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# Optimal control of a rumor propagation model with latent period in emergency event

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**Abstract**

Rumor is an important form of social interaction, and its spreading has a significant impact on human lives. In this paper, a rumor propagation model with latent period and varying population is considered, which assumes an ignorant individual first goes through a latent period after infection before becoming a spreader or a stifler. Agents that read the rumor but have not decided to spread it, stay in the latent period. By means of the Lyapunov function and LaSalle's invariant set theorem, we proved the global asymptotical stable results of the rumor-free equilibrium and the rumor-endemic equilibrium by using the Poincarè-Bendixson property. Then an optimal control problem is formulated, from the perspective of a manager in emergency events, to maximize positive social effects with rumor spreading when the emergency resources are under constraints. Control signals, such as science education and official medial coverage attempt to convert lurkers and spreaders into stiflers. By employing Pontryagin's maximum principle, the optimal solution is acquired when the emergency response incurs nonlinear costs. Finally, we outline some strategies for managers that can contribute to rumor control in an emergency event.

**Keywords:** optimal control; rumor propagation; asymptotical stable; emergency event

## 1 Introduction

Rumors are part of our everyday life, and its spreading has a significant impact on human lives. Hayakawa [1] defines rumor as a kind of social phenomenon that a similar remark spreads on a large scale in a short time through chains of communication. Shibutani [2] regards rumor as collective problem-solving, in which people 'caught' in ambiguous situations trying to caught 'construe a meaningful interpretation ... by pooling their intellectual resources'. Rumors may contain confidential information about public figures or news which concerns important social issues, they can shape the public opinion of a society or a market by affecting the individual beliefs of its members, and its spreading plays a significant role in a variety of human affairs. The effects of rumors have been widely documented in many fields, such as in markets, social organizations, and disasters.

Rumor spreading is the social phenomenon that a remark spreads on a large scale in a short time through a chain of communication. To analyze the spreading and cessation of them, rumor transmissions are often modeled as social contagion processes. The classical models for the spread of rumor were introduced by Daley and Kendal [3] and Maki and

Thompson [4], and then many researchers have used the model extensively in the past for their quantitative studies [5].

In classical models, people are divided into three classes: ignorants (those not aware of the rumor), spreaders (those who are spreading it), and stiflers (those who know the rumor but have ceased communicating it after meeting somebody already informed), and they interact by pairwise contacts. In the Daley-Kendall (D-K) model, spreader-ignorant contact will convert the ignorant to spreader; spreader-spreader contact will convert both spreaders to stiflers, and spreader- stifler contact will stifle the spreader. In the Maki-Thompson (M-T) model, the rumor is spread by directed contact of the spreaders with other individuals. Hence, when a spreader contacts another spreader, only the initiating one becomes a stifler. Pearce [6] and Gani [7] analyzed the probability generating functions in the stochastic rumor models by means of a block-matrix methodology. In addition, Dickinson and Pearce [8] studied stochastic models for more general transient processes including epidemics. Independently of this series of studies, deterministic models for rumor propagation have been studied sporadically. For example, Castillo-Chávez and Song [9] proposed the propagation model for a fanatic behavior based on the models for sexually transmitted diseases and analyzed them qualitatively and numerically. Bettencourt *et al.* [10] have worked on the spreading process of multiple varying ideas. Kawachi [11] proposed and mathematically analyzed deterministic models for rumor transmission, which are extensions of the deterministic D-K model. In Kawachi's other extension model [12], he and his cooperators studied a flexible spreader-ignorant-stifler model where spreader to ignorant and stifler to spreader transitions are possible, while Lebensztayn [13] investigated the case that a new uninterested class of people exists. Huang [14] studied the rumor spreading process with denial and skepticism, two models are established to accommodate skeptics. A number of studies proposed more complex models of rumor spreading based on several classical models of social networks including homogeneous networks, Erdos-Renyi (ER) random graphs, uncorrelated scale-free networks and scale-free networks with assortative degree correlations.

In particular, we have those which were mediated by the internet, such as 'virtual' communities and email networks. Those extended models include a general class of Markov processes for generating time-dependent evolution, and studies of the effects of social landscapes on the spread, through Monte Carlo simulations over small-world and scale-free networks, but major shortcomings of these models were that either they neglected the topological characteristics of social networks or some of these models were not suitable for large-scale spreading process. As for applications, Zanette [15, 16] and Buzna *et al.* [17] established rumor spreading model on the small-world networks and found the existence of a rumor spreading critical value. Moreno [18] studied the stochastic D-K model on scale-free networks and insisted that the uniformity of networks had a significant impact on the dynamic mechanism of rumor spreading. Isham [19] studied the final size distribution of rumors on general networks. Sauerwald [20] studied the relation between the vertex expansion of a graph and the performance of randomized rumor spreading by replacing conductance by vertex expansion. Recently, Zhao *et al.* [21, 22] have put forward a series rumor spreading models, such as the SIRaRu and 2SI2R models, entailing inherited and extended classical rumor spread theory.

Although there are a lot of studies on preventing the spread of disease and computer viruses in human and computer networks through optimal control [23, 24], information epidemics have attracted less attention (see for example [25]). Apart from differences in epidemic models, our objective is to minimize the final size distribution of rumors spread and the negative social effects with rumor spreading in emergency event. In addition, the cost functions used by Behncke [23] and Lashari [24] are linear in control, while our cost is quadratic in control. Zhu *et al.* [26] assumed a time-varying state variable in the cost function, while our formulation considers only the final system state. In this paper, we aim to prevent the rumor diffusion which uses science education and official media coverage as control strategies. The publicity of science education and the advance of people's cultural qualities encourage lurkers to protect themselves from the rumor and attempt to convert lurkers into stiflers. In an emergency situation, the authority plays the role of the manager, putting an announcement in the authoritative media to address rumors, inducting public opinions, and putting emergency under control, reducing the potential for secondary damage. It is imperative that announcements be enacted and executed to stop the production and transmission of the rumors in an emergency event and attempt to convert spreaders into stiflers.

During the past decades, various mathematical models for the propagation of a rumor within a population have been developed. Beyond obvious qualitative parallels there are also important differences between the spread of rumor and diseases. In this paper, we apply models similar to those used in epidemiology to the spread of rumor. By the term 'rumor', we refer generally to any concept that can be transmitted from person to person. It may refer to uncertain information, which may require efforts and apprenticeship to be learned. What is important is that it is possible to tell if someone has adopted the rumor, believed it, and is capable of and/or active in spreading it to others.

A major difference between an information epidemic and a biological epidemic is that in the case of a biological epidemic, the infection rate, and recovery rate are constant throughout the season. The interest level of the population during the emergency period changes as we approach the deadline (poll date or movie release date). We have modeled this by making the effective information spreading rate a time-varying quantity. Previous studies have ignored this characteristic of information epidemics. Another difference between epidemiological models and rumor spreading models is the removal mechanism. The spread of a rumor, unlike a disease, is usually an intentional act on the part of the transmitter and/or the adopter. A core element associated with a rumor is lack of verification, and some rumors that take time to identify, such as those involving consideration or confirmation, require active effort to discern between true and false and one needs to determine whether there is propagation or not. Because network information has always suffered from a lack of credibility, people cannot believe it immediately but are able to believe news from their friends and relatives more easily. Especially, rumors mostly come from a network and then spread in real life mouth to mouth. Many rumors come from a network and are hidden in the depths of one's heart for a period of time before he/she becomes a spreader or stifler in real life. Being a model that is more general than the classical D-K model, the XYZ type needs to be studied to investigate the role of a latent period in rumor propagation. Using a compartmental approach based on a disease infection, we assume that an ignorant individual first goes through a latent period (and is said to become exposed or in the class $W$) after infection before becoming a spreader or stifler.

In this paper, we apply a general model, inspired by epidemiology and informed by our knowledge of the sociology of the spread dynamics, to the diffusion of the rumor. This paper deals with the rumor propagation model with latent period and varying total population. Then an optimal control problem is formulated, from the perspective of a manager in emergency events, to minimize negative social effects with rumor spreading when the emergency resources are under constraints. By employing Pontryagin's maximum principle, the optimal solution is acquired when the emergency response incurs nonlinear costs. We provide a more detailed and realistic description of the rumor spreading process with combination of a latent period mechanism and the D-K model of a rumor. This paper is organized as follows. In Section 2, we review the D-K model and introduce a model with homogeneity in susceptibility and propagation. In Section 3, we discuss and analyze our general rumor propagation model with latent period and having a constant immigration; and we discuss the existence of an equilibrium. In Section 4, we consider the global stability of the rumor-free equilibrium (RFE) and the unique rumor-endemic equilibrium (REE). In Section 5, an optimal control problem is formulated, and the optimal solution is acquired by employing Pontryagin's maximum principle. Finally, we present simulations and conclusions about the model, and we discuss the implications of these results, and from this we conclude what parameters have an impact on each system so that we can come up with suggestions for possible preventative or control methods.

## 2 Review of Daley-Kendall framework

Daley and Kendall published a paper aiming to stochastically model the spread of rumors. They considered a closed homogeneously mixing population of $N$ individuals. At any time an individual can be classified as one of three categories: $X(t)$ denotes those individuals who are ignorant of the rumor; $Y(t)$ denotes those individuals who are actively spreading the rumor; and $Z(t)$ denotes those individuals who know the rumor but have ceased spreading it. For all $t$, $X(t) + Y(t) + Z(t) = N$. They referred to these three types of individuals as ignorants, spreaders, and stiflers, respectively [3]. The rumor is propagated through the closed population by contact between ignorants and spreaders, following the law of mass action. They assume that any spreader involved in any pairwise meeting 'infects' the 'other'. If the 'other' is an ignorant then he/she will become a spreader; if the 'other' is a spreader or a stifler, then the spreader(s) will become a stifler(s). A stifler will never, under any circumstances, infect a susceptible because of the definition of a spreader. Stiflers do not transmit the rumor.

Next, Cintrón-Arias and Castillo-Chávez [27] proposed the following deterministic version of the D-K model:

$$\begin{cases} \frac{dX}{dt} = -\beta \frac{XY}{N}, \\ \frac{dY}{dt} = \beta \frac{XY}{N} - \lambda \frac{Y(Y+Z)}{N}, \\ \frac{dZ}{dt} = \lambda \frac{Y(Y+Z)}{N}. \end{cases} \tag{1}$$

This model has been extremely useful in the interpretation of the D-K because some analytical analysis can be done on this deterministic version of the model. Still, the D-K model makes some other assumptions. There is no inflow to the susceptible class or outflow from any of the classes. The model also assumes that everybody should be a spreader immediately after they learn the rumor, and the process of thinking is virtually ignored. Along the same lines, their model does not take into account the personality of the person who is spreading or receiving the rumor. Finally, it does not allow for people who are 'igno-

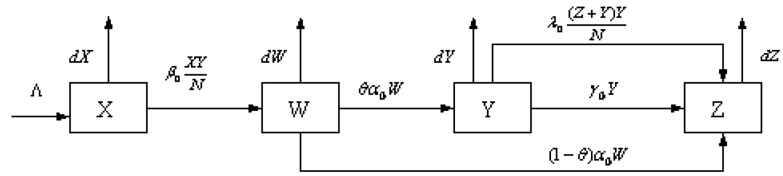


**Figure 1 The basic scheme of population dynamics models for the spread of rumor.**

rant' to hear the rumor and then choose not to spread it. Still, their model was extremely innovative and is still very useful in the modeling and analysis of rumor spreading.

## 3 General rumor propagation model with latent period and having constant immigration

In the following we will concentrate on the model, based on 'homogeneous mixing' with state variables as functions of time, which is more general than the XYZ type model and needs to be studied to investigate the role of latent period in rumor propagation. However, the propagation requires some time for individuals to pass from the infected to the spread state, and we assume that an ignorant individual first goes through a latent period (and is said to become exposed or in the class $W$) after being infected before becoming a spreader or stifler, and the resulting model is of XWYZ type.

We divide the population into four classes: the ignorant class, the latent class, the spreader class, and the stifler class. Each population at time $t$ is denoted by $X(t)$, $W(t)$, $Y(t)$, $Z(t)$, respectively, each of which we call a rumor-class. Those who belong to the ignorant class, whom we call ignorants, do not know about the rumor. Those who belong to the latent class, whom we call lurker, know about the rumor and require active effort to discern between true and false and need to determine whether there is propagation or not; a part of them believe the rumor and become spreaders, others do not believe it and become stiflers. Those who belong to the spreader class, whom we call spreaders, know about the rumor and spread it actively. Those who belong to the stifler class, whom we call stiflers, know about the rumor and do not spread it. The total population size at time $t$ is denoted by $N(t)$, with $N = X + W + Y + Z$. The transfer diagram is depicted in Figure 1.

We assume that no transition of rumor-class happens unless a spreader contacts someone, since the two people who are not spreaders do not talk about the rumor. That is, it is spreaders who are involved in the transition of rumor-class. When a spreader contacts an ignorant, the spreader transmits the rumor at a constant frequency and the ignorant gets to know about it and requires time to discern between true and false and becomes rumor latent. Then the ignorant does not always become a spreader, but may doubt its credibility and consequently becomes a stifler. Therefore, we assume that $\beta_0 XY \triangle t/N$ ignorants change their rumor-class and become exposed during the small interval $(t, t + \triangle t)$, where $\beta_0$ is a positive constant number representing the product of the contact frequency and the probability of transmitting the rumor. $W$ have been infected but are not yet spreading, we assume that $\alpha_0 W$ exposed change their rumor-class and become spreaders at a constant rate $\theta \in (0, 1]$, and others become stiflers at a rate $1 - \theta$, where $\alpha_0$ is a positive constant number representing the proportion of exposed change their rumor-class. When two spreaders contact with each other, both of them transmit the rumor at a constant frequency. Hearing it again and again, the spreader gets bored, gradually loses interest in it, and consequently becomes a stifler. Therefore, we assume that $\lambda_0 Y^2 \triangle t/N$ spreaders become stiflers during the small interval $(t, t + \triangle t)$, where $\lambda_0$ is a positive constant number. When a spreader contacts a stifler, the spreader transmits the rumor at a constant frequency, and after hearing it, the stifler tries to remove it, because the stifler shows no interest in it or denies it. As a result, the spreader becomes a stifler. Therefore, we assume that $\lambda_0 YZ\triangle t/N$ spreaders become stiflers during the small interval $(t, t + \triangle t)$. Any spreader may lose interest in spreading those influenced by official media, and it becomes a stifler at a rate $\gamma_0$.

For the meantime we assume that the rumor is 'constant', that is, the same remark is transmitted at all times. First, we consider the propagation of a constant rumor with variable population size, we assume that the propagation of a constant rumor in a population with constant immigration and emigration. Let $\Lambda$ be the immigration, such as the number of new internet accounts created ('births'), $\mu$ the emigration rate such as the number of internet accounts that are canceled or become void ('deaths'). Thus, the maximum value that $\frac{1}{\mu}$ can take is the average lifespan of the rumor within a generation of researchers in the relevant community. We assume that $\Lambda$, $\mu$ are positive constants, that the newcomers are all ignorant, and that emigration is independent of rumor-class.

The model is described by the following system of differential equations:

$$\begin{cases} \frac{dX}{dt} = \Lambda - \beta_0 \frac{XY}{N} - \mu X, \\ \frac{dW}{dt} = \beta_0 \frac{XY}{N} - \alpha_0 W - \mu W, \\ \frac{dY}{dt} = \theta\alpha_0 W - \lambda_0 \frac{(Y+Z)Y}{N} - \gamma_0 Y - \mu Y, \\ \frac{dZ}{dt} = (1-\theta)\alpha_0 W + \lambda_0 \frac{(Y+Z)Y}{N} + \gamma_0 Y - \mu Z. \end{cases} \tag{2}$$

In the epidemic models used in this study, the demographic dynamics are modeled by $\frac{dN}{dt} = \Lambda - \mu N$, then $N(t)$ varies over time and approaches a stable fixed point, $\frac{\Lambda}{\mu}$, as $t \to \infty$, in other words, the community approaches its 'carrying' capacity. Let $x = \frac{X}{N}$, $y = \frac{Y}{N}$, $z = \frac{Z}{N}$, $\tau = \mu t$, $\beta = \frac{\beta_0}{\mu}$, $\alpha = \frac{\alpha_0}{\mu}$, $\lambda = \frac{\lambda_0}{\mu}$, $\gamma = \frac{\gamma_0}{\mu}$. It is easy to verify that $x$, $w$, $y$, and $z$ satisfy the system of differential equations

$$\begin{cases} \frac{dx}{d\tau} = 1 - \beta xy - x, \\ \frac{dw}{d\tau} = \beta xy - \alpha w - w, \\ \frac{dy}{d\tau} = \theta\alpha w - \lambda(y+z)y - \gamma y - y, \\ \frac{dz}{d\tau} = (1-\theta)\alpha w + \lambda(y+z)y + \gamma y - z, \end{cases} \tag{3}$$

subject to the restriction $x + w + y + z = 1$. Note that the total population size $N(t)$ does not appear in (2); this is a direct result of the homogeneity of the system (1). Also, we rewrite $\tau$ as $t$, determining $z$ from $z = 1 - x - w - y$, this allows us to research system (2) by studying the subsystem

$$\begin{cases} \frac{dx}{dt} = 1 - \beta xy - x, \\ \frac{dw}{dt} = \beta xy - \alpha w - w, \\ \frac{dy}{dt} = \theta\alpha w - \lambda(1 - x - w)y - \gamma y - y. \end{cases} \tag{4}$$

The feasible region for (1) is $R^3_+$, we study (4) in the closed set

$$A = \left\{(x, w, y) \in R^3_+ | 0 \leq x + w + y \leq 1\right\}, \tag{5}$$

where $R^3_+$ denotes the nonnegative cone and its lower dimensional faces. It can be verified that $A$ is positively invariant with respect to (4). We denote by $\partial A$ and $\mathring{A}$ the boundary and the interior of $A$ in $R^3_+$, respectively.

Letting the right side of each of the three differential equations be equal to zero in system (4) gives the equation

$$\begin{cases} 1-\beta xy - x = 0, \\ \beta xy - \alpha w - w = 0, \\ \theta\alpha w - \lambda(1-x-w)y - \gamma y - y = 0. \end{cases} \tag{6}$$

Adding the first and second equation of system (6), and then substituting it into the last equation of system (6), we have

$$\begin{cases} x = 1-(\alpha+1)w, \\ y = \frac{\theta\alpha w}{\lambda\alpha w+(\gamma+1)}. \end{cases} \tag{7}$$

Substituting (7) into the second equation of (6), $w$ satisfies the following equation:

$$\frac{\theta\alpha w}{\lambda\alpha w+(\gamma+1)}\bigl[1-(\alpha+1)w\bigr] = (\alpha+1)w. \tag{8}$$

So the system (4) always has the RFE $P_0 = (1,0,0)$, and unique REE $P^* = (x^*,w^*,y^*)$, where $w^* = \frac{\beta\theta\alpha-(\alpha+1)(\gamma+1)}{\alpha(\alpha+1)(\beta\theta+\lambda)}$, $x^* = 1-(\alpha+1)w^*$, $y^* = \frac{\theta\alpha w^*}{\lambda\alpha w^*+(\gamma+1)}$ if $R_0 = \frac{\beta\theta\alpha}{(\alpha+1)(\gamma+1)} > 1$, where $R_0$ the basic reproduction number of the system (4).

## 4 Global stability of the RFE and the unique REE

**Theorem 4.1** *If $R_0 \le 1$, the RFE $P_0$ is globally asymptotically stable in $A$. $P_0$ is unstable if $R_0 > 1$, the solutions to the system* (4) *starting sufficiently close to $P_0$ in $A$ move away from $P_0$ except that those starting on invariant $x$-axis approach $P_0$ along this axis.*

*Proof* Consider the Lyapunov function:

$$L = \frac{\theta}{\alpha+1}w + \frac{1}{\alpha}y. \tag{9}$$

Its derivative along the solutions to the system (4) is

$$\begin{aligned} L' &= \frac{\beta\theta\alpha xy - (\alpha+1)(\gamma+1)y - \lambda(\alpha+1)(1-x-w)y}{\alpha(\alpha+1)} \\ &\le \frac{[\beta\theta\alpha - (\alpha+1)(\gamma+1)]y}{\alpha(\alpha+1)} = \frac{(\gamma+1)(R_0-1)y}{\alpha} \le 0. \end{aligned} \tag{10}$$

Furthermore, $L' = 0$ only if $w = 0$, $y = 0$. The maximum invariant set in $\{(x,w,y) : L' = 0\}$, is the singleton $\{P_0\}$. The global stability of $P_0$ when $R_0 \le 1$ follows from LaSalle's invariance principle [28]. □

**Theorem 4.2** *If $R_0 > 1$, the REE $P^*$ of the system* (4) *is locally asymptotically stable.*

*Proof* The Jacobian matrix at REE $P^*$ is given by

$$J(P^*) = \begin{pmatrix} -1-\beta y^* & 0 & -\beta x^* \\ \beta y^* & -(\alpha+1) & \beta x^* \\ \lambda y^* & \alpha\theta + \lambda y^* & -\lambda(1-x^*-w^*)-(\gamma+1) \end{pmatrix}. \tag{11}$$

Its characteristic equation is $\det(\omega E - J(P^*)) = 0$, where $E$ is the unit matrix and

$$x^* = 1-(\alpha+1)w^*, \qquad y^* = \frac{\theta\alpha w^*}{\lambda\alpha w^* + (\gamma+1)}.$$

So the characteristic equation becomes $\omega^3 + C_1\omega^2 + C_2\omega + C_3 = 0$, where

$$C_1 = \beta\alpha\theta - (\alpha+1)(\gamma+1) > 0 \quad \text{if } R_0 > 1,$$

$$C_2 = \frac{\alpha[\lambda(1+\alpha)^2 + \beta\theta(2+\alpha^2+\gamma+\alpha(3+\lambda+\gamma))]}{(\alpha+1)^2(\gamma+1)+\lambda\alpha(\alpha+1)} > 0,$$

$$C_3 = 3+\alpha+\beta y^* + \gamma + \lambda(1-w^*-x^*) > 0.$$

We calculate easily $C_1C_2 - C_3 > 0$. According to the Hurwitz criterion, the REE $P^*$ has local asymptotical stability. □

In the following,using the geometrical approach of Li and Muldowney in [29]; we obtain simple sufficient conditions that the rumor steady state $P^*$ is globally asymptotically stable. First, we give a brief outline of this geometrical approach.

**Lemma 4.1** [30] *Consider the following systems*:

$$\nu' = f(\nu), \quad \nu \in A. \tag{12}$$

*If the following conditions are satisfied*:

- (I) *For the system* (12) *there exists a compact absorbing set* $\Gamma \subset A$ *and has a unique equilibrium* $P$ *in* $\mathring{A}$;
- (II) $P$ *is local asymptotically stable*;
- (III) *the system* (12) *satisfies a Poincaré-Bendixson criterion*;
- (IV) *a periodic orbit of the system* (12) *is asymptotically orbitally stable*;

*then the only equilibrium* $P$ *is the globally asymptotically stable in* $\mathring{A}$.

**Lemma 4.2** [31] *A sufficient condition for a periodic orbit* $P = \{P(t) : 0 \le t \le \varpi\}$ *of system* (12) *to be asymptotically orbitally stable with asymptotic phase is that the linear system* $V' = \frac{\partial f^{[2]}}{\partial t}(P(t))V(t)$ *is asymptotically stable, where* $\frac{\partial f^{[2]}}{\partial t}$ *is the second additive compound matrix of the Jacobian matrix* $\frac{\partial f}{\partial t}$ *of* $f$. *The system* (12) *is called the second compound system of the orbit* $P(t)$.

**Lemma 4.3** *Any periodic solution to the system* (4), *if it exists, is asymptotically orbitally stable*.

*Proof* Suppose that the solution $(x(t), w(t), y(t))$ is periodic of least period $\varpi > 0$ such that $(x(0), w(0), y(0)) \in \mathring{A}$. The periodic orbit is $P = \{P(t) : 0 \le t \le \varpi\}$. From (11), the second compound system $\nu' = J^{[2]}(P)\nu$ of the differential system $\vartheta' = J(P)\vartheta$ in the periodic solution is the following periodic linear system:

$$\begin{cases} \frac{dV_1}{dt} = -(\beta y + \alpha + 2)V_1 + \beta x(V_2 + V_3), \\ \frac{dV_2}{dt} = (\theta\alpha + \lambda y)V_1 - [\gamma + 2 + \beta y + \lambda(1 - x - w)]V_2, \\ \frac{dV_3}{dt} = -\lambda y V_1 + \beta y V_2 - [\alpha + \gamma + 2 + \lambda(1 - x - w)]V_3, \end{cases} \tag{13}$$

where

$$J^{[2]}(P) = \begin{pmatrix} -\beta y - \alpha - 2 & \beta x & \beta x \\ \theta\alpha + \lambda y & -\gamma - 2 - \beta y - \lambda(1 - x - w) & 0 \\ -\lambda y & \beta y & -\alpha - \gamma - 2 - \lambda(1 - x - w) \end{pmatrix}. \tag{14}$$

Suppose that $(V_1(t), V_2(t), V_3(t))$ is a solution to (13) and $(x(t), w(t), y(t)) \in P$.
Let

$$U(V_1, V_2, V_3, x, w, y) = \operatorname{Sup}\left\{|V_1|, \frac{w}{y}\bigl(|V_2 + V_3|\bigr)\right\}. \tag{15}$$

From the condition (I) of Lemma 4.1 we find that there exists a constant $\eta > 0$ such that

$$U(V_1, V_2, V_3, x, w, y) \geq \eta\bigl|(V_1, V_2, V_3)\bigr| \tag{16}$$

for all $(V_1, V_2, V_3) \in R^3$ and $(x(t), w(t), y(t)) \in P$. Direct calculations lead to the following differential inequalities:

$$D_+\bigl|V_1(t)\bigr| \leq -(\beta y + \alpha + 2)|V_1| + \frac{\beta xy}{w}\frac{w}{y}\bigl(|V_2| + |V_3|\bigr), \tag{17}$$

$$D_+\bigl|V_2(t)\bigr| \leq (\theta\alpha + \lambda y)|V_1| - \bigl[\gamma + 2 + \beta y + \lambda(1 - x - w)\bigr]|V_2|, \tag{18}$$

$$D_+\bigl|V_3(t)\bigr| \leq -\lambda y|V_1| + \beta y|V_2| - \bigl[\alpha + \gamma + 2 + \lambda(1 - x - w)\bigr]|V_3|. \tag{19}$$

Using (18) and (19), we have

$$\begin{aligned} &D_+\frac{w}{y}\bigl(\bigl|V_2(t)\bigr| + \bigl|V_3(t)\bigr|\bigr) \\ &\quad\leq \frac{w}{y}\theta\alpha\bigl|V_1(t)\bigr| + \frac{w}{y}\left[\frac{w'}{w} - \frac{y'}{y} - \gamma - 2 - \lambda(1 - x - w)\right]\bigl(\bigl|V_2(t)\bigr| + \bigl|V_3(t)\bigr|\bigr), \end{aligned} \tag{20}$$

where $'$ denotes the derivative of a function. Equations (17) and (20) lead to

$$D_+U(t) \leq \operatorname{Sup}\{g_1, g_2\}U(t), \tag{21}$$

where

$$g_1 = -(\beta y + \alpha + 2) + \frac{\beta xy}{w}, \tag{22}$$

$$g_2 = \frac{w}{y}\theta\alpha + \frac{w'}{w} - \frac{y'}{y} - \gamma - 2 - \lambda(1 - x - w). \tag{23}$$

Rewriting (4), we find that

$$\frac{w'}{w} = \frac{\beta xy}{w} - (\alpha + 1), \tag{24}$$

$$\frac{y'}{y} = \frac{\theta\alpha w - \lambda(1 - x - w)y}{y} - (\gamma + 1). \tag{25}$$

Substituting (24) and (25) into (23), we have

$$g_2 = \frac{\beta xy}{w} - (\alpha + 1) - 1. \tag{26}$$

Thus $\mathrm{Sup}\{g_1, g_2\} \leq \frac{w'}{w} - 1$ and

$$\int_0^{\varpi} \mathrm{Sup}\{g_1, g_2\}\, d\varpi \leq \ln w(t)|_0^{\varpi} - \varpi = -\varpi. \tag{27}$$

From (21), this implies that $U(t) \to 0$ as $t \to \infty$, and in turn that $(V_1(t), V_2(t), V_3(t)) \to 0$ as $t \to \infty$. As a result, the second compound system (13) is asymptotically stable and the periodic solution $(x(t), w(t), y(t))$ is asymptotically orbitally stable by Lemma 4.2. □

Now, we are ready to prove the global stability of the endemic equilibrium $P^*$.

**Theorem 4.3** *If* $R_0 > 1$, *then the unique REE* $P^*$ *of the system* (4) *is globally asymptotically stable*.

*Proof* Firstly, from the condition (1) of Lemma 4.1, the uniformly persistent property of the solution of the system (1) can be concluded to [32]. In fact, let $G = \{P_0\}$, Lemma 4.1 implies that, when $R_0 > 1$, $G^s$ is just contained in the $x$-axis and thus just in the boundary of $A$. It also implies that $G^s$ is isolated in $A$. Then, when $R_0 > 1$, the system (4) satisfies the conditions of Theorem 4.3 of [33], namely, (i) the maximal compact invariant set $G$ in the boundary of $A$ is isolated, and (ii) the stable set $G^s$ of $G$ is contained in the boundary of $A$. Therefore, the system (4) is uniformly persistent in $A$ when $R_0 > 1$.

Secondly, let $H = \mathrm{diag}(1, -1, 1)$, $HJ(P)H$ has non-positive off-diagonal elements for $(x(t), w(t), y(t))$. Thus we can verify that the system (4) is competitive with respect to the partial ordering defined by the orthant $S = \{(x, w, y) \in R_+^3 | x \geq 0, w \geq 0, y \geq 0\}$ (see [34]).

From [31], we know system (4) satisfies the Poincaré-Bendixson property. By Lemma 4.3 and Theorem 4.2, we know that the system (4) is satisfied with every condition of Lemma 4.1, so the unique REE of the system (4) is globally asymptotically stable. □

## 5 The optimal control model

Our objective in this section is to extend the initial model to include two intervention methods, called controls, represented as functions of time and assigned reasonable upper and lower bounds, each representing a possible method of rumor intervention.

Historically, rumor outbreaks have tended to reach the attention of authorities only after transmission has been amplified by inadequate infection control. Conversely, as with any breaking news story, information is often fluid and the authoritative media updated the story with the official explanation as soon as it was provided. If there is a rumor spread in emergency event, the official has the obligation to state whether it is true or not, it helps the person to understand the situation better. After the 9-magnitude earthquake in Fukushima, incurring nuclear leakage accidents in 2011, some rumors said that taking materials containing iodine could help ward of nuclear radiation, which led to the public rushing for everything containing iodine, such as Chinese snapping up iodized salt, Americans rushing for iodine pills, Russians hoarding iodine, and Korean residents rushing for seaweed. The science knowledge is that eating iodized salt cannot prevent people from radiation, which encourages lurkers to protect themselves from the rumor and attempt to convert lurkers into stiflers. Official media as authority announcing the news, after receiving the true information, the public will not be confused by the rumors. It is imperative that announcements be enacted and executed to stop the production and transmission of the rumors in emergency event and attempt to convert spreaders into stiflers.

Generally, the lurker individual becomes a spreader when being convinced of the truth of the rumor and then decides to inform others. However, note that 'a convinced' lurker can possibly refuse to spread the rumor, or alternately a spreader can lose interest in the rumor and then decide not to spread the rumor any further. In these two situations, both become stiflers. A stifler is therefore either an individual who knows the rumor but who is not spreading it or a spreader who, with time, loses interest and is no longer spreading the rumor. Furthermore, one can notice that the tendency of accepting a rumor as credible information differs from one lurker to another. This can be explained by the strong background knowledge that some of the lurker individuals possess. These types of lurker individuals, once aware of the rumor, generally raise some reasonable questions and/or logical arguments in order to assess the credibility or the validity of the rumor. Science education is therefore among the factors that also contribute to the cessation of a rumor spreading and is an important aspect that has not been considered in previous studies.

We will integrate the essential components into one XWYZ-type model to accommodate the dynamics of rumor outbreak determined by population-specific parameters such as the effect of contact reduction when infectious and stifler individuals are reported in the official media.

Let $u_\eta$ and $u_\gamma$ be the control variables for science education and official media coverage, respectively, where $\eta = 1 - \theta$. Thus, model (2) now reads

$$\begin{cases} \frac{dx}{dt} = 1 - \beta xy - x, \\ \frac{dw}{dt} = \beta xy - \theta\alpha w - u_\eta \eta\alpha w - w, \\ \frac{dy}{dt} = \theta\alpha w - \lambda(y+z)y - (1-u_\gamma)y - y, \\ \frac{dz}{dt} = u_\eta \eta\alpha w + \lambda(y+z)y + (1-u_\gamma)y - z. \end{cases} \tag{28}$$

The balance of multiple intervention methods can differ between populations. A successful mitigation scheme is one which reduces rumor infectious with minimal cost. A control scheme is assumed to be optimal if it maximizes the objective functional

$$W\bigl(u_\eta(t), u_\gamma(t)\bigr) = \int_{t_0}^{t_f} \bigl[B_0\bigl(x(t)+z(t)\bigr) - B_1 y(t) - B_2\bigl(u_\eta(t)^2 + u_\gamma(t)^2\bigr)\bigr]\,dt. \tag{29}$$

The first two terms represent the benefits of the ignorant and stifler populations. The parameters $B_0$ represent the weight constraints for the ignorant and stifler populations,

$B_1$ and $B_2$ represent the weight constraints for the infected population and the control, respectively. They can also represent balancing coefficients transforming the integral into dollars expended over a finite time period of $[t_0, t_f]$. The goal is to maximize the populations of ignorant and stifler individuals, minimize the population of infectives, and maximize the benefits of official media coverage and vaccination, while minimizing the systemic costs of both rumor vaccination and official media coverage. The terms $B_2 u_\eta(t)^2$ and $B_2 u_\gamma(t)^2$ represent the maximal cost of education, implementation and campaigns on both rumor vaccination and official media coverage. $x(t)$ and $z(t)$ account for the fitness of the ignorant and the stifler groups. We thus seek optimal controls

$$W\big(u_\eta^*(t), u_\gamma^*(t)\big) = \max\big[W\big(u_\eta(t), u_\gamma(t)\big) | \big(u_\eta(t), u_\gamma(t)\big) \in U\big], \tag{30}$$

where $U = \{u_\eta, u_\gamma | u_\eta, u_\gamma$ measurable, $0 \le a_{11} \le u_\eta \le b_{11} \le 1, 0 \le a_{22} \le u_\gamma \le b_{22} \le 1, t \in [t_0, t_f]\}$. The basic framework of this problem is to characterize the optimal control. The existence of an optimal control can be obtained by using a result by Joshi [35].

**Theorem 5.1** *Consider the control problem with the system of equations* (9)-(12). *There exists an optimal control, such that* $\max\{W(u_\eta(t), u_\gamma(t)) | (u_\eta(t), u_\gamma(t)) \in U\} = W(u_\eta^*(t), u_\gamma^*(t))$.

*Proof* To prove this theorem on the existence of an optimal control, we use a result from Fleming and Rishel [36] (Theorem 4.1 pp.68-69), where the following properties must be satisfied: (I) The set of controls and corresponding state variables is nonempty; (II) the control set $U$ is closed and convex; (III) the right-hand side of the state system is bounded above by a linear function in the state and control; (IV) the integrand of the functional is concave on $U$ and is bounded above by $c_2 - c_1(|u_\eta(t)|^2 + |u_\gamma(t)|^2)$, where $c_1 > 0$, $c_2 > 0$, and $k > 1$.

An existence result in Lukes [37] (Theorem 9.2.1) for the system of equations (28) for bounded coefficients is used to give the first condition. The control set is closed and convex by definition. The right-hand side of the state system satisfies Condition III since the state solutions are *a priori* bounded. The integrand in the objective functional, $B_0(x(t) + z(t)) - B_1 y(t) - B_2(u_\eta(t)^2 + u_\gamma(t)^2)$ is concave on $U$. Furthermore, $c_1 > 0$, $c_2 > 0$, and $k > 1$, so

$$B_0\big(x(t) + z(t)\big) - B_1 y(t) - B_2\big(u_\eta(t)^2 + u_\gamma(t)^2\big) \le c_2 - c_1\big(\big|u_\eta(t)\big|^2 + \big|u_\gamma(t)\big|^2\big). \tag{31}$$

Therefore, the optimal control exists, since the left-hand side of (31) is bounded; consequently, the states are bounded.

Since there exists an optimal control for maximizing the functional (29) subject to (28), we use Pontryagin's maximum principle to derive the necessary conditions for this optimal control. Pontryagin's maximum principle introduces adjoint functions that allow us to attach our state system (of differential equations), to our objective functional. After first showing the existence of optimal controls, this principle can be used to obtain the differential equations for the adjoint variables, corresponding boundary conditions, and the characterization of an optimal control $u_\eta^*(t)$, $u_\gamma^*(t)$. This characterization gives a representation of an optimal control in terms of the state and adjoint functions. Also, this principle converts the problem of minimizing the objective functional subject to the state system into minimizing either the Lagrangian or the Hamiltonian with respect to the controls (bounded measurable functions) at each time $t$.

The Lagrangian is defined as

$$\begin{aligned} L &= B_0\big(x(t)+z(t)\big) - B_1 y(t) - B_2\big(u_\eta(t)^2 + u_\gamma(t)^2\big) + \rho_1[1-\beta xy - x] \\ &\quad + \rho_2[\beta xy - \theta\alpha w - u_\eta \eta\alpha w - w] \\ &\quad + \rho_3\big[\theta\alpha w - \lambda(y+z)y - (1-u_\gamma)y - y\big] \\ &\quad + \rho_4\big[u_\eta\eta\alpha w + \lambda(y+z)y + (1-u_\gamma)y - z\big] \\ &\quad + \omega_{11}\big(a_{11} - u_\eta(t)\big) + \omega_{12}\big(u_\eta(t) - b_{11}\big) \\ &\quad + \omega_{21}\big(a_{22} - u_\gamma(t)\big) + \omega_{22}\big(u_\gamma(t) - b_{22}\big), \end{aligned} \tag{32}$$

where $\omega_{11} \geq 0$, $\omega_{12} \geq 0$ are penalty multipliers satisfying $\omega_{11}(a_{11} - u_\eta(t)) + \omega_{12}(u_\eta(t) - b_{11})$ at optimal $u_\eta^*(t)$, and $\omega_{21} \geq 0$, $\omega_{22} \geq 0$ are penalty multipliers satisfying $\omega_{21}(a_{22} - u_\gamma(t)) + \omega_{22}(u_\gamma(t) - b_{22})$ at optimal $u_\gamma^*(t)$.

Given optimal controls $u_\eta^*(t)$ and $u_\gamma^*(t)$, and solutions of the corresponding state system (28), there exist adjoint variables $\rho_i$, for $i = 1,2,3,4$, satisfying the following equation:

$$\begin{cases} \frac{d\rho_1}{dt} = -\frac{\partial L}{\partial x} = -B_0 + (\rho_1 - \rho_2)\beta y + \rho_1, \\ \frac{d\rho_2}{dt} = -\frac{\partial L}{\partial w} = (\rho_2 - \rho_3)\theta\alpha + (\rho_2 - \rho_4)u_\eta\eta\alpha + \rho_2, \\ \frac{d\rho_3}{dt} = -\frac{\partial L}{\partial y} = -B_1 + (\rho_1 - \rho_2)\beta x + (\rho_3 - \rho_4)(2\lambda y + \lambda z + (1-u_\gamma)) + \rho_3, \\ \frac{d\rho_4}{dt} = -\frac{\partial L}{\partial z} = -B_0 + (\rho_3 - \rho_4)\lambda y + \rho_4, \end{cases} \tag{33}$$

with transversality conditions $\rho_i(t_f)$, for $i = 1,2,3,4$. To determine the interior maximum of our Lagrangian, we take the partial derivatives of $L$ with respect to $u_\eta(t)$ and $u_\gamma(t)$, respectively, and set them to zero. Thus,

$$\begin{cases} \frac{\partial L}{\partial u_\eta(t)} = -2B_2 u_\eta^*(t) + (\rho_4 - \rho_2)\eta\alpha w - \omega_{11} + \omega_{12}, \\ \frac{\partial L}{\partial u_\gamma(t)} = -2B_2 u_\gamma^*(t) + (\rho_3 - \rho_4)y - \omega_{21} + \omega_{22}. \end{cases} \tag{34}$$

To determine an explicit expression for our controls $u_\eta^*(t)$ and $u_\gamma^*(t)$ (without $\omega_{11}$, $\omega_{12}$, $\omega_{21}$, $\omega_{22}$), a standard optimality technique is utilized. The following cases are considered to determine the specific characterization of the optimal control.

Case 1: Optimality of $u_\eta^*(t)$

1. We consider the set $\{t | a_{11} \leq u_\eta^*(t) \leq b_{11}\}$, $\omega_{11} = \omega_{12} = 0$. Hence, the optimal control is

$$u_\eta^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2}. \tag{35}$$

2. We consider the set $\{t | a_{11} = u_\eta^*(t)\}$, $\omega_{11} = 0$. We have

$$u_\eta^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w + \omega_{12}}{2B_2} \tag{36}$$

or

$$u_\eta^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2} \leq a_{11} \tag{37}$$

since $\omega_{12} \geq 0$.

3. We consider the set $\{t|b_{11} = u_\eta^*(t)\}$, $\omega_{12} = 0$. We have

$$u_\eta^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w - \omega_{11}}{2B_2} \tag{38}$$

or

$$u_\eta^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2} \geq b_{11}. \tag{39}$$

Combining all the three sub-cases in a compact form gives

$$u_\eta^*(t) = \min\left\{\max\left\{a_{11}\frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2}\right\}, b_{11}\right\}. \tag{40}$$

Case 2: Optimality of $u_\gamma^*(t)$

1. We consider the set $\{t|a_{22} \leq u_\gamma^*(t) \leq b_{22}\}$, $\omega_{21} = \omega_{22} = 0$. Hence, the optimal control is

$$u_\gamma^*(t) = \frac{(\rho_3 - \rho_4)y}{2B_2}. \tag{41}$$

2. We consider the set $\{t|a_{22} = u_\gamma^*(t)\}$, $\omega_{21} = 0$. We have

$$u_\gamma^*(t) = \frac{(\rho_3 - \rho_4)y + \omega_{22}}{2B_2} \tag{42}$$

or

$$u_\gamma^*(t) = \frac{(\rho_3 - \rho_4)y + \omega_{22}}{2B_2} \leq a_{22} \tag{43}$$

since $\omega_{22} \geq 0$.

3. We consider the set $\{t|b_{22} = u_\gamma^*(t)\}$, $\omega_{22} = 0$. We have

$$u_\gamma^*(t) = \frac{(\rho_3 - \rho_4)y - \omega_{21}}{2B_2} \tag{44}$$

or

$$u_\gamma^*(t) = \frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2} \geq b_{22}. \tag{45}$$

Combining all the three sub-cases in a compact form gives

$$u_\gamma^*(t) = \min\left\{\max\left\{a_{22}, \frac{(\rho_4 - \rho_2)\eta\alpha w}{2B_2}\right\}, b_{22}\right\}. \tag{46}$$

The optimality system consists of the state system coupled with the adjoint system, with the initial conditions, the transversality conditions and the characterization of the optimal control:

$$\begin{cases} \frac{dx}{dt} = 1 - \beta xy - x, \\ \frac{dw}{dt} = \beta xy - \theta\alpha w - u_\eta \eta\alpha w - w, \\ \frac{dy}{dt} = \theta\alpha w - \lambda(y+z)y - (1-u_\gamma)y - y, \\ \frac{dz}{dt} = u_\eta \eta\alpha w + \lambda(y+z)y + (1-u_\gamma)y - z, \\ \frac{d\rho_1}{dt} = -B_0 + (\rho_1 - \rho_2)\beta y + \rho_1, \\ \frac{d\rho_2}{dt} = (\rho_2 - \rho_3)\theta\alpha + (\rho_2 - \rho_4)u_\eta \eta\alpha + \rho_2, \\ \frac{d\rho_3}{dt} = -B_1 + (\rho_1 - \rho_2)\beta x + (\rho_3 - \rho_4)(2\lambda y + \lambda z + (1-u_\gamma)) + \rho_3, \\ \frac{d\rho_4}{dt} = -B_0 + (\rho_3 - \rho_4)\lambda y + \rho_4, \end{cases} \tag{47}$$

where $u_\eta^*(t)$ and $u_\gamma^*(t)$, are given by (40) and (46), respectively, with $x(0) = x_0$, $w(0) = w_0$, $y(0) = y_0$, $z(0) = z_0$, and $\rho_i(t_f) = 0$, for $i = 1, \ldots, 4$. Due to the *a priori* boundedness of the state and adjoint functions and the resulting Lipschitz structure of the ODEs, we obtain the uniqueness of the optimal control for small $[t_f]$. The uniqueness of the optimal control follows from the uniqueness of the optimality system. The state system of differential equations and the adjoint system of differential equations together with the control characterization above form the optimality system solved numerically and depicted in the next section. □

## 6 Discussions and simulations

### 6.1 Model simulation without involve control

This section deals with the rumor propagation model without optimal control. It concerns a rumor with latent period; for example, many rumors come from network and are hidden in the depths of one's heart for a period of time before he/she becomes a spreader or stifler in real life. Our main results present the global dynamics of rumor propagation model with latent period and its transformed proportionate system, the process of communication correlations between the two systems in rumor eradication and persistence, and the effects of different management strategies on the rumor control.

Numerical simulations (parameters and variables used in simulations are summarized in Table 1) carried out for system (4) show that the rumor 'dies out' when the basic reproduction number $R_0 < 1$ (the threshold) (Figure 2) and the rumor persists at an 'endemic' level when $R_0 > 1$ (Figure 3), where $R_0 = \frac{\beta\theta}{(1+\frac{1}{\alpha})(1+\gamma)}$.

The threshold $R_0$ is increasing with the rumor propagation coefficient $\beta$, the believe and spread rate $\theta$. If there exist many active members who believe and spread the rumor actively, then $\beta$ and $\theta$ will be sufficiently large. Then $R_0 > 1$ holds easily, and the rumor will persists at an 'endemic' level. Conversely, if the managers induct public views and remind more carefully the public of not relying on rumors, increasing the ability to distinguish, then the parameters $\beta$ and $\theta$ will become smaller, and make the threshold $R_0 < 1$, and the rumor will 'die out'. The threshold $R_0$ for the differential equation model also determines the asymptotic behavior of the infectious fraction $y(t)$. When $R_0 > 1$, $y(t)$ always approaches the rumor-endemic value, $y^* = \frac{\theta\alpha w^*}{\lambda\alpha w^* + (\gamma+1)}$, where $w^* = \frac{\beta\theta\alpha - (\alpha+1)(\gamma+1)}{\alpha(\alpha+1)(\beta\theta+\lambda)}$.

In the special cases when the population size remains constant (*i.e.* $\Lambda = 0$, and $\mu = 0$), the latent period is negligible, and the model (1) reduces to a D-K rumor model with bilinear incidence.

**Table 1 Dimension of parameters and variables used in simulations**

| Variable or parameter | Dimension | Implication |
|---|---|---|
| $x, w, y, z$ | Dimensionless | Proportional population |
| $X, W, Y, Z$ | Hundred thousand Day$^{-1}$ | Population |
| $\Lambda$ | Hundred thousand Day$^{-1}$ | The number of immigration population |
| $\beta_0$ | Unity. Day$^{-1}$ | Rumor propagation coefficient |
| $\lambda_0$ | Unity. Day$^{-1}$ | Rumor stifler coefficient |
| $\alpha_0$ | Dimensionless | Change rate for exposed |
| $\gamma_0$ | Dimensionless | Change rate for spreaders |
| $\theta$ | Dimensionless | Believe and spread rate |
| $\mu$ | Dimensionless | The emigration rate |
| $N$ | Hundred thousand | Total population |

Here $x = \frac{X}{N}, y = \frac{Y}{N}, z = \frac{Z}{N}, \beta = \frac{\beta_0}{\mu}, \alpha = \frac{\alpha_0}{\mu}, \lambda = \frac{\lambda_0}{\mu}, \gamma = \frac{\gamma_0}{\mu}$.

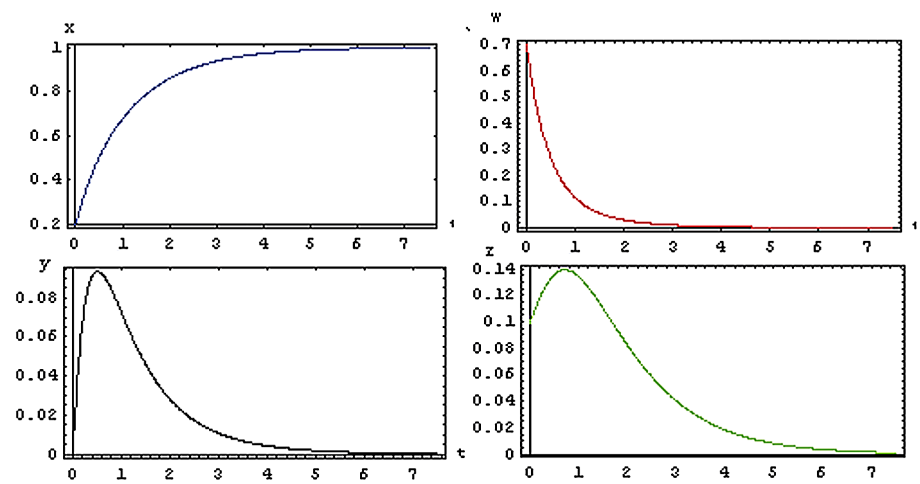


**Figure 2** $\beta_0 = 0.1, \lambda_0 = 0.2, \alpha_0 = 0.11, \gamma_0 = 0.05, \theta = 0.7, \mu = 0.1, R_0 = 0.47$.

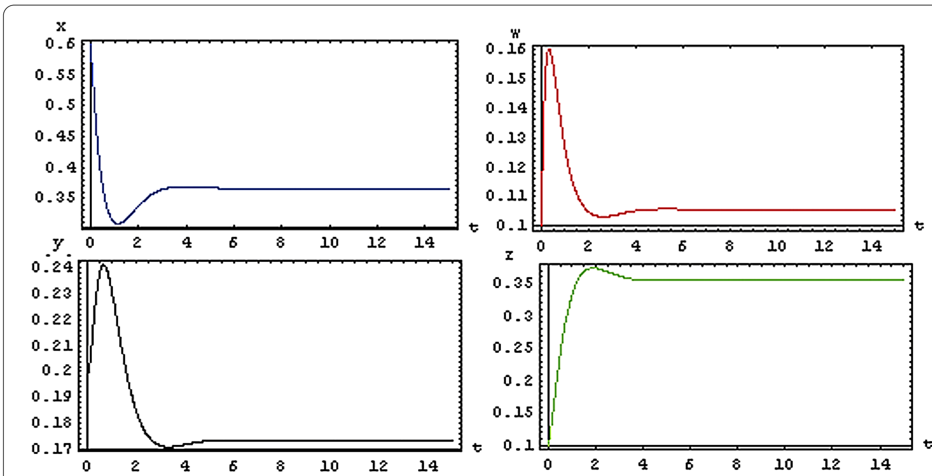


**Figure 3** $\beta_0 = 0.9, \lambda_0 = 0.1, \alpha_0 = 0.5, \gamma_0 = 0.03, \theta = 0.6, \mu = 0.1, R_0 = 3.46$.

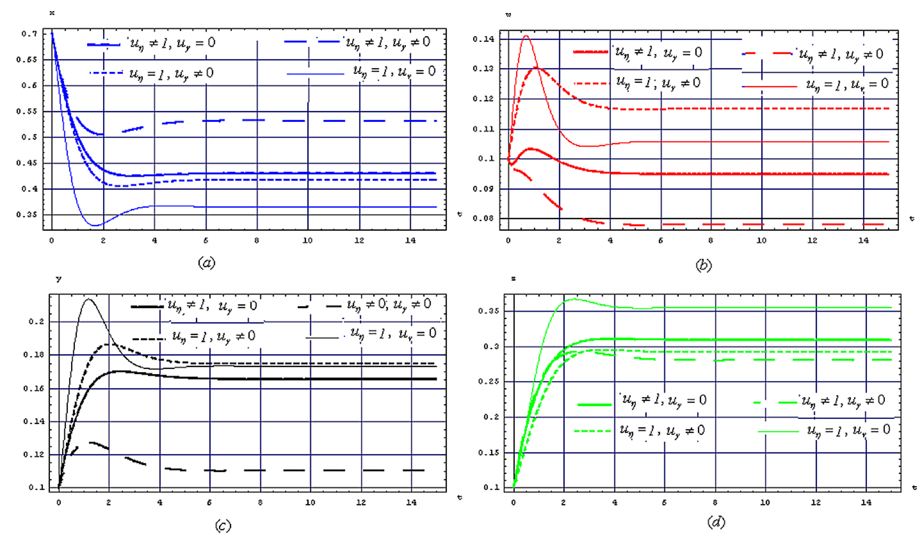


**Figure 4 Variation of population for different situation with control.**

## 6.2 Model simulation with optimal control

In this section, we perform a numerical simulation to verify the analytical results in Section 4 and to further investigate the properties of rumor spreading model with optimal control by using the fourth-order Runge-Kutta iteration algorithm. We will investigate how science education and official media affect a rumor's spreading, four kinds of situations are analyzed, respectively. We have case 1: $u_\eta \neq 1$, $u_\gamma = 0$; case 2: $u_\eta = 1$, $u_\gamma \neq 0$; case 3: $u_\eta \neq 1$, $u_\gamma \neq 0$; case 4: $u_\eta = 1$, $u_\gamma = 0$. $u_\eta = 1$, which means the model does not involve any control variables for science education, and $u_\eta \neq 1$, which means the control variables for science education have influence on the model. Similarly, $u_\gamma = 0$, which means there are not involved any control variables for official media coverage. In particular, the optimal control model (28) reduces to model (2) under the conditions of case 4.

We include four simulations which compare the structure of solutions for the rumor spreading problems in Figure 4(a-d). The data and parameters for the dynamics are chosen in Table 1. It is easy to see that the optimal control and sub-optimal control are much more effective for rumor control.

In Figure 4(a), we report the densities of ignorant population for different situations with control or without, the density of the ignorant population remarkably decreases, but the case 4 shows that the biggest falls in infection rates were during certain periods, the greatest was the impact on rumor spreading. From this, we know that the optimal control for rumor propagation can greatly reduce the spreading scale of the rumor.

In Figures 4(b) and 4(c), we report the densities of latent and infected population for different situation with control or without, we observe that the density of lurker and spreaders in each scenario increases and reaches its peak. This simply comes from the fact that spreaders are infecting other individuals, of whom a fraction gradually progresses into the lurker and spreader class. There is a steady decrease of the density of the spreaders from its peak value to the steady state. During this phase, the density of the stiflers in each corresponding scenario steeply increases and reaches a steady state in Figure 4(d). We clearly observe from Figure 4(b-c) that optimal control can slow down the transmission velocity of the rumor, with the given parameter values, the lowest rate for the lurker and spreader with case 4. The densities of lurkers and spreaders in case 1 reach a smaller peak and then decrease faster than case 2; it means that science education is more effective than official media coverage in rumor control.

## 7 Conclusions

Rumor propagation can have serious consequences; thus the study on how to take effective measures to control its spreading is of great practical significance. In the paper, we discuss the XWYZ model with constant immigration and latent periods. One may assume that an ignorant individual first goes through a latent period after infection before becoming a spreader or stifler. We derive the basic reproduction number $R_0$ and find that it determines the global dynamics of system (4); if $R_0 < 1$, by means of the Lyapunov function and LaSalle's invariant set theorem, we proved the global asymptotical stable results of the RFE $P_0$ in $A$ and the rumor can be eradicated; if $R_0 > 1$, by means of the Poincarè-Bendixson property, we proved the global asymptotical stable results of the unique REE $P^*$ is globally asymptotically stable in the interior of the feasible region, so that the rumor persists at the REE level if it is initially present.

The optimal control problem is formulated; it clearly shows that science education and official media coverage are the most effective form of rumor control, especially, science education is more effective than official media coverage. The science education of a population reflects the degree of vulnerability of a typical individual to any kind of information, and the population's education rate is also an influential factor in rumor spreading cessation. This hypothesis stems from the plausible observations that the more an individual is educated, the stronger is his/her ability to evaluate the credibility of the rumor, and the quicker is his/her disinclination of the rumor. In short, educational influence on rumor spreading needs to be considered when talking about rumor spreading control; thus, we must emphasize the science education for the public and perform the elementary education reform with new educational ideas. Official media coverage was still a significant factor of rumor control. In an emergency situation, official media play the role of leader, leveraging the information and clarifying it, inducting public opinions, and making emergency under control, reducing the potential for secondary damage. Admittedly, popularization of legal knowledge from the citizens to comprehensively improve the quality of emergency eventually enhances the effectiveness of elements of educational purposes. Official media should comply with the principles of timeliness, genuineness, and impartiality. Whether the information is linked up and grasped in time and is abundant, open, and clear, can influence the quality and result of rumor control directly. From the perspective of a manager in emergency events, one is to minimize negative social effects with rumor spreading when the emergency resources are under constraints. By employing Pontryagin's maximum principle, the optimal solution is acquired when the emergency response incurs nonlinear costs.

The techniques developed in this paper are general and can be applied to similar optimal control problems in other areas. Since some rumors spread in a certain group of people, we then can assume that the propagation coefficient is a function of the parameters for special populations and time, which may involve a non-autonomous system instead of an autonomous system, and we leave this for future work.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

HLA, ZJ and LTT designed research; LC and FCJ performed the research; HLA and ZJ analyzed the data; LTT wrote the paper.

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