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Stability of a Jensen type quadratic-additive functional equation under the approximately conditions

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Abstract

We prove the Hyers-Ulam-Rassias stability of the Jensen type quadratic and additive functional equation $9f(\frac{x+y+z}{3}) + 4[f(\frac{x-y}{2}) + f(\frac{y-z}{2}) + f(\frac{z-x}{2})] = 3[f(x) + f(y) + f(z)]$ under the approximately conditions such as even, odd, quadratic, and additive in Banach spaces.

MSC: 39B82; 39B52

Keywords: quadratic functional equation; additive functional equation; mixed type functional equation; stability

1 Introduction

The study of stability problems for functional equations has originally been raised by Ulam [1]: *under what condition does there exist a homomorphism near an approximate homomorphism?* In 1941, Hyers [2] had answered affirmatively the question of Ulam for Banach spaces. The theorem of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The work of Rassias [4] has had a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. The terminology *Hyers-Ulam-Rassias stability* originates from these historical backgrounds and this terminology is also applied to the case of other functional equations (see [5–10]).

In particular, Kannappan [11] introduced the following functional equation:

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x) \tag{1.1}$$

and proved that a function on a real vector space is a solution of (1.1) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x,x) + A(x)$. For this reason, we call (1.1) as the mixed type quadratic and additive functional equation. In addition, Jung [12] investigated stability of (1.1) on restricted domains and applied the result to the study of an interesting asymptotic behavior of the quadratic functions. More generally, Jun and Kim [13] solved the general solutions and proved the stability of the following functional equation, which is a generalization of (1.1):

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \quad (n > 2).$$

Najati and Moghimi [14] introduced another type quadratic and additive functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x)$$

and investigated the stability of this equation in quasi-Banach spaces.

Recently, Lee *et al.* [15] introduced the following Jensen type quadratic and additive functional equation:

$$9f\left(\frac{x+y+z}{3}\right)+4\left[f\left(\frac{x-y}{2}\right)+f\left(\frac{y-z}{2}\right)+f\left(\frac{z-x}{2}\right)\right]=3\left[f(x)+f(y)+f(z)\right] \tag{1.2}$$

and proved the stability of this equation. Also they established the general solutions of (1.2) as follows:

Let X and Y be real vector spaces. A mapping $f:X\to Y$ satisfies (1.2) if and only if there exists a quadratic mapping $Q:X\to Y$ satisfying

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y) \tag{1.3}$$

and an additive mapping $A:X\to Y$ satisfying

$$A(x+y)=A(x)+A(y) \tag{1.4}$$

such that

$$f(x)=Q(x)+A(x)$$

for all $x\in X$.

We refer to [16–26] for stability results of the mixed type and Jensen type functional equations. Also we refer to [27–30] for more stability results of another functional equations.

In this paper, we prove the stability of (1.2) in addition to a suitable condition of the function. In Section 2, we first induce two lemmas which plays a crucial role in the proof of stability theorems. Based on the two lemmas, we evaluate the stability of (1.2) under the approximately even condition and the approximately odd condition in Section 3 and Section 4, respectively. Using similar approaches, we prove the stability of (1.2) under the approximately quadratic condition and the approximately additive condition in Section 5 and Section 6, respectively.

2 Preliminaries

Throughout this paper, we assume that X is a real vector space and Y is a Banach space. For simplicity, given a mapping $f:X\to Y$, we use the abbreviation

$$\begin{aligned}Df(x,y,z):={}&9f\left(\frac{x+y+z}{3}\right)+4\left[f\left(\frac{x-y}{2}\right)+f\left(\frac{y-z}{2}\right)+f\left(\frac{z-x}{2}\right)\right]\\&-3\left[f(x)+f(y)+f(z)\right]\end{aligned}$$

for all $x,y,z\in X$. Here we need the following lemmas.

Lemma 2.1 *Let $\varphi : X \times X \times X \to [0,\infty)$ be a given mapping. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \varphi(x,y,z) \tag{2.1}$$

for all $x,y,z \in X$. We then have

$$\begin{aligned}&\left\| f(x) - \frac{3^n+1}{2\cdot 9^n} f(3^n x) + \frac{3^n-1}{2\cdot 9^n} f(-3^n x) + \frac{9^n-1}{2\cdot 9^n} f(0)\right\| \\ &\quad\leq \sum_{k=1}^{n}\left[\frac{3^{k-1}+1}{2\cdot 9^k}\varphi(3^k x, 3^k x, -3^k x) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi(-3^k x, -3^k x, 3^k x)\right]\end{aligned} \tag{2.2}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof Putting $y = x$, $z = -x$ in (2.1) yields

$$\left\| 9g\left(\frac{x}{3}\right) - 2g(x) + g(-x)\right\| \leq \varphi(x,x,-x) \tag{2.3}$$

for all $x \in X$, where $g(x) := f(x) + \frac{1}{2}f(0)$. Replacing x by $3x$ in (2.3) and dividing by 9 gives

$$\left\| g(x) - \frac{2}{9}g(3x) + \frac{1}{9}g(-3x)\right\| \leq \frac{1}{9}\varphi(3x,3x,-3x) \tag{2.4}$$

for all $x \in X$. We use mathematical induction on n to prove lemma. Note that (2.4) proves the validity of inequality (2.2) for the case $n = 1$. Assume that inequality (2.2) holds for some $n \in \mathbb{N}$. Using (2.4) we have the following relation:

$$\begin{aligned}&\left\| g(x) - \frac{3^{n+1}+1}{2\cdot 9^{n+1}} g(3^{n+1}x) + \frac{3^{n+1}-1}{2\cdot 9^{n+1}} g(-3^{n+1}x)\right\| \\ &\quad\leq \left\| g(x) - \frac{3^n+1}{2\cdot 9^n} g(3^n x) + \frac{3^n-1}{2\cdot 9^n} g(-3^n x)\right\| \\ &\qquad + \frac{3^n+1}{2\cdot 9^n}\left\| g(3^n x) - \frac{2}{9}g(3^{n+1}x) + \frac{1}{9}g(-3^{n+1}x)\right\| \\ &\qquad + \frac{3^n-1}{2\cdot 9^n}\left\| -g(-3^n x) + \frac{2}{9}g(-3^{n+1}x) - \frac{1}{9}g(3^{n+1}x)\right\| \\ &\quad\leq \sum_{k=1}^{n+1}\left(\frac{3^{k-1}+1}{2\cdot 9^k}\varphi(3^k x, 3^k x, -3^k x) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi(-3^k x, -3^k x, 3^k x)\right)\end{aligned}$$

for all $x \in X$. This proves the validity of inequality (2.2) for the case $n + 1$. □

Lemma 2.2 *Let $\varphi : X \times X \times X \to [0,\infty)$ be a given mapping. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \varphi(x,y,z) \tag{2.5}$$

for all $x, y, z \in X$. We then have

$$
\begin{aligned}
&\left\| f(x) - \frac{9^n + 3^n}{2} f\left(\frac{x}{3^n}\right) - \frac{9^n - 3^n}{2} f\left(-\frac{x}{3^n}\right) - \frac{9^n - 1}{2} f(0) \right\| \\
&\quad \le \sum_{k=0}^{n-1} \left[\frac{3 \cdot 9^k + 3^k}{6} \varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3 \cdot 9^k - 3^k}{6} \varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right) \right]
\end{aligned}
\tag{2.6}
$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof Putting $y = x$, $z = -x$ in (2.5) gives

$$
\left\| 9g\left(\frac{x}{3}\right) - 2g(x) + g(-x) \right\| \le \varphi(x, x, -x) \tag{2.7}
$$

for all $x \in X$, where $g(x) := f(x) + \frac{1}{2} f(0)$. Interchanging x with $-x$ in (2.7) we have

$$
\left\| 9g\left(-\frac{x}{3}\right) - 2g(-x) + g(x) \right\| \le \varphi(-x, -x, x) \tag{2.8}
$$

for all $x \in X$. It follows from (2.7) and (2.8) that

$$
\left\| g(x) - 6g\left(\frac{x}{3}\right) - 3g\left(-\frac{x}{3}\right) \right\| \le \frac{2}{3} \varphi(x, x, -x) + \frac{1}{3} \varphi(-x, -x, x) \tag{2.9}
$$

which proves the validity of inequality (2.6) for the case $n = 1$. Applying (2.9) we obtain

$$
\begin{aligned}
&\left\| g(x) - \frac{9^{n+1} + 3^{n+1}}{2} g\left(\frac{x}{3^{n+1}}\right) - \frac{9^{n+1} - 3^{n+1}}{2} g\left(-\frac{x}{3^{n+1}}\right) \right\| \\
&\quad \le \left\| g(x) - \frac{9^n + 3^n}{2} g\left(\frac{x}{3^n}\right) - \frac{9^n - 3^n}{2} g\left(-\frac{x}{3^n}\right) \right\| \\
&\qquad + \frac{9^n + 3^n}{2} \left\| g\left(\frac{x}{3^n}\right) - 6g\left(\frac{x}{3^{n+1}}\right) - 3g\left(-\frac{x}{3^{n+1}}\right) \right\| \\
&\qquad + \frac{9^n - 3^n}{2} \left\| g\left(-\frac{x}{3^n}\right) - 6g\left(-\frac{x}{3^{n+1}}\right) - 3g\left(\frac{x}{3^{n+1}}\right) \right\| \\
&\quad \le \sum_{k=0}^{n} \left[\frac{3 \cdot 9^k + 3^k}{6} \varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3 \cdot 9^k - 3^k}{6} \varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right) \right]
\end{aligned}
$$

for all $x \in X$. This proves the validity of inequality (2.6) for the case $n + 1$. □

In order to prove the stability of (1.2), we suppose that $\varphi : X \times X \times X \to [0, \infty)$ is a mapping satisfying one of the conditions $(\mathcal{A}_1)$ and $(\mathcal{B}_1)$,

$(\mathcal{A}_1)$ $\quad \sum_{k=1}^{\infty} \frac{1}{3^k} \varphi(3^k x, 3^k y, 3^k z) < \infty,$

$(\mathcal{B}_1)$ $\quad \sum_{k=0}^{\infty} 9^k \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) < \infty$

for all $x, y, z \in X$. Also we assume that $\psi : X \to [0, \infty)$ is a mapping satisfying one of the conditions $(\mathcal{A}_2)$ and $(\mathcal{B}_2)$,

$$(\mathcal{A}_2) \quad \lim_{n\to\infty} \frac{\psi(3^n x)}{3^n} = 0,$$

$$(\mathcal{B}_2) \quad \lim_{n\to\infty} 9^n \psi\left(\frac{x}{3^n}\right) = 0$$

for all $x \in X$.

3 Approximately even case

Now we are going to state and prove the stability of (1.2) under the approximately even condition.

Theorem 3.1 *Let $\varphi : X \times X \times X \to [0, \infty)$ satisfy the condition $(\mathcal{A}_1)$ and $\psi : X \to [0, \infty)$ satisfies the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\left\| Df(x, y, z) \right\| \le \varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$ and

$$\left\| f(x) - f(-x) \right\| \le \psi(x) \tag{3.2}$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\begin{aligned} &\left\| f(x) - Q(x) + \frac{1}{2} f(0) \right\| \\ &\quad \le \sum_{k=1}^{\infty} \left[\frac{3^{k-1}+1}{2\cdot 9^k} \varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k} \varphi\left(-3^k x, -3^k x, 3^k x\right) \right] \end{aligned} \tag{3.3}$$

for all $x \in X$.

Proof It follows from Lemma 2.1 and (3.2) that we have

$$\begin{aligned} &\left\| f(x) - \frac{f(3^n x)}{9^n} + \frac{9^n - 1}{2\cdot 9^n} f(0) \right\| \\ &\quad \le \sum_{k=1}^{n} \left[\frac{3^{k-1}+1}{2\cdot 9^k} \varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k} \varphi\left(-3^k x, -3^k x, 3^k x\right) \right] \\ &\qquad + \frac{3^n - 1}{2\cdot 9^n} \psi\left(3^n x\right) \end{aligned} \tag{3.4}$$

for all $x \in X$ and $n \in \mathbb{N}$. By virtue of (3.4), for $n, m \in \mathbb{N}$ with $n > m$, we obtain

$$\begin{aligned} &\left\| \frac{f(3^m x)}{9^m} - \frac{f(3^n x)}{9^n} \right\| \\ &\quad = \frac{1}{9^m} \left\| f\left(3^m x\right) - \frac{f(3^{n-m} \cdot 3^m x)}{9^{n-m}} \right\| \end{aligned}$$

$$\leq \sum_{k=1}^{n-m} \left[\frac{3^{k-1}+1}{2 \cdot 9^{k+m}} \varphi\left(3^{k+m}x, 3^{k+m}x, -3^{k+m}x\right) + \frac{3^{k-1}-1}{2 \cdot 9^{k+m}} \varphi\left(-3^{k+m}x, -3^{k+m}x, 3^{k+m}x\right) \right] + \frac{3^{n-m}-1}{2 \cdot 9^{n}} \psi\left(3^{n}x\right) + \frac{9^{n-m}-1}{2 \cdot 9^{n}} \left\|f(0)\right\| \tag{3.5}$$

for all $x \in X$. Since the right-hand side of inequality (3.5) tends to 0 as $m \to \infty$, the sequence $\{9^{-n}f(3^n x)\}$ is a Cauchy sequence. Completeness of Y allows us to assume that there exists a mapping Q so that

$$Q(x) := \lim_{n\to\infty} 9^{-n} f\left(3^n x\right)$$

for all $x \in X$. Replacing x, y, z by $3^n x$, $3^n y$, $3^n z$ in (3.1) and dividing both sides by 9^n yields

$$\frac{1}{9^n} \left\| Df\left(3^n x, 3^n y, 3^n z\right) \right\| \leq \frac{1}{9^n} \varphi\left(3^n x, 3^n y, 3^n z\right)$$

for all $x, y, z \in X$. Taking the limit in the above inequality we have

$$9Q\left(\frac{x+y+z}{3}\right) + 4\left[Q\left(\frac{x-y}{2}\right) + Q\left(\frac{y-z}{2}\right) + Q\left(\frac{z-x}{2}\right)\right] = 3\left[Q(x) + Q(y) + Q(z)\right] \tag{3.6}$$

for all $x, y, z \in X$. Similarly, we get $Q(-x) = Q(x)$ for all $x \in X$ by (3.2). Putting $y = x$, $z = -x$ in (3.6) and using the evenness of Q we obtain $9Q(\frac{x}{3}) = Q(x)$ for all $x \in X$. Setting $y = 0$, $z = -x$ in (3.6) gives $4Q(\frac{x}{2}) = Q(x)$ for all $x \in X$. Thus, we can rewrite (3.6) as

$$Q(x+y+z) + Q(x-y) + Q(y-z) + Q(z-x) = 3\left[Q(x) + Q(y) + Q(z)\right] \tag{3.7}$$

for all $x, y, z \in X$. Putting $z = 0$ in (3.7) we see that Q satisfies the quadratic functional equation (1.3). Letting $n \to \infty$ in (3.4) we finally obtain the result (3.3).

In order to prove the uniqueness of Q we assume that $Q' : X \to Y$ be another quadratic mapping satisfying (3.3). From the quadratic property of Q and Q' we calculate

$$\begin{aligned} &\left\| Q(x) - Q'(x) \right\| \\ &= \frac{1}{9^n} \left\| Q\left(3^n x\right) - Q'\left(3^n x\right) \right\| \\ &\leq \frac{1}{9^n} \left\| Q\left(3^n x\right) - f\left(3^n x\right) - \frac{1}{2} f(0) \right\| + \frac{1}{9^n} \left\| f\left(3^n x\right) - Q'\left(3^n x\right) + \frac{1}{2} f(0) \right\| \\ &\leq \frac{2}{9^n} \sum_{k=1}^{\infty} \left[\frac{3^{k-1}+1}{2 \cdot 9^k} \varphi\left(3^{k+n}x, 3^{k+n}x, -3^{k+n}x\right) + \frac{3^{k-1}-1}{2 \cdot 9^k} \varphi\left(-3^{k+n}x, -3^{k+n}x, 3^{k+n}x\right) \right] \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$ in the preceding inequality we immediately find the uniqueness of Q. □

As a consequence of the above theorem we have the following corollaries immediately.

Corollary 3.2 *Let $\epsilon \geq 0$, $p < 1$, and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \epsilon\bigl(\|x\|^p + \|y\|^p + \|z\|^p\bigr)$$

for all $x,y,z \in X$ ($x,y,z \in X\backslash\{0\}$ if $p < 0$) and

$$\|f(x) - f(-x)\| \leq \psi(x)$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\left\|f(x) - Q(x) + \frac{1}{2}f(0)\right\| \leq \frac{3^p}{3 - 3^p}\epsilon\|x\|^p$$

for all $x \in X$ ($x \in X\backslash\{0\}$ if $p < 0$).

Corollary 3.3 *Let $\epsilon \geq 0$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \epsilon$$

for all $x,y,z \in X$ and

$$\|f(x) - f(-x)\| \leq \psi(x)$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\left\|f(x) - Q(x) + \frac{1}{2}f(0)\right\| \leq \frac{1}{6}\epsilon$$

for all $x \in X$.

We have the following result, which is analogous to Theorem 3.1 for the functional equation (1.2).

Theorem 3.4 *Let $\varphi : X \times X \times X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_1)$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \varphi(x,y,z) \tag{3.8}$$

for all $x,y,z \in X$ and

$$\|f(x) - f(-x)\| \leq \psi(x) \tag{3.9}$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\begin{aligned} &\|f(x) - Q(x)\| \\ &\quad \le \sum_{k=0}^{\infty} \left[\frac{3 \cdot 9^k + 3^k}{6} \varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k} \right) + \frac{3 \cdot 9^k - 3^k}{6} \varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k} \right) \right] \end{aligned} \tag{3.10}$$

for all $x \in X$.

Proof Putting $x = y = z = 0$ in (3.8) yields $f(0) = 0$. According to Lemma 2.2 and (3.9) we have

$$\begin{aligned} &\left\| f(x) - 9^n f\left(\frac{x}{3^n} \right) \right\| \\ &\quad \le \sum_{k=0}^{n-1} \left[\frac{3 \cdot 9^k + 3^k}{6} \varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k} \right) + \frac{3 \cdot 9^k - 3^k}{6} \varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k} \right) \right] \\ &\qquad + \frac{9^n - 3^n}{2} \psi\left(\frac{x}{3^n} \right) \end{aligned} \tag{3.11}$$

for all $x \in X$ and $n \in \mathbb{N}$. From (3.11) we verify

$$\begin{aligned} &\left\| 9^m f\left(\frac{x}{3^m} \right) - 9^n f\left(\frac{x}{3^n} \right) \right\| \\ &\quad = 9^m \left\| f\left(\frac{x}{3^m} \right) - 9^{n-m} f\left(\frac{x}{3^{n-m} \cdot 3^m} \right) \right\| \\ &\quad \le \sum_{k=0}^{n-m-1} \left[\frac{3 \cdot 9^{k+m} + 3^{k+2m}}{6} \varphi\left(\frac{x}{3^{k+m}}, \frac{x}{3^{k+m}}, -\frac{x}{3^{k+m}} \right) \right. \\ &\qquad \left. + \frac{3 \cdot 9^{k+m} - 3^{k+2m}}{6} \varphi\left(-\frac{x}{3^{k+m}}, -\frac{x}{3^{k+m}}, \frac{x}{3^{k+m}} \right) \right] + \frac{9^n - 3^{n+m}}{2} \psi\left(\frac{x}{3^n} \right) \end{aligned} \tag{3.12}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. Since the right-hand side of inequality (3.12) tends to 0 as $m \to \infty$, we may define a mapping

$$Q(x) := \lim_{n \to \infty} 9^n f\left(\frac{x}{3^n} \right)$$

for all $x \in X$. Replacing x, y, z by $\frac{x}{3^n}$, $\frac{y}{3^n}$, $\frac{z}{3^n}$ in (3.8) and multiplying both sides by 9^n, and after taking the limit in the resulting inequality, we see that Q satisfies (1.2). Using a similar method to the proof of Theorem 3.1 we see that Q is the unique quadratic mapping satisfying (1.3). Now letting $n \to \infty$ in (3.11) we arrive at the desired result (3.10). □

Corollary 3.5 *Let $\epsilon \ge 0$, $p > 2$, and $\psi : X \to [0, \infty)$ satisfy the condition $(\mathcal{B}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x, y, z)\| \le \epsilon \left(\|x\|^p + \|y\|^p + \|z\|^p \right)$$

for all $x, y, z \in X$ *and*

$$\|f(x) - f(-x)\| \le \psi(x)$$

for all $x \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (1.3) *such that*

$$\|f(x) - Q(x)\| \le \frac{3^{p+1}}{3^p - 9}\epsilon \|x\|^p$$

for all $x \in X$.

4 Approximately odd case

In this section, we establish the stability of (1.2) under the approximately odd condition.

Theorem 4.1 *Let* $\varphi : X \times X \times X \to [0,\infty)$ *satisfy the condition* $(\mathcal{A}_1)$ *and* $\psi : X \to [0,\infty)$ *satisfy the condition* $(\mathcal{A}_2)$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x,y,z)\| \le \varphi(x,y,z) \tag{4.1}$$

for all $x, y, z \in X$ *and*

$$\|f(x) + f(-x)\| \le \psi(x) \tag{4.2}$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\begin{aligned} &\left\|f(x) - A(x) + \frac{1}{2}f(0)\right\| \\ &\quad \le \sum_{k=1}^{\infty}\left[\frac{3^{k-1}+1}{2\cdot 9^k}\varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi\left(-3^k x, -3^k x, 3^k x\right)\right] \end{aligned} \tag{4.3}$$

for all $x \in X$.

Proof It follows from Lemma 2.1 and (4.2) that

$$\begin{aligned} &\left\|f(x) - \frac{f(3^n x)}{3^n} + \frac{9^n - 1}{2\cdot 9^n}f(0)\right\| \\ &\quad \le \sum_{k=1}^{n}\left[\frac{3^{k-1}+1}{2\cdot 9^k}\varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi\left(-3^k x, -3^k x, 3^k x\right)\right] \\ &\qquad + \frac{3^n - 1}{2\cdot 9^n}\psi\left(3^n x\right) \end{aligned} \tag{4.4}$$

for all $x \in X$ and $n \in \mathbb{N}$. From (4.4) we figure out

$$\begin{aligned} \left\|\frac{f(3^m x)}{3^m} - \frac{f(3^n x)}{3^n}\right\| &= \frac{1}{3^m}\left\|f\left(3^m x\right) - \frac{f(3^{n-m}\cdot 3^m x)}{3^{n-m}}\right\| \\ &\le \sum_{k=1}^{n-m}\left[\frac{3^{k-1}+1}{2\cdot 3^{2k+m}}\varphi\left(3^{k+m}x, 3^{k+m}x, -3^{k+m}x\right)\right. \end{aligned}$$

$$+\frac{3^{k-1}-1}{2\cdot 3^{2k+m}}\varphi\left(-3^{k+m}x,-3^{k+m}x,3^{k+m}x\right)\Bigg]$$
$$+\frac{3^{n-m}-1}{2\cdot 3^{2n-m}}\psi\left(3^{n}x\right)+\frac{9^{n-m}-1}{2\cdot 3^{2n-m}}\left\|f(0)\right\| \tag{4.5}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. Taking the limit as $m \to \infty$ in (4.5) we verify that the right-hand side of inequality (4.5) tends to 0. Thus, the mentioned sequence is convergent to the mapping A; that is,

$$A(x) := \lim_{n\to\infty} 3^{-n}f\left(3^{n}x\right)$$

for all $x \in X$. Replacing x, y, z by $3^n x$, $3^n y$, $3^n z$ in (4.1) and dividing both sides by 3^n, and after taking the limit in the resulting inequality, we see that

$$9A\left(\frac{x+y+z}{3}\right)+4\left[A\left(\frac{x-y}{2}\right)+A\left(\frac{y-z}{2}\right)+A\left(\frac{z-x}{2}\right)\right]$$
$$=3\left[A(x)+A(y)+A(z)\right] \tag{4.6}$$

for all $x, y, z \in X$. By virtue of (4.2) we have $A(-x) = -A(x)$ for all $x \in X$. Setting $y = x$, $z = -x$ in (4.6) yields $3A(\frac{x}{3}) = A(x)$ for all $x \in X$. Similarly, putting $y = 0$, $z = -x$ in (4.6) gives $2A(\frac{x}{2}) = A(x)$ for all $x \in X$. Thus, we can rewrite (4.6) as

$$3A(x+y+z)+2\left[A(x-y)+A(y-z)+A(z-x)\right]=3\left[A(x)+A(y)+A(z)\right] \tag{4.7}$$

for all $x, y, z \in X$. Replacing z by $-x$ in (4.7) we have

$$A(x+y)+A(x-y)=2A(x) \tag{4.8}$$

for all $x, y \in X$. Replacing x, y by $\frac{x+y}{2}$, $\frac{x-y}{2}$ in (4.8) we see that A satisfies the Cauchy functional equation (1.4). Letting $n \to \infty$ in (4.4) we finally obtain the result (4.3).

To show the uniqueness of A we assume that $A' : X \to Y$ be another quadratic mapping satisfying (4.3). Obviously, we have $A(3^n x) = 3^n A(x)$ and $A'(3^n x) = 3^n A'(x)$ for all $x \in X$. According to the additive property of A and A' we figure out

$$\begin{aligned}
&\left\|A(x)-A'(x)\right\| \\
&\quad=\frac{1}{3^{n}}\left\|A\left(3^{n}x\right)-A'\left(3^{n}x\right)\right\| \\
&\quad\le\frac{1}{3^{n}}\left\|A\left(3^{n}x\right)-f\left(3^{n}x\right)-\frac{1}{2}f(0)\right\|+\frac{1}{3^{n}}\left\|f\left(3^{n}x\right)-A'\left(3^{n}x\right)+\frac{1}{2}f(0)\right\| \\
&\quad\le\frac{2}{3^{n}}\sum_{k=1}^{\infty}\left[\frac{3^{k-1}+1}{2\cdot 9^{k}}\varphi\left(3^{k+n}x,3^{k+n}x,-3^{k+n}x\right)+\frac{3^{k-1}-1}{2\cdot 9^{k}}\varphi\left(-3^{k+n}x,-3^{k+n}x,3^{k+n}x\right)\right]
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in X$, which means the uniqueness of A. □

From the theorem above we obtain the following corollaries immediately.

Corollary 4.2 *Let $\epsilon \geq 0$, $p < 1$, and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \epsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$ ($x, y, z \in X\backslash\{0\}$ if $p < 0$) and

$$\|f(x) + f(-x)\| \leq \psi(x)$$

for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (1.4) *such that*

$$\left\|f(x) - A(x) + \frac{1}{2}f(0)\right\| \leq \frac{3^p}{3 - 3^p}\epsilon\|x\|^p$$

for all $x \in X$ ($x \in X\backslash\{0\}$ if $p < 0$).

Corollary 4.3 *Let $\epsilon \geq 0$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \epsilon$$

for all $x, y, z \in X$ and

$$\|f(x) + f(-x)\| \leq \psi(x)$$

for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (1.4) *such that*

$$\left\|f(x) - A(x) + \frac{1}{2}f(0)\right\| \leq \frac{1}{6}\epsilon$$

for all $x \in X$.

We have the following result, which is analogous to Theorem 4.1.

Theorem 4.4 *Let $\varphi : X \times X \times X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_1)$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \leq \varphi(x,y,z) \tag{4.9}$$

for all $x, y, z \in X$ and

$$\|f(x) + f(-x)\| \leq \psi(x) \tag{4.10}$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\begin{aligned}&\|f(x) - A(x)\| \\ &\quad \le \sum_{k=0}^{\infty}\left[\frac{3\cdot 9^k + 3^k}{6}\varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3\cdot 9^k - 3^k}{6}\varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right)\right]\end{aligned} \tag{4.11}$$

for all $x \in X$.

Proof From Lemma 2.2 and the approximately odd condition (4.10) we have

$$\begin{aligned}&\left\|f(x) - 3^n f\left(\frac{x}{3^n}\right)\right\| \\ &\quad \le \sum_{k=0}^{n-1}\left[\frac{3\cdot 9^k + 3^k}{6}\varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3\cdot 9^k - 3^k}{6}\varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right)\right] \\ &\qquad + \frac{9^n - 3^n}{2}\psi\left(\frac{x}{3^n}\right)\end{aligned} \tag{4.12}$$

for all $x \in X$ and $n \in \mathbb{N}$. By (4.12) we obtain

$$\begin{aligned}&\left\|3^m f\left(\frac{x}{3^m}\right) - 3^n f\left(\frac{x}{3^n}\right)\right\| \\ &\quad = 3^m\left\|f\left(\frac{x}{3^m}\right) - 3^{n-m} f\left(\frac{x}{3^{n-m}\cdot 3^m}\right)\right\| \\ &\quad \le \sum_{k=0}^{n-m-1}\left[\frac{3\cdot 3^{2k+m} + 3^{k+m}}{6}\varphi\left(\frac{x}{3^{k+m}}, \frac{x}{3^{k+m}}, -\frac{x}{3^{k+m}}\right)\right. \\ &\qquad \left. + \frac{3\cdot 3^{2k+m} - 3^{k+m}}{6}\varphi\left(-\frac{x}{3^{k+m}}, -\frac{x}{3^{k+m}}, \frac{x}{3^{k+m}}\right)\right] + \frac{3^{2n-m} - 3^n}{2}\psi\left(\frac{x}{3^n}\right)\end{aligned} \tag{4.13}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. Since the right-hand side of inequality (4.13) tends to 0 as $m \to \infty$, the sequence $\{3^n f(3^{-n}x)\}$ is a Cauchy sequence. Thus, we can define a mapping

$$A(x) := \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$$

for all $x \in X$. Replacing x, y, z by $\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}$ in (4.9) and multiplying both sides by 3^n, and after taking the limit in the resulting inequality, we see that A satisfies (1.2). Using a similar method to the proof of Theorem 4.1 we see that A is the unique additive mapping satisfying (1.4). Taking the limit as $n \to \infty$ in (4.12) we finally obtain the result (4.11). □

Corollary 4.5 *Let* $\epsilon \ge 0, p > 2$, *and* $\psi : X \to [0,\infty)$ *satisfy the condition* $(\mathcal{B}_2)$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x,y,z)\| \le \epsilon\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

for all $x, y, z \in X$ *and*

$$\|f(x) + f(-x)\| \leq \psi(x)$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\|f(x) - A(x)\| \leq \frac{3^{p+1}}{3^p - 9}\epsilon\|x\|^p$$

for all $x \in X$ $(x \in X \backslash \{0\}$ *if* $p < 0)$.

5 Approximately quadratic case

We are going to prove the stability of (1.2) under the approximately quadratic condition.

Theorem 5.1 *Let* $\varphi : X \times X \times X \to [0, \infty)$ *satisfy the condition* $(\mathcal{A}_1)$ *and* $\psi : X \to [0, \infty)$ *satisfy the condition* $(\mathcal{A}_2)$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{5.1}$$

for all $x, y, z \in X$ *and*

$$\|f(3x) - 9f(-x)\| \leq \psi(x) \tag{5.2}$$

for all $x \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *satisfying* (1.3) *such that*

$$\begin{aligned}&\left\|f(x) - Q(x) + \frac{1}{2}f(0)\right\| \\ &\quad\leq \sum_{k=1}^{\infty}\left[\frac{3^{k-1}+1}{2\cdot 9^k}\varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi\left(-3^k x, -3^k x, 3^k x\right)\right]\end{aligned} \tag{5.3}$$

for all $x \in X$.

Proof Setting $y = x$, $z = -x$ in (5.1) gives

$$\left\|9f\left(\frac{x}{3}\right) - 2f(x) + f(-x) + 4f(0)\right\| \leq \varphi(x, x, -x) \tag{5.4}$$

for all $x \in X$ and $n \in \mathbb{N}$. Combining (5.2), (5.4) and Lemma 2.1 we have

$$\begin{aligned}&\left\|f(x) - \frac{f(3^n x)}{9^n} + \frac{9^n + 2\cdot 3^n - 3}{2\cdot 9^n}f(0)\right\| \\ &\quad\leq \sum_{k=1}^{n}\left[\frac{3^{k-1}+1}{2\cdot 9^k}\varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2\cdot 9^k}\varphi\left(-3^k x, -3^k x, 3^k x\right)\right] \\ &\qquad + \frac{3^n-1}{4\cdot 9^n}\varphi\left(-3^n x, -3^n x, 3^n x\right) + \frac{3^n-1}{4\cdot 9^n}\psi\left(3^{n-1}x\right)\end{aligned} \tag{5.5}$$

for all $x \in X$ and $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$ with $n > m$, we verify by (5.5) that

$$\begin{aligned}
&\left\| \frac{f(3^m x)}{9^m} - \frac{f(3^n x)}{9^n} \right\| \\
&\quad = \frac{1}{9^m} \left\| f\left(3^m x\right) - \frac{f(3^{n-m} \cdot 3^m x)}{9^{n-m}} \right\| \\
&\quad \le \sum_{k=1}^{n-m} \left[\frac{3^{k-1}+1}{2 \cdot 9^{k+m}} \varphi\left(3^{k+m}x, 3^{k+m}x, -3^{k+m}x\right) + \frac{3^{k-1}-1}{2 \cdot 9^{k+m}} \varphi\left(-3^{k+m}x, -3^{k+m}x, 3^{k+m}x\right) \right] \\
&\qquad + \frac{3^{n-m}-1}{4 \cdot 9^n} \varphi\left(-3^n x, -3^n x, 3^n x\right) + \frac{3^{n-m}-1}{4 \cdot 9^n} \psi\left(3^{n-1}x\right) \\
&\qquad + \frac{9^{n-m} + 2 \cdot 3^{n-m} - 3}{2 \cdot 9^n} \left\| f(0) \right\|
\end{aligned} \tag{5.6}$$

for all $x \in X$. Since the right-hand side of inequality (5.6) tends to 0 as $m \to \infty$, we can define a mapping

$$Q(x) := \lim_{n\to\infty} 9^{-n} f\left(3^n x\right)$$

for all $x \in X$. Replacing x, y, z by $3^n x$, $3^n y$, $3^n z$ in (5.1) and dividing both sides by 9^n, and after taking the limit in the resulting inequality we see that Q satisfies (1.2). Letting $n \to \infty$ in (5.4) we have the result (5.3). The rest of the proof is similar to that of the proof of Theorem 3.1. □

The following corollaries are immediate consequences of the above theorem.

Corollary 5.2 *Let $\epsilon \ge 0$, $p < 1$, and $\psi : X \to [0, \infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\left\| Df(x, y, z) \right\| \le \epsilon \left(\|x\|^p + \|y\|^p + \|z\|^p \right)$$

for all $x, y, z \in X$ ($x, y, z \in X \backslash \{0\}$ if $p < 0$) and

$$\left\| f(3x) - 9f(-x) \right\| \le \psi(x)$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) such that

$$\left\| f(x) - Q(x) + \frac{1}{2} f(0) \right\| \le \frac{3^p}{3 - 3^p} \epsilon \|x\|^p$$

for all $x \in X$ ($x \in X \backslash \{0\}$ if $p < 0$).

Corollary 5.3 *Let $\epsilon \ge 0$ and $\psi : X \to [0, \infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\left\| Df(x, y, z) \right\| \le \epsilon$$

for all $x, y, z \in X$ and

$$\|f(3x) - 9f(-x)\| \le \psi(x)$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\left\|f(x) - Q(x) + \frac{1}{2}f(0)\right\| \le \frac{1}{6}\epsilon$$

for all $x \in X$.

We have the following result, which is analogous to Theorem 5.1.

Theorem 5.4 *Let $\varphi : X \times X \times X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_1)$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x,y,z)\| \le \varphi(x,y,z) \tag{5.7}$$

for all $x, y, z \in X$ and

$$\|f(3x) - 9f(-x)\| \le \psi(x) \tag{5.8}$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\begin{aligned}
&\|f(x) - Q(x)\| \\
&\quad \le \sum_{k=0}^{\infty}\left[\frac{3\cdot 9^k + 3^k}{6}\varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3\cdot 9^k - 3^k}{6}\varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right)\right]
\end{aligned} \tag{5.9}$$

for all $x \in X$.

Proof It follows from (5.7), (5.8), and Lemma 2.2 that we have

$$\begin{aligned}
&\left\|f(x) - 9^n f\left(\frac{x}{3^n}\right)\right\| \\
&\quad \le \sum_{k=0}^{n-1}\left[\frac{3\cdot 9^k + 3^k}{6}\varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k}\right) + \frac{3\cdot 9^k - 3^k}{6}\varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k}\right)\right] \\
&\qquad + \frac{9^n - 3^n}{4}\varphi\left(-\frac{x}{3^n}, -\frac{x}{3^n}, \frac{x}{3^n}\right) + \frac{9^n - 3^n}{4}\psi\left(\frac{x}{3^{n+1}}\right)
\end{aligned} \tag{5.10}$$

for all $x \in X$ and $n \in \mathbb{N}$. From (5.10) we obtain

$$\begin{aligned}
&\left\|9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right)\right\| \\
&\quad = 9^m\left\|f\left(\frac{x}{3^m}\right) - 9^{n-m} f\left(\frac{x}{3^{n-m}\cdot 3^m}\right)\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n-m-1}\biggl[\frac{3\cdot 9^{k+m}+3^{k+2m}}{6}\varphi\biggl(\frac{x}{3^{k+m}},\frac{x}{3^{k+m}},-\frac{x}{3^{k+m}}\biggr)\\
&\quad+\frac{3\cdot 9^{k+m}-3^{k+2m}}{6}\varphi\biggl(-\frac{x}{3^{k+m}},-\frac{x}{3^{k+m}},\frac{x}{3^{k+m}}\biggr)\biggr]\\
&\quad+\frac{9^n-3^{n+m}}{4}\varphi\biggl(-\frac{x}{3^n},-\frac{x}{3^n},\frac{x}{3^n}\biggr)+\frac{9^n-3^{n+m}}{4}\psi\biggl(\frac{x}{3^{n+1}}\biggr)
\end{aligned}\tag{5.11}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. Note that the right-hand side of inequality (5.11) tends to 0 as $m \to \infty$. This means the sequence $\{9^n f(3^{-n}x)\}$ is a Cauchy sequence. Now we define a mapping

$$Q(x) := \lim_{n\to\infty} 9^n f\biggl(\frac{x}{3^n}\biggr)$$

for all $x \in X$. Replacing x, y, z by $\frac{x}{3^n}$, $\frac{y}{3^n}$, $\frac{z}{3^n}$ in (5.7) and multiplying both sides by 9^n, and after taking the limit in the resulting inequality, we see that Q satisfies (1.2). Letting $n \to \infty$ in (5.10) we have the result (5.9). Using a similar method to the proof of Theorem 3.1 we see that Q is the unique quadratic mapping satisfying (1.3). □

Corollary 5.5 *Let $\epsilon \geq 0$, $p > 2$, and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{B}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\bigl\|Df(x,y,z)\bigr\| \leq \epsilon\bigl(\|x\|^p + \|y\|^p + \|z\|^p\bigr)$$

for all $x, y, z \in X$ and

$$\bigl\|f(3x) - 9f(-x)\bigr\| \leq \psi(x)$$

for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.3) *such that*

$$\bigl\|f(x) - Q(x)\bigr\| \leq \frac{3^{p+1}}{3^p - 9}\epsilon\|x\|^p$$

for all $x \in X$.

6 Approximately additive case

In this section, we establish the stability of (1.2) under the approximately additive condition.

Theorem 6.1 *Let $\varphi : X \times X \times X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_1)$ and $\psi : X \to [0,\infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\bigl\|Df(x,y,z)\bigr\| \leq \varphi(x,y,z)\tag{6.1}$$

for all $x, y, z \in X$ and

$$\bigl\|f(3x) + 3f(-x)\bigr\| \leq \psi(x)\tag{6.2}$$

for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying (1.4) *such that*

$$\begin{aligned}&\left\| f(x) - A(x) + \frac{1}{2}f(0) \right\| \\ &\quad \le \sum_{k=1}^{\infty} \left[\frac{3^{k-1}+1}{2 \cdot 9^k} \varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2 \cdot 9^k} \varphi\left(-3^k x, -3^k x, 3^k x\right) \right] \end{aligned} \tag{6.3}$$

for all $x \in X$.

Proof It follows from (6.1), (6.2), and Lemma 2.1 that we have

$$\begin{aligned}&\left\| f(x) - \frac{f(3^n x)}{3^n} + \frac{9^n + 2 \cdot 3^n - 3}{2 \cdot 9^n} f(0) \right\| \\ &\quad \le \sum_{k=1}^{n} \left[\frac{3^{k-1}+1}{2 \cdot 9^k} \varphi\left(3^k x, 3^k x, -3^k x\right) + \frac{3^{k-1}-1}{2 \cdot 9^k} \varphi\left(-3^k x, -3^k x, 3^k x\right) \right] \\ &\qquad + \frac{3^n - 1}{4 \cdot 9^n} \varphi\left(-3^n x, -3^n x, 3^n x\right) + \frac{3^{n+1} - 3}{4 \cdot 9^n} \psi\left(3^{n-1} x\right) \end{aligned} \tag{6.4}$$

for all $x \in X$ and $n \in \mathbb{N}$. From (6.4) we figure out

$$\begin{aligned}&\left\| \frac{f(3^m x)}{3^m} - \frac{f(3^n x)}{3^n} \right\| \\ &\quad = \frac{1}{3^m} \left\| f\left(3^m x\right) - \frac{f(3^{n-m} \cdot 3^m x)}{3^{n-m}} \right\| \\ &\quad \le \sum_{k=1}^{n-m} \left[\frac{3^{k-1}+1}{2 \cdot 3^{2k+m}} \varphi\left(3^{k+m} x, 3^{k+m} x, -3^{k+m} x\right) + \frac{3^{k-1}-1}{2 \cdot 3^{2k+m}} \varphi\left(-3^{k+m} x, -3^{k+m} x, 3^{k+m} x\right) \right] \\ &\qquad + \frac{3^{n-m} - 1}{4 \cdot 3^{2n-m}} \varphi\left(-3^n x, -3^n x, 3^n x\right) + \frac{3^{n-m+1} - 3}{4 \cdot 3^{2n-m}} \psi\left(3^{n-m} x\right) \\ &\qquad + \frac{9^{n-m} + 2 \cdot 3^{n-m} - 3}{2 \cdot 3^{2n-m}} \left\| f(0) \right\| \end{aligned} \tag{6.5}$$

for all $x \in X$ and $m, n \in \mathbb{N}$ with $n > m$. We remark that the right-hand side of inequality (6.5) tends to 0 as $m \to \infty$. Define a mapping

$$A(x) := \lim_{n \to \infty} 3^{-n} f\left(3^n x\right)$$

for all $x \in X$. Replacing x, y, z by $3^n x$, $3^n y$, $3^n z$ in (6.1) and dividing both sides by 3^n, and after taking the limit in the resulting inequality, we see that A satisfies (1.2). Letting $n \to \infty$ in (6.4) we obtain the result (6.3). The remains of the proof are similar to that of the proof of Theorem 4.1. □

From the theorem above we have the following corollaries.

Corollary 6.2 *Let $\epsilon \ge 0$, $p < 1$, and $\psi : X \to [0, \infty)$ satisfy the condition $(\mathcal{A}_2)$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\left\| Df(x, y, z) \right\| \le \epsilon \left(\|x\|^p + \|y\|^p + \|z\|^p \right)$$

for all $x, y, z \in X$ ($x, y, z \in X \backslash \{0\}$ *if* $p < 0$) *and*

$$\|f(3x) + 3f(-x)\| \leq \psi(x)$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\left\| f(x) - A(x) + \frac{1}{2} f(0) \right\| \leq \frac{3^p}{3 - 3^p} \epsilon \|x\|^p$$

for all $x \in X$ ($x \in X \backslash \{0\}$ *if* $p < 0$).

Corollary 6.3 *Let* $\epsilon \geq 0$ *and* $\psi : X \to [0, \infty)$ *satisfy the condition* $(\mathcal{A}_2)$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x, y, z)\| \leq \epsilon$$

for all $x, y, z \in X$ *and*

$$\|f(3x) + 3f(-x)\| \leq \psi(x)$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\left\| f(x) - A(x) + \frac{1}{2} f(0) \right\| \leq \frac{1}{6} \epsilon$$

for all $x \in X$.

We have the following result, which is analogous to Theorem 6.1.

Theorem 6.4 *Let* $\varphi : X \times X \times X \to [0, \infty)$ *satisfy the condition* $(\mathcal{B}_1)$ *and* $\psi : X \to [0, \infty)$ *satisfy the condition* $(\mathcal{B}_2)$. *Suppose that a mapping* $f : X \to Y$ *satisfies*

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \tag{6.6}$$

for all $x, y, z \in X$ *and*

$$\|f(3x) + 3f(-x)\| \leq \psi(x) \tag{6.7}$$

for all $x \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *satisfying* (1.4) *such that*

$$\begin{aligned} &\|f(x) - A(x)\| \\ &\quad \leq \sum_{k=0}^{\infty} \left[\frac{3 \cdot 9^k + 3^k}{6} \varphi\left(\frac{x}{3^k}, \frac{x}{3^k}, -\frac{x}{3^k} \right) + \frac{3 \cdot 9^k - 3^k}{6} \varphi\left(-\frac{x}{3^k}, -\frac{x}{3^k}, \frac{x}{3^k} \right) \right] \end{aligned} \tag{6.8}$$

for all $x \in X$.

Proof Combining (6.6), (6.7), and Lemma 2.2 we have

$$\begin{aligned}
&\left\|f(x)-3^nf\left(\frac{x}{3^n}\right)\right\| \\
&\quad\le\sum_{k=0}^{n-1}\left[\frac{3\cdot 9^k+3^k}{6}\varphi\left(\frac{x}{3^k},\frac{x}{3^k},-\frac{x}{3^k}\right)+\frac{3\cdot 9^k-3^k}{6}\varphi\left(-\frac{x}{3^k},-\frac{x}{3^k},\frac{x}{3^k}\right)\right] \\
&\qquad+\frac{9^n-3^n}{4}\varphi\left(-\frac{x}{3^n},-\frac{x}{3^n},\frac{x}{3^n}\right)+\frac{3(9^n-3^n)}{4}\psi\left(\frac{x}{3^{n+1}}\right)
\end{aligned}\tag{6.9}$$

for all $x\in X$ and $n\in\mathbb{N}$. We claim that $\{9^nf(3^{-n}x)\}$ is a Cauchy sequence. From (6.9) we obtain

$$\begin{aligned}
&\left\|3^mf\left(\frac{x}{3^m}\right)-3^nf\left(\frac{x}{3^n}\right)\right\| \\
&\quad=3^m\left\|f\left(\frac{x}{3^m}\right)-3^{n-m}f\left(\frac{x}{3^{n-m}\cdot 3^m}\right)\right\| \\
&\quad\le\sum_{k=0}^{n-m-1}\left[\frac{3\cdot 3^{2k+m}+3^{k+m}}{6}\varphi\left(\frac{x}{3^{k+m}},\frac{x}{3^{k+m}},-\frac{x}{3^{k+m}}\right)\right. \\
&\qquad\left.+\frac{3\cdot 3^{2k+m}-3^{k+m}}{6}\varphi\left(-\frac{x}{3^{k+m}},-\frac{x}{3^{k+m}},\frac{x}{3^{k+m}}\right)\right] \\
&\qquad+\frac{3^{2n-m}-3^n}{4}\varphi\left(-\frac{x}{3^n},-\frac{x}{3^n},\frac{x}{3^n}\right)+\frac{3^{2n-m+1}-3^{n+1}}{4}\psi\left(\frac{x}{3^n}\right)
\end{aligned}\tag{6.10}$$

for all $x\in X$ and $m,n\in\mathbb{N}$ with $n>m$. Since the right-hand side of inequality (6.10) tends to 0 as $m\to\infty$, the sequence $\{3^nf(3^{-n}x)\}$ is a Cauchy sequence. Define a mapping

$$A(x):=\lim_{n\to\infty}3^nf\left(\frac{x}{3^n}\right)$$

for all $x\in X$. Replacing x, y, z by $\frac{x}{3^n}$, $\frac{y}{3^n}$, $\frac{z}{3^n}$ in (6.6) and multiplying both sides by 3^n, and after taking the limit in the resulting inequality, we see that A satisfies (1.2). Using a similar method to the proof of Theorem 4.1 we see that A is the unique additive mapping satisfying (1.4). Finally taking the limit as $n\to\infty$ in (6.9) we have the result (6.8). □

Corollary 6.5 *Let $\epsilon\ge 0$, $p>2$, and $\psi:X\to[0,\infty)$ satisfy the condition* ($\mathcal{B}_2$). *Suppose that a mapping $f:X\to Y$ satisfies*

$$\|Df(x,y,z)\|\le\epsilon\left(\|x\|^p+\|y\|^p+\|z\|^p\right)$$

for all $x,y,z\in X$ and

$$\|f(3x)+3f(-x)\|\le\psi(x)$$

for all $x\in X$. Then there exists a unique additive mapping $A:X\to Y$ satisfying (1.4) *such that*

$$\|f(x)-A(x)\|\le\frac{3^{p+1}}{3^p-9}\epsilon\|x\|^p$$

for all $x\in X$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. All authors read and approved the final manuscript.

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References

1. Ulam, SM: Problems in Modern Mathematics. Wiley, New York (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
4. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
5. Czerwik, S: Functional Equations and Inequalities in Several Variables. World Scientific, River Edge (2002)
6. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
7. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
8. Jung, S-M: Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. Springer Optimization and Its Applications. Springer, New York (2011)
9. Kannappan, P: Functional Equations and Inequalities with Applications. Springer, New York (2009)
10. Sahoo, PK, Kannappan, P: Introduction to Functional Equations. CRC Press, Boca Raton (2011)
11. Kannappan, P: Quadratic functional equation and inner product spaces. Results Math. **27**, 368-372 (1995)
12. Jung, S-M: On the Hyers-Ulam stability of the functional equations that have the quadratic property. J. Math. Anal. Appl. **222**, 126-137 (1998)
13. Jun, K-W, Kim, H-M: On the stability of an *n*-dimensional quadratic and additive functional equation. Math. Inequal. Appl. **9**, 153-165 (2006)
14. Najati, A, Moghimi, MB: Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces. J. Math. Anal. Appl. **337**, 399-415 (2008)
15. Lee, Y-S, Na, J, Woo, H: On the stability of a mixed type quadratic-additive functional equation. Adv. Differ. Equ. **2013**, Article ID 198 (2013)
16. Eshaghi Gordji, M, Azadi Kenary, H, Rezaei, H, Lee, YW, Kim, GH: Solution and Hyers-Ulam-Rassias stability of generalized mixed type additive-quadratic functional equations in fuzzy Banach spaces. Abstr. Appl. Anal. **2012**, Article ID 953938 (2012)
17. Janfada, M, Shourvazi, R: Solutions and the generalized Hyers-Ulam-Rassias stability of a generalized quadratic-additive functional equation. Abstr. Appl. Anal. **2011**, Article ID 326951 (2011)
18. Lee, Y-H, Jung, S-M, Themistocles, M: Stability of the *n*-dimensional mixed-type additive and quadratic functional equation in non-Archimedean normed spaces. Abstr. Appl. Anal. **2012**, Article ID 401762 (2012)
19. Lee, Y-S: Stability of a quadratic functional equation in the spaces of generalized functions. J. Inequal. Appl. **2008**, Article ID 210615 (2008)
20. Nakmahachalasint, P: On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations. Int. J. Math. Math. Sci. **2007**, Article ID 63239 (2007)
21. Najati, A, Eskandani, GZ: A fixed point method to the generalized stability of a mixed additive and quadratic functional equation in Banach modules. J. Differ. Equ. Appl. **16**, 773-788 (2010)
22. Towanlong, W, Nakmahachalasint, P: An *n*-dimensional mixed-type additive and quadratic functional equation and its stability. Science Asia **35**, 381-385 (2009)
23. Lee, YW: On the stability of a quadratic Jensen type functional equation. J. Math. Anal. Appl. **270**, 590-601 (2002)
24. Kenary, HA, Cho, YJ: Stability of mixed additive-quadratic Jensen type functional equation in various spaces. Comput. Math. Appl. **61**, 2704-2724 (2011)
25. Xu, TZ, Rassias, JM, Xu, WX: Generalized Hyers-Ulam stability of a general mixed additive-cubic functional equation in quasi-Banach spaces. Acta Math. Sin. **28**, 529-560 (2012)
26. Xu, TZ: Stability of multi-Jensen mappings in non-Archimedean normed spaces. J. Math. Phys. **53**, 023507 (2012)
27. Ravi, K, Senthil Kumar, BV: Ulam-Gavruta-Rassias stability of Rassias reciprocal functional equation. Glob. J. Math. Math. Sci. **3**(1-2), 57-79 (2010)
28. Ravi, K, Rassias, JM, Senthil Kumar, BV: Ulam stability of reciprocal difference and adjoint functional equations. Aust. J. Math. Anal. Appl. **8**(1), 13 (2011)
29. Ravi, K, Rassias, JM, Senthil Kumar, BV: Ulam stability of generalized reciprocal functional equation in several variables. Int. J. Appl. Math. Stat. **19**, D10 (2010)
30. Ravi, K, Rassias, JM, Senthil Kumar, BV: A fixed point approach to the generalized Hyers-Ulam stability of reciprocal difference and adjoint functional equations. Thai J. Math. **8**(3), 469-481 (2010)