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Homoclinic orbits for asymptotically linear discrete Hamiltonian systems

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Abstract
We study the existence of homoclinic solutions for the following second-order self-adjoint discrete Hamiltonian system: $\triangle[p(n)\triangle u(n-1)] - L(n)u(n) + \nabla W(n,u(n)) = 0$, where $p(n)$, $L(n)$, and $W(n,x)$ are N-periodic in n, and $\nabla W(n,x)$ is asymptotically linear in x as $|x| \to \infty$.
MSC: 39A11; 58E05; 70H05

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1 Introduction

Discrete Hamiltonian systems can be applied in many areas, such as physics, chemistry, and so on. For more discussions on discrete Hamiltonian systems, we refer the reader to [1, 2]. In this paper, we consider the second-order self-adjoint discrete Hamiltonian system

$$\triangle\big[p(n)\triangle u(n-1)\big] - L(n)u(n) + \nabla W\big(n,u(n)\big) = 0, \tag{1.1}$$

where $n \in \mathbb{Z}$, $u \in \mathbb{R}^{\mathcal{N}}$, $\triangle u(n) = u(n+1) - u(n)$ is the forward difference, $p, L : \mathbb{Z} \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ and $W : \mathbb{Z} \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$.

As usual, we say that a solution $u(n)$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

In recent years, several authors studied homoclinic orbits for system (1.1) or its special forms via critical point theory. For example, see [3–18]. We emphasize that in all these papers the nonlinear term was assumed to be superlinear or sublinear at infinity. To the best of our knowledge, the existence of homoclinics for asymptotically linear discrete Hamiltonian systems has not been previously studied.

In this paper, we assume that $p(n)$ and $L(n)$ are N-periodic $\mathcal{N} \times \mathcal{N}$ real symmetric matrices. Let $\mathcal{A}$ is an operator defined as follows:

$$(\mathcal{A}u)(n) = \triangle\big[p(n)\triangle u(n-1)\big] - L(n)u(n), \quad \forall n \in \mathbb{Z}.$$

Then it is easy to check that $\mathcal{A}$ is a bounded self-adjoint operator in $l^2(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$, where $l^2(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$ is defined in Section 2. By the Floquet theorem, it is easy to verify that $\mathcal{A}$ has only continuous spectrum $\sigma(\mathcal{A})$, which is a union of bounded closed intervals.

When $p(n)$ and $L(n)$ are positive definite, $\sigma(\mathcal{A}) \subset (0,+\infty)$. In this case, the mountain pass theorem of Ambrosetti and Rabinowitz is a very useful tool for finding critical points of the energy functionals associated to (1.1). However, when $p(n)$ or $L(n)$ is not positive definite, 0 is a saddle point rather than a local minimum of the functional associated to (1.1), which is strongly indefinite. This case is difficult because the mountain-pass reduction of the definite case is not available, and it is not known if the Palais-Smale sequences are bounded. We choose this case as the object of the present paper.

To state our results, we first introduce the following assumptions:

(PL) $p(n)$ and $L(n)$ are N-periodic $\mathcal{N} \times \mathcal{N}$ real symmetric matrices, and

$$\sup\left[\sigma(\mathcal{A}) \cap (-\infty, 0)\right] := \underline{\Lambda} < 0 < \bar{\Lambda} := \inf\left[\sigma(\mathcal{A}) \cap (0,\infty)\right]; \tag{1.2}$$

(W1) $W(n,x)$ is continuously differentiable in x for every $n \in \mathbb{Z}$, $W(n,0) = 0$, $W(n,x) \geq 0$, and $W(n,x)$ is N-periodic in n;

(W2) $\nabla W(n,x) = o(|x|)$ as $|x| \to 0$ uniformly for $n \in \mathbb{Z}$;

(W3) $W(n,x) = \frac{1}{2}M(n)x \cdot x + W_\infty(n,x)$, where $M(n)$ is an N-periodic $\mathcal{N} \times \mathcal{N}$ real symmetric matrix, $\inf_{n\in\mathbb{Z},|x|=1} M(n)x \cdot x > \bar{\Lambda}$, $\nabla W_\infty(n,x) = o(|x|)$ as $|x| \to \infty$, uniformly for $n \in \mathbb{Z}$;

(W4) $\widetilde{W}(n,x) := \frac{1}{2}\nabla W(n,x) \cdot x - W(n,x) \geq 0$, $\forall (n,x) \in \mathbb{Z} \times \mathbb{R}^{\mathcal{N}}$, and there exists a $\delta_0 \in (0, \Lambda_0)$ with $\Lambda_0 = \min\{-\underline{\Lambda}, \bar{\Lambda}\}$ such that

$$\frac{|\nabla W(n,x)|}{|x|} \geq \Lambda_0 - \delta_0 \quad \Rightarrow \quad \widetilde{W}(n,x) \geq \delta_0.$$

Now, we are ready to state the main result of this paper.

Theorem 1.1 *Assume that p, L, and W satisfy* (PL), (W1), (W2), (W3), *and* (W4). *Then system* (1.1) *possesses a nontrivial homoclinic solution.*

Remark 1.2 The following functions satisfy (W1)-(W4):

$$W(n,x) = a(n)|x|^2\left[1 - \frac{1}{\ln(e + |x|)}\right], \tag{1.3}$$

$$W(n,x) = \int_0^{|x|} \alpha(n,s)s\,\mathrm{d}s, \tag{1.4}$$

where $a(n)$ and $\alpha(n,s)$ are N-periodic positive function in n, $\alpha(n,s)$ is non-decreasing for $s \in [0,\infty)$, $\alpha(n,s) \to 0$ as $s \to 0$ and $\alpha(n,s) \to b(n)$ as $s \to \infty$ with $\inf_{\mathbb{Z}} b > \bar{\Lambda}$, uniformly in $n \in \mathbb{Z}$.

2 Proof of theorem

Let

$$S = \left\{\left\{u(n)\right\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^{\mathcal{N}}, n \in \mathbb{Z}\right\}.$$

As usual, for $1 \leq q < \infty$, set

$$l^q\left(\mathbb{Z}, \mathbb{R}^{\mathcal{N}}\right) = \left\{u \in S : \sum_{n\in\mathbb{Z}}\left|u(n)\right|^q < \infty\right\}$$

and

$$l^{\infty}(\mathbb{Z},\mathbb{R}^{\mathcal{N}}) = \left\{u \in S : \sup_{n\in\mathbb{Z}}\left|u(n)\right| < \infty\right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}}\left|u(n)\right|^q\right)^{1/q}, \quad \forall\, u \in l^q(\mathbb{Z},\mathbb{R}^{\mathcal{N}});$$

$$\|u\|_{\infty} = \sup_{n\in\mathbb{Z}}\left|u(n)\right|, \quad \forall u \in l^{\infty}(\mathbb{Z},\mathbb{R}^{\mathcal{N}}),$$

respectively. In particular, $l^2(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$ is a Hilbert space with the following inner product:

$$(u,v)_{l^2} = \sum_{n\in\mathbb{Z}} u(n)\cdot v(n), \quad \forall u,v \in l^2(\mathbb{Z},\mathbb{R}^{\mathcal{N}}).$$

Let $\{\mathcal{E}(\lambda) : -a_0 \leq \lambda \leq b_0\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [19, Theorem 4.3.3]).

As in [20], let $E = l^2(\mathbb{Z},\mathbb{R}^{\mathcal{N}})$ and

$$E^- = \mathcal{E}(0)E, \qquad E^+ = \left[\mathrm{id} - \mathcal{E}(0)\right]E.$$

For any $u \in E$, it is easy to see that

$$u^- := \mathcal{E}(0)u \in E^-, \qquad u^+ := \left[\mathrm{id} - \mathcal{E}(0)\right]u \in E^+, \qquad u = u^- + u^+, \tag{2.1}$$

and

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \qquad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall u \in E. \tag{2.2}$$

Let

$$(u,v) = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{l^2}, \quad \forall u,v \in E. \tag{2.3}$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| = \left\||\mathcal{A}|^{1/2}u\right\|_2, \quad \forall u \in E. \tag{2.4}$$

By virtue of (2.1)-(2.4), one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{l^2}$ and $(\cdot,\cdot)$. Moreover,

$$-\underline{\Lambda}\left\|u^-\right\|_2^2 \leq \left\|u^-\right\|^2 \leq a_0\left\|u^-\right\|_2^2, \qquad \bar{\Lambda}\left\|u^+\right\|_2^2 \leq \left\|u^+\right\|^2 \leq b_0\left\|u^+\right\|_2^2, \quad \forall u \in E, \tag{2.5}$$

and

$$\Lambda_0\|u\|_2^2 \leq \|u\|^2 \leq \max\{a_0,b_0\}\|u\|_2^2, \quad \forall u \in E. \tag{2.6}$$

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in C^1(X,\mathbb{R})$, φ is said to be weakly sequentially lower semi-continuous if for any $u_k \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_k)$, and φ' is said to be weakly sequentially continuous if $\lim_{k\to\infty}\langle\varphi'(u_k), v\rangle = \langle\varphi'(u), v\rangle$ for each $v \in X$.

Lemma 2.1 ([21, Theorem 2.1]) *Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X,\mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(LS1) $\psi \in C^1(X,\mathbb{R})$ *is bounded from below and weakly sequentially lower semi-continuous*;

(LS2) ψ' *is weakly sequentially continuous*;

(LS3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf \varphi\left(S_\rho^+\right) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \left\{u \in X^+ : \|u\| = \rho\right\}, \qquad Q = \left\{se + v : v \in X^-, s \ge 0, \|se + v\| \le r\right\}.$$

Then for some $c \ge \kappa$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \left\|\varphi'(u_n)\right\|\left(1 + \|u_n\|\right) \to 0. \tag{2.7}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$ sequence.

Now we define a functional Φ on E by

$$\Phi(u) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left[p(n+1)\triangle u \cdot \triangle u + L(n)u \cdot u\right] - \sum_{n\in\mathbb{Z}} W(n,u). \tag{2.8}$$

For any $u \in E$, there exists an $n_0 \in \mathbb{N}$ such that $|u(n)| \le 1$ for $|n| \ge n_0$. Hence, under assumptions (PL), (W1), and (W2), the functional Φ is of class $C^1(E,\mathbb{R})$. Moreover,

$$\left\langle\Phi'(u), v\right\rangle = \sum_{n\in\mathbb{Z}}\left[p(n+1)\triangle u \cdot \triangle v + L(n)u \cdot v\right] - \sum_{n\in\mathbb{Z}} \nabla W(n,u) \cdot v, \quad \forall u, v \in E. \tag{2.9}$$

By virtue of (2.1), (2.2), (2.3), and (2.4), one has

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \sum_{n\in\mathbb{Z}} W(n,u), \quad \forall u \in E, \tag{2.10}$$

and

$$\left\langle\Phi'(u), v\right\rangle = \left(u^+, v\right) - \left(u^-, v\right) - \sum_{n\in\mathbb{Z}} \nabla W(n,u) \cdot v, \quad \forall u, v \in E. \tag{2.11}$$

Furthermore, the critical points of Φ in E are solutions of system (1.1) with $u(\pm\infty) = 0$; see [6, 10].

Let

$$\Psi(u) = \sum_{n\in\mathbb{Z}} W(n,u), \quad \forall u \in E. \tag{2.12}$$

Then, by standard arguments, we can prove the following two lemmas.

Lemma 2.2 *Suppose that* (PL), (W1), *and* (W2) *are satisfied. Then* Ψ *is nonnegative, weakly sequentially lower semi-continuous, and* Ψ' *is weakly sequentially continuous.*

Lemma 2.3 *Suppose that* (PL), (W1), *and* (W2) *are satisfied. Then there is a* $\rho > 0$ *such that* $\kappa := \inf \Phi(S_\rho^+) > 0$, *where* $S_\rho^+ = \partial B_\rho \cap E^+$.

Let $m_0 := \inf_{n\in\mathbb{Z},|x|=1} M(n)x \cdot x$. Then (W3) implies that $m_0 > \bar{\Lambda}$. Since $\sigma(\mathcal{A})$ is a union of closed intervals, we can choose $e \in [\mathcal{E}(m_1) - \mathcal{E}(\bar{\Lambda})]E \subseteq E^+$, where $\bar{\Lambda} < m_1 < m_0$. Thus,

$$\bar{\Lambda}\|e\|_2^2 \leq \|e\|^2 \leq m_1\|e\|_2^2 < m_0\|e\|_2^2 \leq \sum_{n\in\mathbb{Z}} M(n)e \cdot e. \tag{2.13}$$

Lemma 2.4 *Suppose that* (PL), (W1), (W2), *and* (W3) *are satisfied. Then there is a* $r_0 > 0$ *such that* $\sup \Phi(\partial Q) \leq 0$, *where*

$$Q = \{w + se : w \in E^-, s \geq 0, \|w + se\| \leq r_0\}. \tag{2.14}$$

Proof Obviously, $\Phi(w) \leq 0$ for $w \in E^-$. It is sufficient to show that $\Phi(w + te) \leq 0$ for $t \geq 0$, $w \in E^-$ and $\|w + te\| \geq r$ for large $r > 0$. Arguing indirectly, assume that for some sequence $\{w_k + t_k e\} \subset E^- \oplus \mathbb{R}^+ e$ with $\|w_k + t_k e\| \to \infty$, $\Phi(w_k + t_k e) \geq 0$ for all $k \in \mathbb{N}$. Set $v_k = (w_k + t_k e)/\|w_k + t_k e\| = v_k^- + s_k e$, then $\|v_k^- + s_k e\| = 1$. Passing to a subsequence, we may assume that $v_k \rightharpoonup v$ in E, then $v_k(n) \to v(n)$ for all $n \in \mathbb{Z}$, $v_k^- \rightharpoonup v^-$ in E, $s_k \to s$, and

$$\frac{\Phi(w_k + t_k e)}{\|w_k + t_k e\|^2} = \frac{s_k^2}{2}\|e\|^2 - \frac{1}{2}\|v_k^-\|^2 - \sum_{n\in\mathbb{Z}} \frac{W(n, w_k + t_k e)}{\|w_k + t_k e\|^2} \geq 0. \tag{2.15}$$

Clearly, (2.15) yields $s > 0$. By virtue of (2.13), there exists a finite set $\Pi \subset \mathbb{Z}$ such that

$$s^2\|e\|^2 - \|v^-\|^2 - \sum_{n\in\Pi} M(n)(se + v^-) \cdot (se + v^-) < 0. \tag{2.16}$$

From (W3) and (2.15), one has

$$\begin{aligned}
0 &\leq \frac{s_k^2}{2}\|e\|^2 - \frac{1}{2}\|v_k^-\|^2 - \sum_{n\in\Pi} \frac{W(n, w_k + t_k e)}{\|w_k + t_k e\|^2} \\
&= \frac{s_k^2}{2}\|e\|^2 - \frac{1}{2}\|v_k^-\|^2 - \frac{1}{2}\sum_{n\in\Pi} M(n)v_k \cdot v_k - \sum_{n\in\Pi} \frac{W_\infty(n, w_k + t_k e)}{\|w_k + t_k e\|^2}.
\end{aligned}$$

Clearly, $|W_\infty(n,x)| \leq c_0|x|^2$ for some $c_0 > 0$ and $W_\infty(n,x)/|x|^2 \to 0$ as $|x| \to \infty$. Since $v_k \rightharpoonup v$ in E, then $v_k(n) \to v(n)$ for $n \in \Pi$. Hence, one has

$$\begin{aligned}\sum_{n\in\Pi}\frac{W_\infty(n,w_k+t_ke)}{\|w_k+t_ke\|^2} &= \sum_{n\in\Pi}\frac{W_\infty(n,w_k+t_ke)}{|w_k+t_ke|^2}|v_k| \\ &\le c_0\sum_{n\in\Pi,|v(n)|=0}|v_k| + \sum_{n\in\Pi,|v(n)|\neq 0}\frac{W_\infty(n,w_k+t_ke)}{|w_k+t_ke|^2}|v_k| \\ &= o(1).\end{aligned}$$

Hence

$$0\le s^2\|e\|^2-\|v^-\|^2-\sum_{n\in\Pi}M(n)(se+v^-)\cdot(se+v^-),$$

a contradiction to (2.16). □

Lemma 2.5 *Suppose that* (PL), (W1), (W2) *and* (W3) *are satisfied. Then there exist a constant* $c>0$ *and a sequence* $\{u_k\}\subset E$ *satisfying*

$$\Phi(u_k)\to c,\qquad \|\Phi'(u_k)\|(1+\|u_k\|)\to 0. \tag{2.17}$$

Proof Lemma 2.5 is a direct corollary of Lemmas 2.1, 2.2, 2.3, and 2.4. □

Lemma 2.6 *Suppose that* (PL), (W1), (W2), (W3), *and* (W4) *are satisfied. Then any sequence* $\{u_k\}\subset E$ *satisfying* (2.17) *is bounded in* E.

Proof In view of (2.17), there exists a constant $C_0>0$ such that

$$C_0\ge\Phi(u_k)-\frac{1}{2}\langle\Phi'(u_k),u_k\rangle=\sum_{n\in\mathbb{Z}}\widetilde{W}(n,u_k). \tag{2.18}$$

To prove the boundedness of $\{u_k\}$, arguing by contradiction, suppose that $\|u_k\|\to\infty$. Let $v_k=u_k/\|u_k\|$. Then $\|v_k\|=1$. Passing to a subsequence, we may assume that $v_k\rightharpoonup v$ in E, then $v_k(n)\to v(n)$ for all $n\in\mathbb{Z}$. Let

$$\Pi_k=\left\{n\in\mathbb{Z}:\frac{|\nabla W(n,u_k)|}{|u_k|}\le\Lambda_0-\delta_0\right\}.$$

Then by using $\Lambda_0\|v_k\|_2^2\le\|v_k\|^2$, one has

$$\begin{aligned}\sum_{n\in\Pi_k}\frac{|\nabla W(n,u_k)|}{|u_k|}|v_k||v_k^+-v_k^-| &\le(\Lambda_0-\delta_0)\sum_{n\in\Pi_k}|v_k||v_k^+-v_k^-| \\ &\le(\Lambda_0-\delta_0)\|v_k\|_2^2\le 1-\frac{\delta_0}{\Lambda_0}.\end{aligned} \tag{2.19}$$

If $\delta:=\limsup_{k\to\infty}\|v_k\|_\infty=0$, then it follows from (W3), (W4), and (2.18) that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}\setminus\Pi_k}\frac{|\nabla W(n,u_k)|}{|u_k|}|v_k||v_k^+-v_k^-| \\ &\quad\le\|v_k\|_\infty\|v_k^+-v_k^-\|_\infty\sum_{n\in\mathbb{Z}\setminus\Pi_k}\frac{|\nabla W(n,u_k)|}{|u_k|}\end{aligned}$$

$$\begin{aligned} &\le C_1 \|v_k\|_\infty \sum_{n\in\mathbb{Z}\setminus\Pi_k} \widetilde{W}(n,u_k) \\ &\le C_2 \|v_k\|_\infty = o(1). \end{aligned} \tag{2.20}$$

From (2.10), (2.11), (2.19), and (2.20), one gets

$$\begin{aligned} 1+o(1) &= \frac{\|u_k\|^2 - \langle \Phi'(u_k), u_k^+ - u_k^- \rangle}{\|u_k\|^2} \\ &\le \sum_{n\in\mathbb{Z}} \frac{|\nabla W(n,u_k)|}{|u_k|} |v_k| \big|v_k^+ - v_k^-\big| \\ &= \sum_{n\in\Pi_k} \frac{|\nabla W(n,u_k)|}{|u_k|} |v_k| \big|v_k^+ - v_k^-\big| + \sum_{n\in\mathbb{Z}\setminus\Pi_k} \frac{|\nabla W(n,u_k)|}{|u_k|} |v_k| \big|v_k^+ - v_k^-\big| \\ &\le 1 - \frac{\delta_0}{\Lambda_0} + o(1), \end{aligned} \tag{2.21}$$

a contradiction. Thus $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $n_k \in \mathbb{Z}$ such that

$$\big|v_k(n_k)\big| = \|v_k\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and m_k with $0 \le m_k \le N-1$ such that $n_k = i_k N + m_k$. Let $\tilde{v}_k(n) = v_k(n + i_k N)$, then

$$\big|\tilde{v}_k(m_k)\big| > \frac{\delta}{2}, \quad \forall k \in N. \tag{2.22}$$

Now we define $\tilde{u}_k(n) = u_k(n + i_k N)$. Since $p(n)$, $L(n)$, and $W(n,x)$ are N-periodic in n, then $\tilde{u}_k/\|u_k\| = \tilde{v}_k$ and $\|\tilde{u}_k\| = \|u_k\|$. Passing to a subsequence, we have $\tilde{v}_k \rightharpoonup \tilde{v}$ in E, then $\tilde{v}_k(n) \to \tilde{v}(n)$ for all $n \in \mathbb{Z}$. Obviously, (2.22) implies that $\tilde{v}(n) \neq 0$ for some $n \in \{0,1,\ldots,N-1\}$. Let

$$E_0 = \big\{u \in E : \big\{n \in \mathbb{Z} : \big|u(n)\big| > 0\big\} \text{ is finite set}\big\}.$$

For any $\phi \in E_0$, there exists an $n_0 \in \mathbb{N}$ such that $\phi(n) = 0$ for all $|n| > n_0$. Setting $\phi_k(n) = \phi(n - i_k N)$, then it follows from (W3) and (2.9) that

$$\begin{aligned} &\frac{\langle \Phi'(u_k), \phi_k \rangle}{\|u_k\|} \\ &\quad = \sum_{n\in\mathbb{Z}} \left[p(n+1)\triangle v_k \cdot \triangle\phi_k + L(n) v_k \cdot \phi_k - \frac{\nabla W(n,u_k)\cdot\phi_k}{\|u_k\|} \right] \\ &\quad = \sum_{n\in\mathbb{Z}} \left[p(n+1)\triangle v_k \cdot \triangle\phi_k + \big(L(n) - M(n)\big) v_k \cdot \phi_k - \frac{\nabla W_\infty(n,u_k)\cdot\phi_k}{\|u_k\|} \right] \\ &\quad = \sum_{n\in\mathbb{Z}} \left[p(n+1)\triangle \tilde{v}_k \cdot \triangle\phi + \big(L(n) - M(n)\big) \tilde{v}_k \cdot \phi - \frac{\nabla W_\infty(n,\tilde{u}_k)\cdot\phi}{\|\tilde{u}_k\|} \right]. \end{aligned} \tag{2.23}$$

Note that

$$\left|\sum_{n\in\mathbb{Z}}\frac{\nabla W_\infty(n,\tilde{u}_k)\cdot\phi}{\|\tilde{u}_k\|}\right| = \left|\sum_{|n|\le n_0}\frac{\nabla W_\infty(n,\tilde{u}_k)\cdot\phi}{\|\tilde{u}_k\|}\right| \le \sum_{|n|\le n_0}\frac{|\nabla W_\infty(n,\tilde{u}_k)|}{|\tilde{u}_k|}|\tilde{v}_k||\phi| = o(1).$$

Hence, it follows from (2.17) and (2.23) that

$$\sum_{n\in\mathbb{Z}}\left[p(n+1)\triangle\tilde{v}_k\cdot\triangle\phi + \big(L(n)-M(n)\big)\tilde{v}_k\cdot\phi\right] = o(1), \tag{2.24}$$

which yields

$$\sum_{n\in\mathbb{Z}}\left[p(n+1)\triangle\tilde{v}\cdot\triangle\phi + \big(L(n)-M(n)\big)\tilde{v}\cdot\phi\right] = 0. \tag{2.25}$$

This shows that $\tilde{v}$ is an eigenfunction of the operator $\mathcal{B}$, where

$$(\mathcal{B}u)(n) = \triangle\left[p(n)\triangle u(n-1)\right] - \big(L(n)-M(n)\big)u(n), \quad \forall n\in\mathbb{Z}.$$

But $\mathcal{B}$ has only continuous spectrum in E. This contradiction shows that $\{u_n\}$ is bounded. □

Proof of Theorem 1.1 In view of Lemmas 2.5 and 2.6, there exists a bounded sequence $\{u_k\}\subset E$ satisfying (2.17). Thus there exists a constant $C_3>0$ such that

$$\sqrt{\Lambda_0}\|u_k\|_\infty \le \sqrt{\Lambda_0}\|u_k\|_2 \le \|u_k\| \le C_3, \quad \forall k\in\mathbb{N}. \tag{2.26}$$

Hence, by (W1) and (W2), there exists a constant $C_4>0$ such that

$$\left|\widetilde{W}(n,x)\right| \le \frac{c\Lambda_0}{2C_3^2}|x|^2 + C_4|x|^3, \quad \forall (n,x)\in\mathbb{Z}\times\mathbb{R}^{\mathcal{N}}, |x|\le\frac{C_3}{\sqrt{\lambda_0}}. \tag{2.27}$$

If $\delta := \limsup_{k\to\infty}\|u_k\|_\infty = 0$, then

$$\sum_{n\in\mathbb{Z}}\left|u_k(n)\right|^3 \le \|u_k\|_\infty\sum_{n\in\mathbb{Z}}\left|u_k(n)\right|^2 \le \frac{C_3^2}{\Lambda_0}\|u_k\|_\infty = o(1). \tag{2.28}$$

From (2.10), (2.11), (2.17), (2.26), (2.27), and (2.28), one has

$$\begin{aligned} c &= \Phi(u_k) - \frac{1}{2}\left\langle\Phi'(u_k),u_k\right\rangle + o(1)\\ &= \sum_{n\in\mathbb{Z}}\widetilde{W}(n,u_k) + o(1)\\ &\le \frac{c\Lambda_0}{2C_3^2}\sum_{n\in\mathbb{Z}}\left|u_k(n)\right|^2 + C_4\sum_{n\in\mathbb{Z}}\left|u_k(n)\right|^3 + o(1)\\ &\le \frac{c}{2} + o(1). \end{aligned}$$

This contradiction shows that $\delta>0$.

Going if necessary to a subsequence, we may assume the existence of $n_k \in \mathbb{Z}$ such that

$$\left|u_k(n_k)\right| = \|u_k\|_\infty > \frac{\delta}{2}.$$

Choose integers i_k and m_k with $0 \le m_k \le N-1$ such that $n_k = i_k N + m_k$. Let $v_k(n) = u_k(n + i_k N)$, then

$$\left|v_k(m_k)\right| > \frac{\delta}{2}, \quad \forall k \in \mathbb{N}. \tag{2.29}$$

Since $p(n)$, $L(n)$, and $W(n,x)$ are N-periodic in n, we have $\|v_k\| = \|u_k\|$ and

$$\Phi(v_k) \to c, \quad \left\|\Phi'(v_k)\right\|\left(1 + \|v_k\|\right) \to 0. \tag{2.30}$$

Passing to a subsequence, we have $v_k \rightharpoonup v$ in E, $v_k(n) \to v(n)$ for all $n \in \mathbb{Z}$. Obviously, (2.29) implies that $v \neq 0$. It is easy to show that $\Phi'(v) = 0$. □

Competing interests
The author declares that they have no competing interests.

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