{on } \gamma_p^h, p=2,4, \tag{43}$$

$$\epsilon_h^2=B\epsilon_h^2+(B\nu-\nu) \quad \text{on } \Pi^h, \qquad \epsilon_h^2=0 \quad \text{on } \gamma_j^h, j=1,2,3,4. \tag{44}$$

By Lemma 3.2 and by the maximum principle, for the solution of system (42), (43), we have

$$\max_{(x,y)\in\overline{\Pi}^h}\left|\epsilon_h^1\right|\le\max_{q=1,3}\max_{(x,y)\in\gamma_q^h}\left|\Psi_{qh}(u_h)-\nu\right|\le c_4h^4. \tag{45}$$

The solution ϵ_h^2 of system (44) is the error of the approximate solution obtained by the finite difference method for problem (30), when the boundary values satisfy the conditions

$$\Psi_j\in C^{4,\lambda}(\gamma_j), \quad 0<\lambda<1, j=1,2,3,4, \tag{46}$$

$$\Psi_j^{(2q)}(s_j)=(-1)^q\Psi_{j-1}^{(2q)}(s_j), \quad q=0,1. \tag{47}$$

Since the function $\nu=\frac{\partial u}{\partial x}$ is harmonic on Π with the boundary functions Ψ_j, $j=1,2,3,4$, on the basis of (46), (47), and Remark 15 in [9], we have

$$\max_{(x,y)\in\overline{\Pi}^h}\left|\epsilon_h^2\right|\le c_6h^4. \tag{48}$$

By (41), (45), and (48) inequality (38) follows. □

4 Approximation of the pure second derivatives

We denote $\omega = \frac{\partial^2 u}{\partial x^2}$. The function ω is harmonic on Π, on the basis of Lemma 2.1 is continuous on $\overline{\Pi}$ and is a solution of the following Dirichlet problem:

$$\Delta\omega = 0 \quad \text{on } \Pi, \qquad \omega = F_j \quad \text{on } \gamma_j, j = 1,2,3,4, \tag{49}$$

where

$$F_\tau = \frac{\partial^2 \varphi_\tau}{\partial x^2}, \quad \tau = 2,4, \tag{50}$$

$$F_\nu = -\frac{\partial^2 \varphi_\nu}{\partial y^2}, \quad \nu = 1,3. \tag{51}$$

From the continuity of the function ω on $\overline{\Pi}$ and from (2), (3) and (50), (51) it follows that

$$F_j \in C^{4,\lambda}(\gamma_j), \quad 0 < \lambda < 1, j = 1,2,3,4, \tag{52}$$

$$F_j^{(2q)}(s_j) = (-1)^q F_{j-1}^{(2q)}(s_j), \quad q = 0,1, j = 1,2,3,4. \tag{53}$$

Let ω_h be a solution of the finite difference problem

$$\omega_h = B\omega_h \quad \text{on } \Pi^h, \qquad \omega_h = F_j \quad \text{on } \gamma_j^h \cup \dot{\gamma}_j, j = 1,2,3,4, \tag{54}$$

where F_j, $j = 1,2,3,4$, are the functions determined by (50) and (51).

Theorem 4.1 *The following estimation holds*:

$$\max_{\overline{\Pi}^h} |\omega_h - \omega| \le ch^4, \tag{55}$$

where $\omega = \frac{\partial^2 u}{\partial x^2}$, *u is the solution of problem* (1) *and* ω_h *is the solution of the finite difference problem* (54).

Proof On the basis of conditions (52) and (53), the exact solution of problem (49) belongs to the class of functions $\widetilde{C}^{4,\lambda}(\overline{\Pi})$ (see [9]). Therefore, inequality (55) follows from the results in [9] (see Remark 15), as the case of the Dirichlet problem. □

5 Numerical example

Let $\Pi = \{(x,y) : -1 < x < 1, 0 < y < 1\}$, and let γ be the boundary of Π. We consider the following problem:

$$\Delta u = 0 \quad \text{on } \Pi, \qquad u = p(x,y) \quad \text{on } \gamma_j, j = 1,2,3,4, \tag{56}$$

where

$$p(x,y) = \left(x^2 + y^2\right)^{\frac{181}{60}} \cos\left(\frac{181}{30} \arctan\left(\frac{y}{x}\right)\right) \tag{57}$$

is the exact solution of this problem.

Let U be the exact solution and U_h be its approximate values on $\overline{\Pi}^h$ of the Dirichlet problem on the rectangular domain Π. We denote $\|U - U_h\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h} |U - U_h|$, $\Re_U^m = \frac{\|U-U_{2^{-m}}\|_{\overline{\Pi}^h}}{\|U-U_{2^{-(m+1)}}\|_{\overline{\Pi}^h}}$.

In Table 1 and in Table 2, the maximum errors and the convergence order of the approximations of the first and pure second derivatives of problem (56) for different step sizes h are presented.

The results show that the approximate solutions converge as $O(h^4)$.

The shapes of $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ and their approximations are demonstrated in Figure 1 and Figure 2, respectively.

Table 1 The approximate results for the first derivative

h	$\|v - v_h\|$	$\Re_v^m$
$\frac{1}{8}$	2.29999606476432500965 7E−2	
$\frac{1}{16}$	1.89405910456816052510 4E−3	12.14
$\frac{1}{32}$	1.34488079370147455378 3E−4	14.08
$\frac{1}{64}$	8.96066324997764498692 7E−6	15.01
$\frac{1}{128}$	5.79639386387354269277 4E−7	15.46

Table 2 The approximate results for the pure second derivative

h	$\|\omega - \omega_h\|$	$\Re_\omega^m$
$\frac{1}{8}$	3.14905992859754377287 8E−6	
$\frac{1}{16}$	1.93105811905271941445 1E−7	16.31
$\frac{1}{32}$	1.18048536972734204801 9E−8	16.36
$\frac{1}{64}$	7.21121714049905302202 5E−10	16.37
$\frac{1}{128}$	4.40432649216250726439 2E−11	16.37

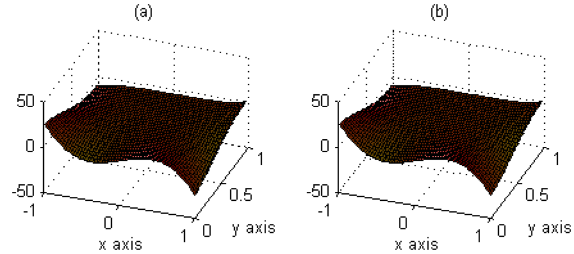

Figure 1 The graph of the approximate (a) and exact (b) solutions of $\frac{\partial u}{\partial x}$.

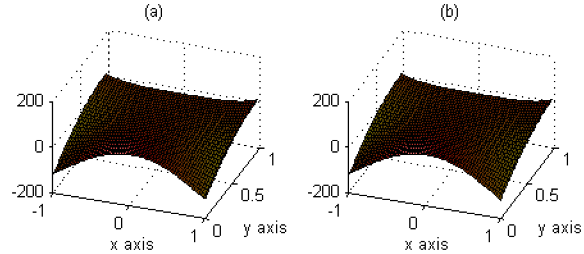

Figure 2 The graph of the approximate (a) and exact (b) solutions of $\frac{\partial^2 u}{\partial x^2}$.

6 Conclusion

The obtained results can be used to highly approximate the derivatives for the solution of Laplace's equation by the finite difference method, in some version of domain decomposition methods, in composite grid methods, and in the combined methods for solving Laplace's boundary value problems on polygons (see [10–13]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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