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Some identities of Barnes-type special polynomials

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Abstract

In this paper, we consider Barnes-type special polynomials and give some identities of their polynomials which are derived from the bosonic p-adic integral or the fermionic p-adic integral on $\mathbb{Z}_p$.

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1 Introduction

As is known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1–29])}. \tag{1}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

For $a_1, a_2, \ldots, a_r \neq 0 \in \mathbb{C}_p$, the Barnes-Bernoulli polynomials are defined by the generating function to be

$$\prod_{i=1}^{r}\left(\frac{t}{e^{a_i t}-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1, a_2, \ldots, a_r)\frac{t^n}{n!}.$$

When $x = 0$, $B_n(0|a_1, a_2, \ldots, a_r) = B_n(a_1, a_2, \ldots, a_r)$ are called Barnes Bernoulli numbers (see [14–33]).

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is defined as $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_0(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_0\left(x+p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x) \quad \text{(see [16, 21])}. \end{aligned} \tag{2}$$

From (2), we have

$$I_0(f_1) = I_0(f) + f'(0), \tag{3}$$

where $f_1(x) = f(x+1)$.

By using iterative method, we get

$$I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i), \tag{4}$$

where $f_n(x) = f(x+n)$ $(n \in \mathbb{N})$.

As is well known, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned} I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}\bigl(x + p^N\mathbb{Z}_p\bigr) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \quad \text{(see [26, 27])}. \end{aligned} \tag{5}$$

From (5), we can derive

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l). \tag{6}$$

In particular, $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \quad \text{(see [26, 27])}. \tag{7}$$

The purpose of this paper is to investigate several special polynomials related to Barnes-type polynomials and give some identities including Witt's formula of their polynomials.

Finally, we give some identities of mixed-type Bernoulli and Euler polynomials.

2 Barnes-type polynomials

Let $a_1, a_2, \dots, a_r \neq 0 \in \mathbb{C}_p$. Then, by (3), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1 + a_2x_2 + \cdots + a_rx_r + x)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r) \\ &\quad = \left(\prod_{i=1}^r a_i\right)\left(\frac{t^r}{(e^{a_1t}-1)(e^{a_2t}-1)\cdots(e^{a_rt}-1)}\right)e^{xt} \\ &\quad = \left(\prod_{i=1}^r a_i\right)\sum_{n=0}^{\infty} B_n(x|a_1, a_2, \dots, a_r)\frac{t^n}{n!}. \end{aligned} \tag{8}$$

From (8), we obtain the following Witt's formula for the Barnes-Bernoulli polynomials.

Theorem 1 *For $a_1, a_2, \ldots, a_r \neq 0 \in \mathbb{C}_p$, we have*

$$B_n(x|a_1,\ldots,a_r) = \left(\prod_{i=1}^{r} a_i\right)^{-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r + x)^n \, d\mu_0(x_1) \cdots d\mu_0(x_r).$$

Note that

$$\begin{aligned}(a_1x_1 + \cdots + a_rx_r)^n &= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} a_1^{l_1}x_1^{l_1} \cdots a_r^{l_r}x_r^{l_r} \\ &= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} \left(\prod_{i=1}^{r} a_i^{l_i}\right) x_1^{l_1} \cdots x_r^{l_r}. \end{aligned} \tag{9}$$

By (9) and Theorem 1, we obtain the following corollary.

Corollary 2 *For $n \geq 2$, we have*

$$B_n(a_1,\ldots,a_r) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} \left(\prod_{i=1}^{r} a_i^{l_i-1}\right) B_{l_1} \cdots B_{l_r},$$

where $B_n = B_n(1)$ is the nth Bernoulli number.

From (2), we can easily derive the following integral equation:

$$\int_{\mathbb{Z}_p} f(x) \, d\mu_0(x) = \frac{1}{d} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} f(a + dx) \, d\mu_0(x), \tag{10}$$

where $d \in \mathbb{N}$.

By (10), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1x_1+a_2x_2+\cdots+a_rx_r+x)t} \, d\mu_0(x) \\ &= \frac{1}{d^r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(l_1a_1+\cdots+l_ra_r+a_1\,dx_1+\cdots+a_r\,dx_r+x)t} \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} \sum_{n=0}^{\infty} \frac{d^n}{d^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{l_1a_1 + \cdots + l_ra_r}{d} \right. \\ &\quad \left. + a_1x_1 + \cdots + a_rx_r + \frac{x}{d}\right)^n d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^n}{n!}. \end{aligned} \tag{11}$$

By Theorem 1 and (11), we get

$$B_n(x|a_1,\ldots,a_r) = d^{n-r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} B_n\left(\frac{l_1a_1 + \cdots + l_ra_r + x}{d} \middle| a_1,\ldots,a_r\right). \tag{12}$$

Therefore, by (12), we obtain the following distribution relation for a Barnes-type Bernoulli polynomial.

Theorem 3 *For $n \geq 0$, we have*

$$B_n(x|a_1,\ldots,a_r) = d^{n-r}\sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1} B_n\left(\frac{l_1a_1+\cdots+l_ra_r+x}{d}\Big|a_1,\ldots,a_r\right).$$

From (4), we note that

$$\int_{\mathbb{Z}_p} e^{a_1(x_1+n)t}\,d\mu_0(x_1) - \int_{\mathbb{Z}_p} e^{a_1x_1t}\,d\mu_0(x_1) = a_1t\sum_{l=0}^{n-1} e^{a_1lt}. \tag{13}$$

By (13), we get

$$\int_{\mathbb{Z}_p} e^{a_1x_1t}\,d\mu_0(x_1) = \frac{a_1t}{e^{a_1nt}-1}\sum_{l=0}^{n-1} e^{a_1lt}. \tag{14}$$

From (14), we can derive

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&\quad= \left(\frac{a_1t}{e^{na_1t}-1}\right)\cdots\left(\frac{a_rt}{e^{na_rt}-1}\right)\sum_{l_1,\ldots,l_r=0}^{n-1} e^{(a_1l_1+\cdots+a_rl_r)t}\\
&\quad= \left(\prod_{i=1}^r a_i\right)\sum_{l_1,\ldots,l_r=0}^{n-1}\left(\sum_{k=0}^\infty B_k(na_1,\ldots,na_r)\frac{t^k}{k!}\right)\sum_{j=0}^\infty (a_1l_1+\cdots+a_rl_r)^j\frac{t^j}{j!}\\
&\quad= \left(\prod_{i=1}^r a_i\right)\sum_{m=0}^\infty\left\{\sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^m (a_1l_1+\cdots+a_rl_r)^j B_{m-j}(na_1,\ldots,na_r)\binom{m}{j}\right\}\frac{t^m}{m!}.
\end{aligned} \tag{15}$$

By (15), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (a_1x_1+\cdots+a_rx_r)^m\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&\quad= \left(\prod_{i=1}^r a_i\right)\sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^m (a_1l_1+\cdots+a_rl_r)^j B_{m-j}(na_1,\ldots,na_r)\binom{m}{j},
\end{aligned} \tag{16}$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z} \geq 0$.

Therefore, by Theorem 1 and (16), we obtain the following theorem.

Theorem 4 *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ with $m \geq 0$, we have*

$$B_m(a_1,\ldots,a_r) = \sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^m (a_1l_1+\cdots+a_rl_r)^j B_{m-j}(na_1,\ldots,na_r)\binom{m}{j}.$$

Moreover,

$$B_m(x|a_1,\ldots,a_r) = \sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{j=0}^m (a_1l_1+\cdots+a_rl_r+x)^j B_{m-j}(na_1,\ldots,na_r)\binom{m}{j}.$$

From (15), we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r)t}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\frac{a_1t}{e^{na_1t}-1}\right)\cdots\left(\frac{a_rt}{e^{na_rt}-1}\right)\sum_{l_1,\ldots,l_r=0}^{n-1} e^{(a_1l_1+\cdots+a_rl_r)t}\\
&=\sum_{l_1,\ldots,l_r=0}^{n-1}\frac{a_1t\cdots a_rt}{(e^{na_1t}-1)\cdots(e^{na_rt}-1)}e^{(\frac{a_1l_1+\cdots+a_rl_r}{n})nt}\\
&=\left(\prod_{i=1}^r a_i\right)\sum_{l_1,\ldots,l_r=0}^{n-1}\sum_{m=0}^{\infty}B_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\ldots,a_r\right)n^m\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left(\prod_{i=1}^r a_i\right)n^m\sum_{l_1,\ldots,l_r=0}^{n-1}B_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\ldots,a_r\right)\frac{t^m}{m!}.
\end{aligned}\tag{17}$$

Thus, by (17), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a_1x_1+\cdots+a_rx_r)^m\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\prod_{i=1}^r a_i\right)n^m\sum_{l_1,\ldots,l_r=0}^{n-1}B_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\ldots,a_r\right),
\end{aligned}\tag{18}$$

where $n\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$.

Therefore, by Theorem 1 and (17), we obtain the following theorem.

Theorem 5 *For $n\in\mathbb{N}$ and $m\geq 0$, we have*

$$B_m(a_1,\ldots,a_r)=n^m\sum_{l_1,\ldots,l_r=0}^{n-1}B_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\ldots,a_r\right).$$

Moreover,

$$B_m(x|a_1,\ldots,a_r)=n^m\sum_{l_1,\ldots,l_r=0}^{n-1}B_m\left(\frac{a_1l_1+\cdots+a_rl_r+x}{n}\Big|a_1,\ldots,a_r\right).$$

Remark Let $a_1=1$ and $r=1$. Then we have

$$\int_{\mathbb{Z}_p}e^{(x+n)t}\,d\mu_0(x)-\int_{\mathbb{Z}_p}e^{xt}\,d\mu_0(x)=t\sum_{l=0}^{n-1}e^{lt}.$$

Thus, we have

$$\sum_{m=0}^{\infty}\left\{\int_{\mathbb{Z}_p}(x+n)^m\,d\mu_0(x)-\int_{\mathbb{Z}_p}x^m\,d\mu_0(x)\right\}\frac{t^m}{m!}(x)=t\sum_{l=0}^{n-1}\sum_{m=0}^{\infty}l^m\frac{t^m}{m!}.\tag{19}$$

By (19), we get

$$\frac{1}{m+1}\left\{\int_{\mathbb{Z}_p}(x+n)^{m+1}\,d\mu_0(x)-\int_{\mathbb{Z}_p}x^{m+1}\,d\mu_0(x)\right\}=\sum_{l=0}^{n-1}l^m, \tag{20}$$

where $n\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$.

It is easy to show that

$$\int_{\mathbb{Z}_p}e^{(x+y)t}\,d\mu_0(y)=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{21}$$

where $B_n(x)$ is the nth Bernoulli polynomial.

Thus, by (21), we get

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_0(y)=B_n(x)\quad(n\geq 0). \tag{22}$$

From (20) and (21), we note that

$$\frac{1}{m+1}\left\{B_{m+1}(n)-B_{m+1}\right\}=\sum_{l=0}^{n-1}l^m,$$

where $m\in\mathbb{Z}\geq 0$ and $n\in\mathbb{N}$.

From (5) and (6), we can derive the following equation:

$$\int_{\mathbb{Z}_p}e^{(x+1)t}\,d\mu_{-1}(x)+\int_{\mathbb{Z}_p}e^{xt}\,d\mu_{-1}(x)=2.$$

Thus, we have

$$\int_{\mathbb{Z}_p}e^{(x+y)t}\,d\mu_{-1}(y)=\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},$$

where $E_n(x)$ is the nth Euler polynomial.

Witt's formula for the Euler polynomials is given by

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y)=E_n(x)\quad(n\geq 0)\text{ (see [26, 27]).} \tag{23}$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers.

For $r\in\mathbb{N}$, the generating function of higher-order Euler polynomials can be derived from the multivariate p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r+x)t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)&=\left(\frac{2}{e^t+1}\right)^r e^{xt}\\&=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=E_n^{(r)}(x)\quad(n\in\mathbb{Z}\geq 0).\tag{24}$$

It is easy to show that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&\quad=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\ldots,l_r}\int_{\mathbb{Z}_p}x_1^{l_1}\,d\mu_{-1}(x_1)\cdots\int_{\mathbb{Z}_p}x_{r-1}^{l_{r-1}}\,d\mu_{-1}(x_{r-1})\\&\qquad\times\int_{\mathbb{Z}_p}(x_r+x)^{l_r}\,d\mu_{-1}(x_r)\\&\quad=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\ldots,l_r}E_{l_1}E_{l_2}\cdots E_{l_{r-1}}E_{l_r}(x).\end{aligned}\tag{25}$$

From (24) and (25), we have

$$E_n^{(r)}(x)=\sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\ldots,l_r}E_{l_1}\cdots E_{l_{r-1}}E_{l_r}(x).\tag{26}$$

When $x=0$, $E_n^{(r)}=E_n^{(r)}(0)$ are called the higher-order Euler numbers.

From (6), we note that

$$\int_{\mathbb{Z}_p}e^{(x+n)t}\,d\mu_{-1}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p}e^{xt}\,d\mu_{-1}(x)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}e^{lt}\quad(n\in\mathbb{N}).\tag{27}$$

Thus, by (27), we get

$$\int_{\mathbb{Z}_p}e^{xt}\,d\mu_{-1}(x)=\frac{2}{e^{nt}+(-1)^{n-1}}\sum_{l=0}^{n-1}(-1)^{n-1-l}e^{lt},\tag{28}$$

and

$$\begin{aligned}&\sum_{m=0}^{\infty}\left\{\int_{\mathbb{Z}_p}(x+n)^m\,d\mu_{-1}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p}x^m\,d\mu_{-1}(x)\right\}\frac{t^m}{m!}\\&\quad=\sum_{m=0}^{\infty}\left\{2\sum_{l=0}^{n-1}(-1)^{n-1-l}l^m\right\}\frac{t^m}{m!}\quad(n\in\mathbb{N}).\end{aligned}\tag{29}$$

By comparing the coefficients on the both sides of (29), we get

$$\int_{\mathbb{Z}_p}(x+n)^m\,d\mu_{-1}(x)+(-1)^{n-1}\int_{\mathbb{Z}_p}x^m\,d\mu_{-1}(x)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}l^m,\tag{30}$$

where $n\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$.

Therefore, by (23) and (30), we obtain the following lemma.

Lemma 6 *For $m \geq 0$, $n \in \mathbb{N}$, we have*

$$E_m(n) + (-1)^{n-1}E_m = 2\sum_{l=0}^{n-1}(-1)^{n-1-l}l^m.$$

Let us assume that $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Then, by (28), we get

$$\int_{\mathbb{Z}_p} e^{xt}\,d\mu_{-1}(x) = \frac{2}{e^{nt}+1}\sum_{l=0}^{n-1}(-1)^l e^{lt}. \tag{31}$$

Now, we consider the multivariate p-adic fermionic integral on $\mathbb{Z}_p$ related to the higher-order Euler numbers as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r)t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&\quad= \left(\frac{2}{e^{nt}+1}\right)^r \sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}e^{(l_1+\cdots+l_r)t}\\
&\quad= \left(\sum_{l=0}^{\infty}E_l^{(r)}n^l\frac{t^l}{l!}\right)\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}\sum_{k=0}^{\infty}(l_1+\cdots+l_r)^k\frac{t^k}{k!}\\
&\quad= \sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}\binom{m}{k}(l_1+\cdots+l_r)^k n^{m-k}E_{m-k}^{(r)}\right\}\frac{t^m}{m!}\\
&\quad= \sum_{m=0}^{\infty}\left\{\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}\sum_{k=0}^{m}\binom{m}{k}n^{m-k}E_{m-k}^{(r)}(l_1+\cdots+l_r)^k\right\}\frac{t^m}{m!}.
\end{aligned}\tag{32}$$

Thus, from (32), we can derive the following equation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1\cdots+x_r)^m\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&\quad= \sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}\sum_{k=0}^{m}(-1)^{l_1+\cdots+l_r}\binom{m}{k}E_{m-k}^{(r)}n^{m-k}(l_1+\cdots+l_r)^k,
\end{aligned}\tag{33}$$

where $m \geq 0$ and $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Therefore, by (24) and (33), we obtain the following theorem.

Theorem 7 *For $m \geq 0$ and $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, we have*

$$E_m^{(r)} = \sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}\sum_{k=0}^{m}(-1)^{l_1+\cdots+l_r}\binom{m}{k}E_{m-k}^{(r)}n^{m-k}(l_1+\cdots+l_r)^k,$$

Moreover,

$$E_m^{(r)}(x) = \sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}\sum_{k=0}^{m}(-1)^{l_1+\cdots+l_r}\binom{m}{k}E_{m-k}^{(r)}n^{m-k}(l_1+\cdots+l_r+x)^k.$$

From (32), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r)t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\left(\frac{2}{e^{nt}+1}\right)^r\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}e^{(l_1+\cdots+l_r)t}\\
&=\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}\left(\frac{2}{e^{nt}+1}\right)^r e^{(\frac{l_1+\cdots+l_r}{n})nt}\\
&=\sum_{m=0}^{\infty}\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}E_m^{(r)}\left(\frac{l_1+\cdots+l_r}{n}\right)n^m\frac{t^m}{m!},
\end{aligned}\tag{34}$$

where $n\in\mathbb{N}$ with $n\equiv 1\ (\operatorname{mod} 2)$.

Thus, by (34), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r)^m\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}E_m^{(r)}\left(\frac{l_1+\cdots+l_r}{n}\right)n^m,
\end{aligned}\tag{35}$$

where $m\in\mathbb{Z}\geq 0$, $n\in\mathbb{N}$ with $n\equiv 1\ (\operatorname{mod} 2)$.

Therefore by (24) and (35), we obtain the following theorem.

Theorem 8 *For $m\in\mathbb{Z}\geq 0$, $n\in\mathbb{N}$ with $n\equiv 1\ (\operatorname{mod} 2)$, we have*

$$E_m^{(r)}=n^m\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}E_m^{(r)}\left(\frac{l_1+\cdots+l_r}{n}\right),$$

Moreover,

$$E_m^{(r)}(x)=n^m\sum_{l_1=0}^{n-1}\cdots\sum_{l_r=0}^{n-1}(-1)^{l_1+\cdots+l_r}E_m^{(r)}\left(\frac{l_1+\cdots+l_r+x}{n}\right).$$

For $a_1,a_2,\ldots,a_r\in\mathbb{C}_p\backslash\{0\}$, let us consider the Barnes-type multiple Euler polynomials as follows:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r+x)t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\left(\frac{2}{e^{a_1t}+1}\right)\times\cdots\times\left(\frac{2}{e^{a_rt}+1}\right)e^{xt}\\
&=\sum_{n=0}^{\infty}E_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}.
\end{aligned}\tag{36}$$

When $x=0$, $E_n(a_1,\ldots,a_r)=E_n(0|a_1,\ldots,a_r)$ is called the nth Barnes-type Euler number.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we observe that

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \sum_{a=0}^{d-1}(-1)^a \int_{\mathbb{Z}_p} f(a+dx)\,d\mu_{-1}(x). \tag{37}$$

From (37), we can derive the following equation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(a_1x_1+\cdots+a_rx_r)t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&\quad= \sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1}(-1)^{l_1+\cdots+l_r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\{a_1l_1+\cdots+a_rl_r+(a_1x_1+\cdots+a_rx_r)d\}t}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&\quad= \sum_{n=0}^{\infty} d^n \sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1}(-1)^{l_1+\cdots+l_r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\biggl(\frac{a_1l_1+\cdots+a_rl_r}{d} \\
&\qquad + a_1x_1+\cdots+a_rx_r\biggr)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{t^n}{n!}.
\end{aligned} \tag{38}$$

By (38), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a_1x_1+\cdots+a_rx_r)^n\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\
&\quad= d^n \sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1}(-1)^{l_1+\cdots+l_r}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\biggl(\frac{a_1l_1+\cdots+a_rl_r}{d} \\
&\qquad + a_1x_1+\cdots+a_rx_r\biggr)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).
\end{aligned} \tag{39}$$

Therefore, by (36) and (39), we obtain the following theorem.

Theorem 9 *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, $n \geq 0$, we have*

$$E_n(a_1,\ldots,a_r) = d^n \sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1}(-1)^{l_1+\cdots+l_r}E_n\biggl(\frac{a_1l_1+\cdots+a_rl_r}{d}\biggm| a_1,\ldots,a_r\biggr).$$

Moreover,

$$E_n(x|a_1,\ldots,a_r) = d^n \sum_{l_1=0}^{d-1}\cdots\sum_{l_r=0}^{d-1}(-1)^{l_1+\cdots+l_r}E_n\biggl(\frac{x+a_1l_1+\cdots+a_rl_r}{d}\biggm| a_1,\ldots,a_r\biggr).$$

Remark Note that

$$\begin{aligned}
E_n(x|a_1,\ldots,a_r) &= \sum_{l=0}^{n}\binom{n}{l}x^l E_{n-l}(a_1,\ldots,a_r) \\
&= \sum_{l=0}^{n}\binom{n}{l}x^{n-l} E_l(a_1,\ldots,a_r).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r \\
&\quad + b_1y_1 + \cdots + b_sy_s)^n \, d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n(a_1x_1 + \cdots + a_rx_r | b_1, \ldots, b_s) \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(b_1, \ldots, b_s) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r)^l \, d\mu_0(x_1) \cdots d\mu_0(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(b_1, \ldots, b_s) B_l(a_1, \ldots, a_r). \qquad (40)
\end{aligned}$$

Now, we define mixed-type Barnes-type Euler and Bernoulli numbers as follows:

$$\begin{aligned}
&EB_n(b_1, \ldots, b_s; a_1, \ldots, a_r) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1 + \cdots + a_rx_r \\
&\quad + b_1y_1 + \cdots + b_sy_s)^n \, d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \, d\mu_0(x_1) \cdots d\mu_0(x_r), \qquad (41)
\end{aligned}$$

where $a_1, \ldots, a_r, b_1, \ldots, b_s \neq 0$.

By (40) and (41), we get

$$EB_n(b_1, \ldots, b_s; a_1, \ldots, a_r) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}(b_1, \ldots, b_s) B_l(a_1, \ldots, a_r).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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