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Invariant curves for a delay differential equation with a piecewise constant argument

Jinghua Liu*

*Correspondence:
jhliu977916@163.com
Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China
School of Mathematics and Computation Science, Lingnan Normal University, Zhanjiang, Guangdong 524048, P.R. China

Abstract
In order to understand the dynamics of a second order delay differential equation with a piecewise constant argument, we investigate invariant curves of the derived planar mapping from the equation. All invariant curves are given in this paper.

Keywords: difference equation; invariant curve; piecewise construction; characteristic root; dual equation

1 Introduction

The study of differential equations with piecewise constant argument (EPCA) initiated in [1, 2]. These equations represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations, hence, they are of importance in control theory and in certain biomedical models [3]. In this paper the second order delay differential equation with a piecewise constant argument

$$x''(t) + g\big(x([t])\big) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1}$$

where $x''(t)$ denotes the second order derivative of $x(t)$, $[t]$ denotes the greatest integer less than or equal to t, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous or at least piecewise continuous function, is considered. In 1987, Aftabizadeh *et al.* discussed the oscillatory and periodic properties of the solutions of (1) in [4]. In 1989, Gyori and Ladas investigated linearized oscillations of the solutions of (1) in [5]. Later, Wiener and Cooke considered oscillations of the solutions of systems of two differential equations with piecewise constant arguments in [6].

The invariant curve [7–11] is another interesting problem in the study of dynamics because it can be used to reduce a system to a 1-dimensional one. The problem of invariant curves is actually a part of the research on invariant manifolds. In 1997, Ng and Zhang studied the nonlinear C^1 invariant curve of planar mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$,

$$G(x,y) = \left(y, 2y - x - \frac{1}{2}\big(g(y) + g(x)\big)\right), \tag{2}$$

derived from (1) in [12] when g is nonlinear and gave the conditions that G has linear invariant curves when g is linear. In 2003, Yang *et al.* investigated nonlinear C^0 invariant curves of (2) when g is nonlinear in [13]. So far, nonlinear invariant curves of (2) when g is linear have not been studied. So it is very interesting to look for nonlinear invariant curves

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of (2) when g is linear. In this paper all the invariant curves of the planar mapping G are given including the linear and nonlinear ones when g is linear.

2 Main results

We discuss invariant curves of the form $y = f(x)$ for the planar mapping (2). Its invariant curves of the form $y = f(x)$ satisfy $f(y) = 2y - x - \frac{1}{2}(g(y) + g(x))$, which leads to the iterative functional equation

$$f(f(x)) = 2f(x) - x - \frac{1}{2}(g(f(x)) + g(x)), \quad \forall x \in \mathbb{R}. \tag{3}$$

Considering linear g and $g(x) = ax + b$, we compute that

$$f(f(x)) - \left(2 - \frac{a}{2}\right)f(x) + \left(1 + \frac{a}{2}\right)x = -b, \quad \forall x \in \mathbb{R}. \tag{4}$$

Thus, the invariant curves of planar mapping G with $g(x) = ax + b$ can be obtained by solving functional (4). We mainly discuss the generic cases $a \notin \{-2, 4\}$, but leave the special cases $a = -2$ and $a = 4$ to the last part of this section. For generic $a \notin \{-2, 4\}$, (4) with $b = 0$ is of the form discussed in [14, 15]. In order to apply the results of [14], we let

$$r_1 := \frac{(4-a) - (a^2 - 16a)^{\frac{1}{2}}}{4}, \qquad r_2 := \frac{(4-a) + (a^2 - 16a)^{\frac{1}{2}}}{4}, \tag{5}$$

which are the roots of the characteristic polynomial $P(r) := r^2 - (2 - \frac{a}{2})r + 1 + \frac{a}{2}$.

From (5) we see that the characteristic roots r_1, r_2 of (4) have the following possibilities:

(C1) $0 < r_1 < 1 < r_2$, if and only if $-2 < a < 0$.

(C2) $r_1 = r_2 = 1$, if and only if $a = 0$.

(C3) $r_1 < 0 < r_2 \neq 1$ and $r_1 \neq -r_2$, if and only if $a < -2$.

(C4) $r_1 = r_2 < 0$, if and only if $a = 16$.

(C5) $r_1 < r_2 < -1$, if and only if $a > 16$.

Note that the case $r_2 > r_1 > 1$ is not listed because the case $r_2 > r_1 > 1$ implies $\frac{(4-a)-(a^2-16a)^{\frac{1}{2}}}{4} > 1$, *i.e.*, $-a > (a^2 - 16a)^{\frac{1}{2}}$, which does not hold, and that the case $0 < r_1 < r_2 < 1$ is not listed because $0 < r_1 < r_2 < 1$ implies $0 < \frac{(4-a)+(a^2-16a)^{\frac{1}{2}}}{4} < 1$, *i.e.*, $0 < a < 4$, which contradicts the requirement that $\Delta = a^2 - 16a \geq 0$, and that the case $0 < a < 16$ is not listed because in this case (4) with $b = 0$ has no continuous solutions, neither r_1 nor r_2 is real, by [14]. Since we consider $a \notin \{-2, 4\}$, none of the case $r_1 = 0$, the case $r_2 = 0$, and the case $r_1 = -r_2 \neq 0$ is listed. Corresponding to the above list, we have the following results.

Theorem 2.1 (i) *If* $-2 < a < 0$, *then a continuous solutions* ϕ *of* (4) *with* $b = 0$ *is either of the piecewise linear form that* $f(x) := r_i x$ *for* $x > 0$, *or* $:= 0$ *for* $x = 0$, *or* $:= r_j x$ *for* $x < 0$, *where* $i, j = 1, 2$, *or given by*

$$f(x) := \begin{cases} f_n(x), & x \in [x_n, x_{n+1}), n = 0, 1, 2, \ldots, \\ f_{-n}^{-1}(x), & x \in [x_{-n}, x_{-n+1}), n = 1, 2, \ldots, \end{cases}$$

where $x_n = \frac{r_2^n}{r_2 - r_1}(x_1 - r_1 x_0) + \frac{r_1^n}{r_2 - r_1}(-x_1 + r_2 x_0)$, $n \in \mathbb{Z}$, *with an arbitrarily chosen* $x_0 \in (-\infty, +\infty)$ *and* $x_1 \in [r_1 x_0, r_2 x_0]$, *and* $f_n(x) = (r_1 + r_2)x - r_1 r_2 f_{n-1}^{-1}(x)$ *for all* $x \in [x_n, x_{n+1})$, $n = 1, 2, \ldots$, $f_{-n-1}(x) = (\frac{1}{r_1} + \frac{1}{r_2})x - \frac{1}{r_1 r_2} f_{-n}^{-1}(x)$ *for all* $x \in [x_{-n}, x_{-n+1})$, $n = 1, 2, \ldots$, *and* $f_{-1}(x) =$

$(\frac{1}{r_1}+\frac{1}{r_2})x-\frac{1}{r_1r_2}f_0(x)$, $x\in[x_0,x_1)$, with the arbitrarily chosen functions f_0 such that $f_0(x_0)=x_1$, $f_0(x_1)=x_2$, and $r_1\le\frac{f_0(x)-f_0(y)}{x-y}\le r_2$ for all $x,y\in[x_0,x_1)$. (ii) *If $a=0$, then* (4) *with $b=0$ has a unique continuous solution f and $f(x)=x+\beta$, where $\beta\in\mathbb{R}$ is an arbitrary constant.*

Proof The proof is a simple application of well-known results in [14]. The result (i) is given by Theorem 2 of [14], where the characteristic roots r_1, r_2 satisfy $r_2>1>r_1>0$ as shown in (C1). We can deduce the result (ii) from Theorem 8 of [14], where $r_1=r_2=1$ as shown in (C2). The proof is completed. □

Theorem 2.2 (i) *If $a<-2$, then* (4) *with $b=0$ only has two continuous solutions f and $f(x)=r_1x$ or r_2x.* (ii) *If $a=16$,* (4) *with $b=0$ just has a continuous solution $f(x)=-3x$.* (iii) *If $a>16$, all continuous solutions f of* (4) *with $b=0$ are given by*

$$f(x):=\begin{cases} f_{2n}(x), & x\in[x_{-2n},x_{-2n+2}), n=0,1,2,\ldots,\\ f_{2n+1}(x), & x\in[x_{-2n+3},x_{-2n+1}), n=0,1,2,\ldots,\\ 0, & x=0,\\ f^{-1}_{-2n}(x), & x\in[x_{2n},x_{2n+2}), n=1,2,\ldots,\\ f^{-1}_{-2n+1}(x), & x\in[x_{2n+3},x_{2n+1}), n=1,2,\ldots,\end{cases}$$

where the sequence $\{x_n\}$ is defined by $x_n=\frac{r_2^n}{r_2-r_1}(x_1-r_1x_0)+\frac{r_1^n}{r_2-r_1}(-x_1+r_2x_0)$, $n\in\mathbb{Z}$, with an arbitrarily chosen $x_0\in(0,+\infty)$ and $x_1\in[r_1x_0,r_2x_0]$, and $f_{2n-1}(x)=(r_1+r_2)x-r_1r_2f^{-1}_{2n-2}(x)$, $x\in[x_{-2n+5},x_{-2n+3})$, $n=1,2,\ldots$, $f_{2n}(x)=(r_1+r_2)x-r_1r_2f^{-1}_{2n-1}(x)$, $x\in[x_{-2n},x_{-2n+2})$, $n=1,2,\ldots$, $f_{-2n}(x)=(\frac{1}{r_1}+\frac{1}{r_2})x-\frac{1}{r_1r_2}f^{-1}_{-2n+1}(x)$, $x\in[x_{2n+3},x_{2n+1})$, $n=1,2,\ldots$, $f_{-2n-1}(x)=(\frac{1}{r_1}+\frac{1}{r_2})x-\frac{1}{r_1r_2}f^{-1}_{-2n}(x)$, $x\in[x_{2n},x_{2n+2})$, $n=1,2,\ldots$, and $f_{-1}(x)=(\frac{1}{r_1}+\frac{1}{r_2})x-\frac{1}{r_1r_2}f_0(x)$, $x\in[x_0,x_2)$, with an arbitrarily chosen continuous function f_0 on $[x_0,x_2)$ such that $f_0(x_0)=x_1$, $f_0(x_2)=x_3$, and $r_1\le\frac{f_0(x)-f_0(y)}{x-y}\le r_2$, $\forall x,y\in[x_0,x_2)$.

Proof Firstly, we consider (i). By Theorem 5 in [14], (4) with $b=0$ only has two continuous solutions f and $f(x)=r_1x$ or r_2x, where the characteristic roots r_1, r_2 satisfy $r_1<0<r_2\neq1$ and $r_1\neq-r_2$ as shown in (C3). Next, we consider (ii). By Theorem 6 in [14], (4) with $b=0$ just has a continuous solution $f(x)=-3x$, where the characteristic roots r_1, r_2 satisfy $r_1=r_2=-3$ as shown in (C4). Finally, we consider (iii). In order to piecewise construct all solutions of (4) with $b=0$ we need a partition for the interval $(-\infty,\infty)$. For this purpose we consider a homogeneous linear difference equation

$$x_{n+2}-\left(2-\frac{a}{2}\right)x_{n+1}+\left(1+\frac{a}{2}\right)x_n=0, \tag{6}$$

which has the same coefficients as (4) with $b=0$ correspondingly. Its characteristic equation is

$$r^2-\left(2-\frac{a}{2}\right)r+1+\frac{a}{2}=0, \tag{7}$$

which has two characteristic roots r_1 and r_2 satisfying $r_1<r_2<-1$ as shown in (C5). Thus, (4) and (6) can be, respectively, rewritten as

$$f\big(f(x)\big)-(r_1+r_2)f(x)+r_1r_2x=0, \tag{8}$$

$$x_{n+2}-(r_1+r_2)x_{n+1}+r_1r_2x_n=0,\quad n=0,1,2,\ldots. \tag{9}$$

If f is a solution of (4) with $b = 0$, we easily see that f is invertible. In fact, if $f(x_1) = f(x_2)$, then $f(f(x_1)) = f(f(x_2))$. Thus, $x_1 = x_2$ by (4) because $a \neq -2$, which implies that f is one to one. Next we only need to show that $f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to +\infty$ as $x \to -\infty$ because $f(x) \to \pm\infty$ as $x \to \pm\infty$, then the left-hand side of (4) with $b = 0$ tends to $\pm\infty$ by $a > 16$, but the right-hand side is equal to 0. Otherwise, $f(x)$ has a finite limit as $x \to \infty$, then $f(f(x)) - (2 - \frac{a}{2})f(x)$ converges to a finite limit by the continuity of f on the whole of $\mathbb{R}$, but $(1 + \frac{a}{2})x$ does not, which contradicts the requirement that $f(f(x)) - (2 - \frac{a}{2})f(x) = -(1 + \frac{a}{2})x$. Thus, we rewrite (4) in the following equivalent form:

$$f^{-1}(f^{-1}(x)) - \left(2 - \frac{a}{2}\right)f^{-1}(x) + \frac{a}{2}x = 0, \tag{10}$$

which is called the dual equation to (4) with $b = 0$. Solving the homogeneous linear difference (9) with arbitrarily chosen real initial values x_0 and x_1, we obtain

$$x_n = \frac{r_2^n}{r_2 - r_1}(x_1 - r_1x_0) + \frac{r_1^n}{r_2 - r_1}(-x_1 + r_2x_0), \quad n \in \mathbb{Z}. \tag{11}$$

Let $x_0 = x$ and $x_{n+1} = f(x_n)$ in (11), we have

$$f^n(x) = \frac{r_2^n}{r_2 - r_1}\big(f(x) - r_1x\big) + \frac{r_1^n}{r_2 - r_1}\big(-f(x) + r_2x\big), \quad n \in \mathbb{Z}.$$

Furthermore, we can obtain

$$\Delta f^n(x,y) = \frac{r_2^n}{r_2 - r_1}\big(\Delta f(x,y) - r_1\big) + \frac{r_1^n}{r_2 - r_1}\big(-\Delta f(x,y) + r_2\big), \tag{12}$$

$$f^{n+1}(x) - f^n(x) = r_2^n\frac{r_2 - 1}{r_2 - r_1}\big(f(x) - r_1x\big) + r_1^n\frac{r_1 - 1}{r_2 - r_1}\big(-f(x) + r_2x\big), \tag{13}$$

where $\Delta f^n(x,y) = \frac{f^n(x)-f^n(y)}{x-y}$ for any $x \neq y$ and $n \in \mathbb{Z}$. From (12) we can see that

$$\lim_{n\to+\infty}\frac{\Delta f^n(x,y)}{r_2^n} = \frac{(\Delta f(x,y) - r_1)}{r_2 - r_1},$$
$$\lim_{n\to-\infty}\frac{\Delta f^n(x,y)}{r_1^n} = \frac{(-\Delta f(x,y) + r_2)}{r_2 - r_1}.$$

Since f is strictly monotonic, $\Delta f^n(x,y) > 0$ for even n, which implies $\Delta f(x,y) - r_1 \geq 0$ and $-\Delta f(x,y) + r_2 \geq 0$, that is,

$$r_1 \leq \Delta f(x,y) \leq r_2. \tag{14}$$

Moreover, we can see that $f(0) = 0$ from (13). In what follows, we arbitrarily choose $x_0 \in (0,+\infty)$ and $x_1 \in [r_1x_0, r_2x_0]$ and define a sequence $\{x_n\}$, $n \in \mathbb{Z}$, by (11). The sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{x_{-2n}\}$ and $\{x_{-2n+1}\}$, where $n = 0,1,2,\ldots$, are strictly monotone such that $x_{2n} \to +\infty$, $x_{2n+1} \to -\infty$, $x_{-2n} \to 0$, and $x_{-2n+1} \to 0$ as $n \to \infty$. Thus, the sequence $\{x_n\}$, $n \in \mathbb{Z}$, is a partition of the interval $(-\infty,\infty)$. Next we arbitrarily choose a continuous function defined in the interval $[x_0,x_2)$, satisfying $f_0(x_0) = x_1$, $f_0(x_2) = x_3$, and condition (14). We can recursively define the homeomorphisms $f_{2n-1} : [x_{-2n+5}, x_{-2n+3}) \to [x_{-2n}, x_{-2n+2})$, $n = 1, 2, \ldots$, and $f_{2n} : [x_{-2n}, x_{-2n+2}) \to [x_{-2n+3}, x_{-2n+1})$, $n = 1, 2, \ldots$, such that

$$f_{2n-1}(x_{-2n+5}) = x_{-2n+2}, \qquad f_{2n-1}(x_{-2n+3}) = x_{-2n}, \tag{15}$$

$$f_{2n}(x_{-2n}) = x_{-2n+1}, \qquad f_{2n}(x_{-2n+2}) = x_{-2n+3}, \tag{16}$$

$$r_1 \le \Delta f_{2n-1}(x, y) \le r_2, \quad \forall x, y \in [x_{-2n+5}, x_{-2n+3}), \tag{17}$$

$$r_1 \le \Delta f_{2n}(x, y) \le r_2, \quad \forall x, y \in [x_{-2n}, x_{-2n+2}). \tag{18}$$

In fact, for f_{2n} defined satisfying (16) and (18), we let

$$f_{2n+1}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n}^{-1}(x), \quad \forall x \in [x_{-2n+3}, x_{-2n+1}).$$

Obviously, $f_{2n+1}(x_{-2n+3}) = x_{-2n}$ and $f_{2n+1}(x_{-2n+1}) = x_{-2n-2}$. Making use of (18), we have $\frac{1}{r_2} \le \frac{f_{2n}^{-1}(x) - f_{2n}^{-1}(y)}{x-y} \le \frac{1}{r_1}$ for $x, y \in [x_{-2n}, x_{-2n+2})$. It is easy to deduce that

$$r_1 \le \Delta f_{2n+1}(x, y) \le r_2, \quad \forall x, y \in [x_{-2n+3}, x_{-2n+1}).$$

Furthermore, we again let

$$f_{2n+2}(x) = (r_1 + r_2)x - r_1 r_2 f_{2n+1}^{-1}(x), \quad \forall x \in [x_{-2n-2}, x_{-2n}).$$

By the same argument we can see that

$$f_{2n+2}(x_{-2n-2}) = x_{-2n-1}, \qquad f_{2n+2}(x_{-2n}) = x_{-2n+1}, \tag{19}$$

$$r_1 \le \Delta f_{2n+2}(x, y) \le r_2, \quad \forall x, y \in [x_{-2n-2}, x_{-2n}). \tag{20}$$

By induction both f_{2n-1} and f_{2n} are well defined. Similarly, we can also recursively define the homeomorphisms $f_{-2n+1} : [x_{2n-2}, x_{2n}) \to [x_{2n+3}, x_{2n+1})$, $n = 1, 2, \ldots$, and $f_{-2n} : [x_{2n+3}, x_{2n+1}) \to [x_{2n}, x_{2n+2})$, $n = 1, 2, \ldots$. By the properties of the dual (10) we can obtain

$$f_{-2n+1}(x_{2n-2}) = x_{2n+1}, \qquad f_{-2n+1}(x_{2n}) = x_{2n+3},$$

$$f_{-2n}(x_{2n+3}) = x_{2n+2}, \qquad f_{-2n}(x_{2n+1}) = x_{2n},$$

$$\frac{1}{r_2} \le \Delta f_{-2n+1}(x, y) \le \frac{1}{r_1}, \quad \forall x, y \in [x_{2n-2}, x_{2n}),$$

$$\frac{1}{r_2} \le \Delta f_{-2n}(x, y) \le \frac{1}{r_1}, \quad \forall x, y \in [x_{2n+3}, x_{2n+1}).$$

Therefore,

$$f_{-2n+1}^{-1}(x_{2n+1}) = x_{2n-2}, \qquad f_{-2n+1}^{-1}(x_{2n+3}) = x_{2n},$$

$$f_{-2n}^{-1}(x_{2n+2}) = x_{2n+3}, \qquad f_{-2n}^{-1}(x_{2n}) = x_{2n+1},$$

$$r_1 \le \Delta f_{-2n+1}^{-1}(x, y) \le r_2, \quad \forall x, y \in [x_{2n+3}, x_{2n+1}),$$

$$r_1 \le \Delta f_{-2n}^{-1}(x, y) \le r_2, \quad \forall x, y \in [x_{2n}, x_{2n+2}).$$

Thus, we can define

$$f(x) := \begin{cases} f_{2n}(x), & x \in [x_{-2n}, x_{-2n+2}), n = 0, 1, 2, \ldots, \\ f_{2n+1}(x), & x \in [x_{-2n+3}, x_{-2n+1}), n = 0, 1, 2, \ldots, \\ 0, & x = 0, \\ f_{-2n}^{-1}(x), & x \in [x_{2n}, x_{2n+2}), n = 1, 2, \ldots, \\ f_{-2n+1}^{-1}(x), & x \in [x_{2n+3}, x_{2n+1}), n = 1, 2, \ldots. \end{cases}$$

f is continuous on $\mathbb{R}$ because $f_{2n}(x_{-2n}) = x_{-2n+1} = f_{2n+2}(x_{-2n})$, $f_{2n+1}(x_{-2n+1}) = x_{-2n-2} = f_{2n+3}(x_{-2n+1})$, where $n = 0, 1, 2, \ldots$, $f_1^{-1}(x_0) = f_{-1}(x_0)$, $f_{-2n}^{-1}(x_{2n+2}) = x_{2n+3} = f_{-2n-2}^{-1}(x_{2n+2})$, and $f_{-2n+1}^{-1}(x_{2n+3}) = x_{2n+2} = f_{-2n-1}^{-1}(x_{2n+3})$, where $n = 1, 2, 3, \ldots$. we can easily check that f defined in Theorem 2.2 satisfies (4) with $b = 0$ in $\mathbb{R}$. In fact, if $x \in [x_{-2n}, x_{-2n+2})$, $n = 0, 1, 2, \ldots$, $f^2(x) = f_{2n+1}(f_{2n}(x)) = (r_1 + r_2)f_{2n}(x) - r_1r_2x = (r_1 + r_2)f(x) - r_1r_2x$, *i.e.*, $f^2(x) - (r_1 + r_2)f(x) - r_1r_2x = 0$. Similarly, we can also check that f satisfies (4) with $b = 0$ for $x \in [x_{-2n+3}, x_{-2n+1})$, $x \in [x_{2n+3}, x_{2n+1})$, $x \in [x_{2n}, x_{2n+2})$ and $x = 0$, where $n = 1, 2, 3, \ldots$. The proof is completed. □

Remark In the case that $b \neq 0$, as indicated in [16] for (2) therein, (4) can be reduced equivalently to the equation

$$\tilde{f}(\tilde{f}(x)) - \left(2 - \frac{a}{2}\right)\tilde{f}(x) + \left(1 + \frac{a}{2}\right)x = 0, \tag{21}$$

the same type of equation as the one considered in Theorems 2.1 and 2.2 with vanishing b, by the replacement $\tilde{f}(x) = f(x + \xi) - \xi$, where $\xi = \frac{-b}{(1-r_1)(1-r_2)}$, if its characteristic roots r_1, r_2 are both real but neither of them is equal to 1. In this case solutions can be found from Theorems 2.1 and 2.2. So (4) with $b \neq 0$ can be reduced to (21) except for the case $a = 0$. For the case of $a = 0$ and $b \neq 0$, (4) has no real continuous solutions. In fact, by induction and (4) we can obtain $f^n(x) = nf(x) - (n-1)x - \frac{n(n+1)}{2}b$, $n \in \mathbb{Z}$. Furthermore, we have $f^{n+1}(x) - f^n(x) = f(x) - x - (n+1)b$, $n \in \mathbb{Z}$. For an arbitrary $x \in \mathbb{R}$, $f^{n+1}(x) - f^n(x)$ has the same sign when n takes the values N and $-N$, where N is a large positive integer, because f is strictly monotonic. But $f(x) - x - (n+1)b$ has not, which contradicts the requirement $f^{n+1}(x) - f^n(x) = f(x) - x - (n+1)b$, $n \in \mathbb{Z}$.

In what follows, we consider the case that either $a = -2$ or $a = 4$, which is not generic.

For $a = -2$, (4) is of the form $f^2(x) - 3f(x) = -b$, from which we get with the replacement $y = f(x)$: $f(x) = 3x - b$.

For $a = 4$, (4) is of the form

$$f^2(x) = -3x - b, \tag{22}$$

which is the problem of iterative roots of the linear function $F(x) := -3x - b$. By the theory of iterative roots, as shown in [8], we know (22) has no real continuous solutions.

Competing interests

The author declares that he has no competing interests.

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