Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness of positive periodic solutions for a first-order functional differential equation

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Abstract
In this article, we establish sufficient conditions for the existence and uniqueness of positive periodic solutions for a class of first-order functional differential equations. Our analysis relies on some fixed point theorems for mixed monotone operators. Our results can not only guarantee the existence of unique positive periodic solutions, but also be applied to construct an iterative scheme for approximating them. Some examples are given to illustrate our main results.

MSC: 26A33; 34B18; 34B27

Keywords: functional differential equation; positive periodic solution; existence and uniqueness; fixed point theorem for mixed monotone operators

1 Introduction

This article will investigate the existence and uniqueness of positive periodic solutions for the following first-order functional differential equation:

$$y'(t) = -\delta(t)y(t) + f\bigl(t, y\bigl(t-\tau(t)\bigr), y\bigl(t-\tau(t)\bigr)\bigr) + g\bigl(t, y\bigl(t-\tau(t)\bigr)\bigr), \tag{1.1}$$

where $T > 0$, $\delta, \tau : \mathbf{R} \to \mathbf{R}$ are continuous T-periodic functions and $\delta(t) > 0$ for $t \in \mathbf{R}$, $f : \mathbf{R}^3 \to \mathbf{R}$ and $g : \mathbf{R}^2 \to R$.

Functional differential equations with periodic delays appear in a number of ecological, economical, control and physiological models. During the past decades, there has been a significant development in the question of periodic solutions for ordinary and partial differential equations, see the papers [1–10] and the references therein. In these papers, most of the authors have investigated the existence of positive periodic solutions for functional differential equations. Recently, Kang, Shi and Wang [5] studied the following first-order functional differential equation:

$$y'(t) = -a(t)y(t) + f\bigl(t, y\bigl(t-\tau(t)\bigr)\bigr), \tag{1.2}$$

where $a, \tau \in C(\mathbf{R}, \mathbf{R})$, $f \in C(\mathbf{R}^2, \mathbf{R})$ are T-periodic functions with $T > 0$, and they established the existence of maximal and minimal periodic solutions for (1.2) by using the method of lower and upper solutions. As we know, in most of the existing works, in order to establish the existence of positive periodic solutions, a key condition is that the

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existence of upper-lower solutions must be assumed. However, it is difficult to verify the existence of upper-lower solutions for concrete functional differential equations. In addition, few papers can be found in the literature on the existence and uniqueness of positive periodic solutions for four-point fractional differential equations. Motivated by the works [11, 12], in our paper, we will use three fixed point theorems for mixed monotone operators to study the existence and uniqueness of positive periodic solutions for problem (1.1). Our results can not only guarantee the existence of unique positive periodic solutions, but also be applied to construct an iterative scheme for approximating them.

With this context in mind, the outline of this paper is as follows. In Section 2, we shall recall certain results from the theory of mixed monotone operators and some definitions, notations of an ordered Banach space. In Section 3, we shall provide some conditions under which problem (1.1) will have a unique positive periodic solution. Finally, in Section 4, we shall provide three examples which explain the applicability of our main results.

2 Preliminaries

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and three fixed point theorems for mixed monotone operators which will be used later. For the convenience of readers, we refer them to [11–14] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$.

P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is the infimum of such constants, it is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ax \geq Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.1 (see [11, 14]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x and decreasing in y, *i.e.*, $u_i, v_i \in P$, $i = 1, 2$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.2 An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tA(x), \quad \forall t \in (0, 1), x \in P. \tag{2.1}$$

Definition 2.3 Let $D = P$ and β be a real number with $0 \leq \beta < 1$. An operator $A : D \to D$ is said to be β-concave if it satisfies

$$A(tx) \geq t^{\beta} A(x), \quad \forall t \in (0, 1), x \in D. \tag{2.2}$$

To prove our results, we need the following fixed point theorems for mixed monotone operators, which were given and proved in [11, 12].

Lemma 2.4 (see [11]) *Let P be normal and $A : P \times P \to P$ be a mixed monotone operator. Suppose that*:

(a) *there exist $v > \theta$ and $c > 0$ such that $\theta < A(v,\theta) \leq v$, $A(\theta, v) \geq cA(v,\theta)$;*

(b) *for any $0 < a < b < 1$, there exists $\eta = \eta(a,b) > 0$ such that*

$$A\bigl(tx, t^{-1}y\bigr) \geq t(1+\eta)A(x,y), \quad \forall a \leq t \leq b, \theta \leq y \leq x \leq v. \tag{2.3}$$

Then A has a unique fixed point x^* in $[\theta, v]$ with $x^* > \theta$. Moreover, for any initial values $x_0, y_0 \in [\theta, v]$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 2.5 (see Theorem 2.1 in [12]) *Let $h > \theta$ and $\beta \in (0,1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \geq t^{\beta}A(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{2.4}$$

$B : P \to P$ is an increasing sub-homogeneous operator. Assume that:

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \geq \delta_0 Bx$, $\forall x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*

(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution x^* in P_h;*

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Lemma 2.6 (see Theorem 2.4 in [12]) *Let $h > \theta$ and $\beta \in (0,1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \geq tA(x,y), \quad \forall t \in (0,1), x, y \in P. \tag{2.5}$$

$B : P \to P$ is an increasing β-concave operator. Assume that:

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x,y) \leq \delta_0 Bx$, $\forall x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*

(2) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation* $A(x,x) + Bx = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

3 Main results

In this section, we apply Lemmas 2.4-2.6 to study problem (1.1), and we obtain some new results on the existence and uniqueness of positive periodic solutions. The method used here is relatively new in the literature and so are the existence and uniqueness results regarding functional differential equations.

In our considerations we will work in the Banach space

$$E = \{x \in C(\mathbf{R}, \mathbf{R}) : x(t + T) = x(t)\}$$

endowed with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Notice that this space can be equipped with a partial order given by

$$x, y \in E, \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t), \quad t \in [0, T].$$

Define a cone $P = \{x \in E \mid x(t) \geq 0, t \in [0, T]\}$. It is easy to see that the cone P is normal. Consider the operator $Q : P \to E$ by

$$(Qy)(t) = \int_t^{t+T} G(t,s)\big[f\big(s, y(s - \tau(s)), y(s - \tau(s))\big) + g\big(s, y(s - \tau(s))\big)\big]\, ds, \tag{3.1}$$

where

$$G(t,s) = \frac{e^{\int_t^s \delta(u)\, du}}{e^{\int_0^T \delta(u)\, du} - 1}. \tag{3.2}$$

Now define, as in [5],

$$0 < m \equiv \min_{0 \leq t,s \leq T} G(t,s) \leq \max_{0 \leq t,s \leq T} G(t,s) \equiv M < \infty. \tag{3.3}$$

Firstly, we will use Lemma 2.4 to establish the existence and uniqueness of a positive periodic solution for problem (1.1).

For convenience, we make the assumptions:

(C_1) $f(t,x,y) \in C(\mathbf{R} \times [0,\infty) \times [0,\infty), [0,\infty))$ is T-periodic with respect to the first variable and increasing with respect to the second variable and decreasing with respect to the third variable;

(C_2) $g(t,y) \in C(\mathbf{R} \times [0,\infty), [0,\infty))$ is T-periodic with respect to the first variable and decreasing with respect to the second variable;

(C$_3$) $g(t,0) \not\equiv 0$ for $t \in \mathbf{R}$;
(C$_4$) for any $0 < p < q < 1$, there exist $\beta_1, \beta_2 \in (0,\infty)$, depending on p and q, such that

$$f\bigl(t, \lambda u, \lambda^{-1} v\bigr) \geq \frac{\lambda}{1-\lambda^{\beta_1}} f(t,u,v), \quad \forall t \in R, p \leq \lambda \leq q, u, v \in [0,\infty),$$
$$g\bigl(t, \lambda^{-1} v\bigr) \geq \frac{\lambda}{1-\lambda^{\beta_2}} g(t,v), \quad \forall t \in R, p \leq \lambda \leq q, v \in [0,\infty).$$

Lemma 3.1 (see [1]) *Assume that* (C$_1$), (C$_2$) *hold, then Q maps P into P.*

Following the approach in [5], we can easily prove the following lemma. So we omit the proof.

Lemma 3.2 *$y(t)$ is a positive T-periodic solution of problem* (1.1) *if and only if $y(t)$ is a positive solution of the operator equation $y(t) = (Qy)(t)$, where Q is given as in* (3.1).

Theorem 3.3 *Assume that* (C$_1$)-(C$_4$) *are satisfied and there exists $R_1 > 0$ such that*

$$MT(M_1 + M_2) \leq R_1, \tag{3.4}$$

where M is given as in (3.3) *and $M_1 = \max_{t\in\mathbf{R}} f(t,R_1,0)$, $M_2 = \max_{t\in\mathbf{R}} g(t,0)$.*

Then problem (1.1) *has a unique positive periodic solution u^* in $P_{[\theta,R_1]} = \{y \in P \mid 0 \leq y(t) \leq R_1, t \in \mathbf{R}\}$ with $u^*(t) \not\equiv 0$, $t \in \mathbf{R}$. Moreover, for any initial values $x_0, y_0 \in P$, constructing successively the sequences*

$$x_{n+1}(t) = \int_t^{t+T} G(t,s)\bigl[f\bigl(s, x_n\bigl(s-\tau(s)\bigr), y_n\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, y_n\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds,$$
$$n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_t^{t+T} G(t,s)\bigl[f\bigl(s, y_n\bigl(s-\tau(s)\bigr), x_n\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, x_n\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds,$$
$$n = 0,1,2,\ldots,$$

where $G(t,s)$ is given as in (3.2), *we have that $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.*

Proof To begin with, from Lemma 3.2, problem (1.1) has an integral formulation given by

$$y(t) = \int_t^{t+T} G(t,s)\bigl[f\bigl(s, y\bigl(s-\tau(s)\bigr), y\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, y\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds,$$

where $G(t,s)$ is given as in (3.2). Define an operator $S : P \times P \to E$ by

$$S(u,v)(t) = \int_t^{t+T} G(t,s)\bigl[f\bigl(s, u\bigl(s-\tau(s)\bigr), v\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, v\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds.$$

From Lemma 3.2, it is easy to prove that u is the solution of problem (1.1) if and only if u is the fixed point of S. From (C$_1$), (C$_2$) and (3.3), we know that $S : P \times P \to P$. In the sequel we check that S satisfies all the assumptions of Lemma 2.4.

Step 1. We prove that S is a mixed monotone operator. In fact, for $u_i, v_i \in P$, $i = 1, 2$, with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in \mathbf{R}$ and by (C_1), (C_2),

$$\begin{aligned} S(u_1, v_1)(t) &= \int_t^{t+T} G(t,s)\bigl[f\bigl(s, u_1\bigl(s-\tau(s)\bigr), v_1\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, v_1\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds \\ &\geq \int_t^{t+T} G(t,s)\bigl[f\bigl(s, u_2\bigl(s-\tau(s)\bigr), v_2\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, v_2\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds \\ &= S(u_2, v_2)(t), \end{aligned}$$

that is to say, $S(u_1, v_1) \geq S(u_2, v_2)$ for $u_1 \geq u_2$, $v_1 \leq v_2$.

Step 2. We show that S satisfies condition (a) of Lemma 2.4. Now, we set $v(t) = R_1$ for $t \in \mathbf{R}$. Then $v > \theta$. On the one hand, from (3.3) and (C_1)-(C_3), we obtain

$$S(v, \theta)(t) = \int_t^{t+T} G(t,s)\bigl[f(s, R_1, 0) + g(s, 0)\bigr]\,ds > 0,$$

that is to say, $S(v, \theta) > \theta$. From (3.4), for any $t \in \mathbf{R}$, we have

$$S(v, \theta)(t) \leq MT(M_1 + M_2) \leq R_1 = v(t),$$

i.e., $S(v, \theta) \leq v$. On the other hand, we take $c \in (0, 1)$ such that

$$c \leq \frac{m_1 + m_2}{M_1 + M_2},$$

where $m_1 = \min_{t\in\mathbf{R}} f(t, 0, R_1)$, $m_2 = \min_{t\in\mathbf{R}} g(t, R_1)$. For any $t \in \mathbf{R}$,

$$\begin{aligned} S(\theta, v)(t) &= \int_t^{t+T} G(t,s)\bigl[f(s, 0, R_1) + g(s, R_1)\bigr]\,ds \\ &\geq \int_t^{t+T} G(t,s)(m_1 + m_2)\,ds \\ &\geq \int_t^{t+T} G(t,s)c(M_1 + M_2)\,ds \\ &\geq \int_t^{t+T} G(t,s)c\bigl[f(s, R_1, 0) + g(s, 0)\bigr]\,ds \\ &= cS(v, \theta)(t), \end{aligned}$$

that is, $S(\theta, v) \geq cS(v, \theta)$. Hence, condition (a) of Lemma 2.4 holds.

Step 3. We show that S satisfies condition (b) of Lemma 2.4. Let $0 < a < b < 1$ and

$$\eta = \min\left\{\frac{1}{1 - a^{\beta_1}} - 1, \frac{1}{1 - a^{\beta_2}} - 1\right\} > 0.$$

For any $a \leq \lambda \leq b$, $x, y \in P$ and $t \in \mathbf{R}$, by (C_4) we have

$$\begin{aligned} &S\bigl(\lambda x, \lambda^{-1} y\bigr)(t) \\ &\quad= \int_t^{t+T} G(t,s)\bigl[f\bigl(s, \lambda x\bigl(s-\tau(s)\bigr), \lambda^{-1} y\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, \lambda^{-1} y\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds \end{aligned}$$

$$\begin{aligned}
&\geq \int_t^{t+T} G(t,s)\left[\frac{\lambda}{1-\lambda^{\beta_1}} f\big(s,x\big(s-\tau(s)\big),y\big(s-\tau(s)\big)\big) + \frac{\lambda}{1-\lambda^{\beta_2}} g\big(s,y\big(s-\tau(s)\big)\big)\right] ds \\
&\geq \int_t^{t+T} G(t,s)\left[\frac{\lambda}{1-a^{\beta_1}} f\big(s,x\big(s-\tau(s)\big),y\big(s-\tau(s)\big)\big) + \frac{\lambda}{1-a^{\beta_2}} g\big(s,y\big(s-\tau(s)\big)\big)\right] ds \\
&\geq \int_t^{t+T} G(t,s)\big[\lambda(1+\eta) f\big(s,x\big(s-\tau(s)\big),y\big(s-\tau(s)\big)\big) + \lambda(1+\eta) g\big(s,y\big(s-\tau(s)\big)\big)\big] ds \\
&= \lambda(1+\eta)S(x,y)(t).
\end{aligned}$$

That is to say, $S(\lambda x, \lambda^{-1}y) \geq \lambda(1+\eta)S(x,y)$, $\forall a \leq \lambda \leq b$, $x, y \in P$. Therefore, condition (b) of Lemma 2.4 is satisfied. Hence, the conclusion of Theorem 3.3 follows from Lemma 2.4. □

Secondly, we will use Lemma 2.5 to study the existence and uniqueness of positive periodic solutions for problem (1.1).

For convenience, we make the assumptions:

(C_5) $g(t,x) \in C(\mathbf{R} \times [0,\infty), [0,\infty))$ is T-periodic with respect to the first variable and increasing with respect to the second variable;

(C_6) $g(t,\lambda u) \geq \lambda g(t,u)$ for $\lambda \in (0,1)$, $t \in [0,T]$, $u \in [0,\infty)$, and there exists a constant $\beta \in (0,1)$ such that $f(t,\lambda u,\lambda^{-1}v) \geq \lambda^{\beta} f(t,u,v)$, $\forall t \in [0,T]$, $\lambda \in (0,1)$, $u, v \in [0,\infty)$;

(C_7) there exists a constant $\delta_0 > 0$ such that $f(t,u,v) \geq \delta_0 g(t,u)$, $t \in [0,T]$, $u, v \in [0,\infty)$;

(C_8) $f(t,mT,MT) > 0$, $g(t,mT) > 0$ for any $t \in \mathbf{R}$, where m, M is given as in (3.3).

Theorem 3.4 *Let* $h(t) = \int_t^{t+T} G(t,s)\,ds$, $t \in \mathbf{R}$. *Suppose that* (C_1) *and* (C_5)-(C_8) *hold. Then*:

(1) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$ *and*

$$\begin{aligned}
u_0(t) &\leq \int_t^{t+T} G(t,s)\big[f\big(s,u_0\big(s-\tau(s)\big),v_0\big(s-\tau(s)\big)\big) + g\big(s,u_0\big(s-\tau(s)\big)\big)\big] ds, \\
&\quad t \in [0,T], \\
v_0(t) &\geq \int_t^{t+T} G(t,s)\big[f\big(s,v_0\big(s-\tau(s)\big),u_0\big(s-\tau(s)\big)\big) + g\big(s,v_0\big(s-\tau(s)\big)\big)\big] ds, \\
&\quad t \in [0,T],
\end{aligned}$$

where $G(t,s)$ *is given as in* (3.2);

(2) *problem* (1.1) *has a unique positive solution* u^* *in* P_h;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$\begin{aligned}
x_{n+1}(t) &= \int_t^{t+T} G(t,s)\big[f\big(s,x_n\big(s-\tau(s)\big),y_n\big(s-\tau(s)\big)\big) + g\big(s,x_n\big(s-\tau(s)\big)\big)\big] ds, \\
&\quad n = 0,1,2,\ldots, \\
y_{n+1}(t) &= \int_t^{t+T} G(t,s)\big[f\big(s,y_n\big(s-\tau(s)\big),x_n\big(s-\tau(s)\big)\big) + g\big(s,y_n\big(s-\tau(s)\big)\big)\big] ds, \\
&\quad n = 0,1,2,\ldots,
\end{aligned}$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof We first need to assert that $h \in P$. It is clear that $h \in C(\mathbf{R}, \mathbf{R})$, and we have

$$\begin{aligned} h(t+T) &= \int_{t+T}^{t+2T} G(t+T, s)\, ds = \int_{t}^{t+T} G(t+T, u+T)\, du \\ &= \int_{t}^{t+T} G(t, u)\, du = h(t). \end{aligned}$$

Hence, $h \in E$. Note that $G(t,s) > 0$, so we have $h \in P$. Also, $mT \leq h(t) \leq MT$, $t \in \mathbf{R}$. From Lemma 3.2, problem (1.1) has an integral formulation given by

$$u(t) = \int_{t}^{t+T} G(t,s)\bigl[f\bigl(s, u\bigl(s-\tau(s)\bigr), u\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\bigr]\, ds.$$

Define two operators $A : P \times P \to E$ and $B : P \to E$ by

$$\begin{aligned} A(u,v)(t) &= \int_{t}^{t+T} G(t,s) f\bigl(s, u\bigl(s-\tau(s)\bigr), v\bigl(s-\tau(s)\bigr)\bigr)\, ds, \\ (Bu)(t) &= \int_{t}^{t+T} G(t,s) g\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\, ds. \end{aligned}$$

It is easy to prove that u is the solution of problem (1.1) if and only if $u = A(u,u) + Bu$. From (C_1) and (C_5), we know that $A : P \times P \to P$ and $B : P \to P$. In the sequel we check that A, B satisfy all the assumptions of Lemma 2.5.

Step 1. We prove that A is mixed monotone and B is increasing. In fact, for $u_i, v_i \in P$, $i = 1,2$, with $u_1 \geq u_2$, $v_1 \leq v_2$, we know that $u_1(t) \geq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0,1]$, and by (C_1) and (3.3),

$$\begin{aligned} A(u_1,v_1)(t) &= \int_{t}^{t+T} G(t,s) f\bigl(s, u_1\bigl(s-\tau(s)\bigr), v_1\bigl(s-\tau(s)\bigr)\bigr)\, ds \\ &\geq \int_{t}^{t+T} G(t,s) f\bigl(s, u_2\bigl(s-\tau(s)\bigr), v_2\bigl(s-\tau(s)\bigr)\bigr)\, ds = A(u_2,v_2)(t). \end{aligned}$$

That is, $A(u_1,v_1) \geq A(u_2,v_2)$ for $u_1 \geq u_2$, $v_1 \leq v_2$. Further, it follows from (C_5) and (3.3) that B is increasing.

Step 2. We show that A satisfies condition (2.4) and B is sub-homogeneous. For any $\lambda \in (0,1)$ and $u, v \in P$, by (C_6) we have

$$\begin{aligned} A\bigl(\lambda u, \lambda^{-1} v\bigr)(t) &= \int_{t}^{t+T} G(t,s) f\bigl(s, \lambda u\bigl(s-\tau(s)\bigr), \lambda^{-1} v\bigl(s-\tau(s)\bigr)\bigr)\, ds \\ &\geq \lambda^{\beta} \int_{t}^{t+T} G(t,s) f\bigl(s, u\bigl(s-\tau(s)\bigr), v\bigl(s-\tau(s)\bigr)\bigr)\, ds = \lambda^{\beta} A(u,v)(t). \end{aligned}$$

That is, $A(\lambda u, \lambda^{-1} v) \geq \lambda^{\beta} A(u,v)$ for $\lambda \in (0,1)$, $u, v \in P$. So the operator A satisfies (2.4). Also, for any $\lambda \in (0,1)$, $u \in P$, from (C_6) we know that

$$B(\lambda u)(t) = \int_{t}^{t+T} G(t,s) g\bigl(s, \lambda u\bigl(s-\tau(s)\bigr)\bigr)\, ds \geq \lambda \int_{t}^{t+T} G(t,s) g\bigl(s, u\bigl(s-\tau(s)\bigr)\bigr)\, ds = \lambda Bu(t),$$

that is, $B(\lambda u) \geq \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. That is, the operator B is sub-homogeneous.

Step 3. We show that $A(h,h) \in P_h$ and $Bh \in P_h$. Set

$$r_1 = \min_{t\in[0,T]} f(t,mT,MT), \qquad r_2 = \max_{t\in[0,T]} f(t,MT,mT).$$

Then, from (C_8), we obtain $0 < r_1 \le r_2$. So

$$A(h,h)(t) = \int_t^{t+T} G(t,s)f\bigl(s,h\bigl(s-\tau(s)\bigr),h\bigl(s-\tau(s)\bigr)\bigr)\,ds \ge r_1 \int_t^{t+T} G(t,s) = r_1 h(t), \quad (3.5)$$

$$A(h,h)(t) = \int_t^{t+T} G(t,s)f\bigl(s,h\bigl(s-\tau(s)\bigr),h\bigl(s-\tau(s)\bigr)\bigr)\,ds \le r_2 \int_t^{t+T} G(t,s) = r_2 h(t), \quad (3.6)$$

that is to say, $r_1 h \le A(h,h) \le r_2 h$, *i.e.*, $A(h,h) \in P_h$. Similarly, from (C_8) we can get $Bh \in P_h$. Hence, condition (i) of Lemma 2.5 is satisfied.

Step 4. We show that condition (ii) of Lemma 2.5 is satisfied. For $u,v \in P$ and any $t \in [0,T]$, from (C_7) we have

$$\begin{aligned} A(u,v)(t) &= \int_t^{t+T} G(t,s)f\bigl(s,u\bigl(s-\tau(s)\bigr),v\bigl(s-\tau(s)\bigr)\bigr)\,ds \\ &\ge \delta_0 \int_t^{t+T} G(t,s)g\bigl(s,u\bigl(s-\tau(s)\bigr)\bigr)\,ds = \delta_0 Bu(t). \end{aligned}$$

Then we get $A(u,v) \ge \delta_0 Bu$ for $u,v \in P$.

Hence, all the conditions of Lemma 2.5 are satisfied. Applying Lemma 2.5 we have that there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$, $u_0 \le A(u_0,v_0) + Bu_0 \le A(v_0,u_0) + Bv_0 \le v_0$; the operator $A(u,u) + Bu = u$ has a unique solution u^* in P_h, and, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1},y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1},x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$u_0(t) \le \int_t^{t+T} G(t,s)\bigl[f\bigl(s,u_0\bigl(s-\tau(s)\bigr),v_0\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s,u_0\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds, \quad t \in [0,T],$$

$$v_0(t) \ge \int_t^{t+T} G(t,s)\bigl[f\bigl(s,v_0\bigl(s-\tau(s)\bigr),u_0\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s,v_0\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds, \quad t \in [0,T];$$

problem (1.1) has a unique positive solution u^* in P_h, and, for $x_0, y_0 \in P_h$, the sequences

$$\begin{aligned} x_{n+1}(t) &= \int_t^{t+T} G(t,s)\bigl[f\bigl(s,x_n\bigl(s-\tau(s)\bigr),y_n\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s,x_n\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds, \\ &\quad n = 0,1,2,\ldots, \\ y_{n+1}(t) &= \int_t^{t+T} G(t,s)\bigl[f\bigl(s,y_n\bigl(s-\tau(s)\bigr),x_n\bigl(s-\tau(s)\bigr)\bigr) + g\bigl(s,y_n\bigl(s-\tau(s)\bigr)\bigr)\bigr]\,ds, \\ &\quad n = 0,1,2,\ldots, \end{aligned}$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Finally, we will use Lemma 2.6 to study the existence and uniqueness of positive periodic solutions of problem (1.1).

For convenience, we make the assumptions:

(C_9) there exists a constant $\beta \in (0,1)$ such that $g(t,\lambda u) \geq \lambda^\beta g(t,u)$, $\forall t \in [0,T]$, $\lambda \in (0,1)$, $u \in [0,\infty)$, and $f(t,\lambda u,\lambda^{-1}v) \geq \lambda f(t,u,v)$ for $\lambda \in (0,1)$, $t \in [0,T]$, $u,v \in [0,\infty)$;

(C_{10}) there exists a constant $\delta_0 > 0$ such that $f(t,u,v) \leq \delta_0 g(t,u)$, $t \in [0,T]$, $u,v \in [0,\infty)$.

Theorem 3.5 *Suppose that* (C_1), (C_5), (C_8)-(C_{10}) *hold. Then*:

(1) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 < v_0$ *and*

$$u_0(t) \leq \int_t^{t+T} G(t,s)\left[f\left(s,u_0\left(s-\tau(s)\right),v_0\left(s-\tau(s)\right)\right) + g\left(s,u_0\left(s-\tau(s)\right)\right)\right] ds,$$
$$t \in [0,T],$$
$$v_0(t) \geq \int_t^{t+T} G(t,s)\left[f\left(s,v_0\left(s-\tau(s)\right),u_0\left(s-\tau(s)\right)\right) + g\left(s,v_0\left(s-\tau(s)\right)\right)\right] ds,$$
$$t \in [0,T],$$

where $h(t) = \int_t^{t+T} G(t,s)\,ds$, $t \in \mathbf{R}$, *and* $G(t,s)$ *is given as in* (3.2);

(2) *problem* (1.1) *has a unique positive solution* u^* *in* P_h;

(3) *for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_t^{t+T} G(t,s)\left[f\left(s,x_n\left(s-\tau(s)\right),y_n\left(s-\tau(s)\right)\right) + g\left(s,x_n\left(s-\tau(s)\right)\right)\right] ds,$$
$$n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_t^{t+T} G(t,s)\left[f\left(s,y_n\left(s-\tau(s)\right),x_n\left(s-\tau(s)\right)\right) + g\left(s,y_n\left(s-\tau(s)\right)\right)\right] ds,$$
$$n = 0,1,2,\ldots,$$

we have $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Sketch of the proof Consider two operators A, B defined in the proof of Theorem 3.4. Similarly, from (C_1) and (C_5) we obtain that $A : P \times P \to P$ is a mixed monotone operator and $B : P \to P$ is increasing. From (C_9) we have

$$A\left(\lambda u,\lambda^{-1}v\right) \geq \lambda A(u,v), \qquad B(\lambda u) \geq \lambda^\beta Bu, \quad \text{for } \lambda \in (0,1), u,v \in P.$$

Also, we can obtain (3.5) and (3.6). So, $A(h,h) \in P_h$ and $Bh \in P_h$. Hence, condition (i) of Lemma 2.6 is satisfied. For $u,v \in P$ and any $t \in [0,T]$, from (C_{10}) we have

$$\begin{aligned} A(u,v)(t) &= \int_t^{t+T} G(t,s) f\left(s,u\left(s-\tau(s)\right),v\left(s-\tau(s)\right)\right) ds \\ &\leq \delta_0 \int_t^{t+T} G(t,s) g\left(s,u\left(s-\tau(s)\right)\right) ds = \delta_0 Bu(t). \end{aligned}$$

Then we get $A(u,v) \leq \delta_0 Bu$ for $u,v \in P$.

An application of Lemma 2.6 implies that there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 < v_0$, $u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0$; the operator $A(u,u) + Bu = u$ has a unique solution u^* in P_h, and, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$u_0(t) \leq \int_t^{t+T} G(t,s)\big[f\big(s, u_0\big(s-\tau(s)\big), v_0\big(s-\tau(s)\big)\big) + g\big(s, u_0\big(s-\tau(s)\big)\big)\big]\,ds, \quad t \in [0,T],$$
$$v_0(t) \geq \int_t^{t+T} G(t,s)\big[f\big(s, v_0\big(s-\tau(s)\big), u_0\big(s-\tau(s)\big)\big) + g\big(s, v_0\big(s-\tau(s)\big)\big)\big]\,ds, \quad t \in [0,T];$$

problem (1.1) has a unique positive solution u^* in P_h, and, for $x_0, y_0 \in P_h$, the sequences

$$x_{n+1}(t) = \int_t^{t+T} G(t,s)\big[f\big(s, x_n\big(s-\tau(s)\big), y_n\big(s-\tau(s)\big)\big) + g\big(s, x_n\big(s-\tau(s)\big)\big)\big]\,ds,$$
$$n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_t^{t+T} G(t,s)\big[f\big(s, y_n\big(s-\tau(s)\big), x_n\big(s-\tau(s)\big)\big) + g\big(s, y_n\big(s-\tau(s)\big)\big)\big]\,ds,$$
$$n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Remark 3.6 Comparing Theorems 3.3-3.5 with the main results in [5], we provide some alternative approaches to study a similar type of problems under different conditions. Our results can guarantee the existence of a unique positive periodic solution without assuming the existence of upper-lower solutions. Our results are seldom seen in the literature.

4 Examples

First, we present one example to illustrate Theorem 3.3.

Example 4.1 Consider the following equation:

$$y'(t) = F\big(t, y(t), y\big(t-\tau(t)\big)\big). \tag{4.1}$$

In this example, we choose $T = \pi$,

$$F\big(t, y(t), y\big(t-\tau(t)\big)\big) = -\frac{1}{\pi}y(t) + \sin^2 t + \frac{y(t-\frac{\pi}{500})}{1+y(t-\frac{\pi}{500})} + \cos^2 t + \frac{2}{1+y(t-\frac{\pi}{500})} + 2,$$

and take

$$\delta(t) = \frac{1}{\pi}, \qquad \tau(t) = \frac{\pi}{500},$$
$$f(t,x,y) = \sin^2 t + 1 + \frac{x}{1+x} + \frac{1}{1+y}, \qquad g(t,y) = \cos^2 t + 1 + \frac{1}{1+y}.$$

Then $f(t,x,y) \in C(\mathbf{R} \times [0,\infty) \times [0,\infty),[0,\infty))$ is π-periodic with respect to the first variable, increasing with respect to the second variable and decreasing with respect to the third variable. $g(t,y) \in C(\mathbf{R} \times [0,\infty),[0,\infty))$ is π-periodic with respect to the first variable and decreasing with respect to the second variable. Moreover, for any $0 < p < q < 1$, we take

$$\beta_1 \geq \log_q\left(1 - \frac{3q}{1+2q}\right), \qquad \beta_2 \geq \log_q\left(1 - \frac{2q}{1+q}\right). \tag{4.2}$$

For any $p \leq \lambda \leq q$, $t \in R$, $x,y \in [0,\infty)$, we have

$$f\left(t,\lambda x,\lambda^{-1}y\right) = \sin^2 t + 1 + \frac{\lambda x}{1+\lambda x} + \frac{\lambda}{\lambda + y} \geq \frac{\lambda}{q}\left(\sin^2 t + 1\right) + \lambda\left(\frac{x}{1+x} + \frac{1}{1+y}\right).$$

We show that

$$f\left(t,\lambda x,\lambda^{-1}y\right) \geq \frac{\lambda}{1-\lambda^{\beta_1}} f(t,x,y), \tag{4.3}$$

if and only if we can prove

$$\frac{1}{q}\left(\sin^2 t + 1\right) + \frac{x}{1+x} + \frac{1}{1+y} \geq \frac{1}{1-\lambda^{\beta_1}}\left(\sin^2 t + 1 + \frac{x}{1+x} + \frac{1}{1+y}\right),$$

that is to say,

$$\left(\sin^2 t + 1\right)\left(\frac{1}{q} - \frac{1}{1-\lambda^{\beta_1}}\right) \geq \left(\frac{1}{1-\lambda^{\beta_1}} - 1\right)\left(\frac{x}{1+x} + \frac{1}{1+y}\right). \tag{4.4}$$

From (4.2) we get

$$\frac{1}{q} - \frac{1}{1-\lambda^{\beta_1}} \geq 2\left(\frac{1}{1-\lambda^{\beta_1}} - 1\right).$$

Hence, (4.4) is satisfied. So, (4.3) holds. Similarly, we obtain that

$$g\left(t,\lambda^{-1}y\right) = \cos^2 t + 1 + \frac{\lambda}{\lambda + y} \geq \frac{\lambda}{q}\left(\cos^2 t + 1\right) + \frac{\lambda}{1+y} \geq \frac{\lambda}{1-\lambda^{\beta_2}} g(t,y).$$

Further, $g(t,0) = \cos^2 t + 1 \geq 1 \not\equiv 0$. Finally, from (3.2) we have

$$G(t,s) = \frac{e^{\int_t^s \delta(u)\,du}}{e^{\int_0^T \delta(u)\,du} - 1} = \frac{e^{\frac{s-t}{\pi}}}{e^{\frac{T}{\pi}} - 1} \leq \frac{e^{\frac{T}{\pi}}}{e^{\frac{T}{\pi}} - 1} = \frac{e}{e-1} \equiv M.$$

We take $R_1 > 0$ such that

$$R_1 \geq 7e\pi.$$

And thus

$$M_1 = \max_{t\in\mathbf{R}} f(t,R_1,0) = \max_{t\in\mathbf{R}}\left(\sin^2 t + 1 + \frac{R_1}{R_1+1} + 1\right) \leq 4,$$
$$M_2 = \max_{t\in\mathbf{R}} g(t,0) = \max_{t\in\mathbf{R}}\left(\cos^2 t + 1 + 1\right) = 3.$$

Then

$$MT(M_1 + M_2) \le \frac{e}{e-1}\pi(4+3) = \frac{7e\pi}{e-1} \le 7e\pi \le R_1.$$

Hence, (3.4) holds. Therefore, all of the conditions of Theorem 3.3 are satisfied. An application of Theorem 3.3 implies that problem (4.1) has a unique positive periodic solution in $P_{[\theta,R_1]} = \{y \in P \mid 0 \le y(t) \le R_1, t \in \mathbf{R}\}$.

Next, we present another example to illustrate Theorem 3.4.

Example 4.2 Consider the following equation:

$$y'(t) = F\bigl(t, y(t), y\bigl(t - \tau(t)\bigr)\bigr). \tag{4.5}$$

In this example, we choose $T = \pi$,

$$\begin{aligned} F\bigl(t, y(t), y\bigl(t - \tau(t)\bigr)\bigr) = {} & -\frac{1}{4\pi}y(t) + \sin^2 t + y\left(t - \frac{\pi}{500}\right)^{\frac{1}{4}} + y\left(t - \frac{\pi}{500}\right)^{-\frac{1}{3}} + 3 \\ & + \cos^2 t + \frac{y(t - \frac{\pi}{500})}{1 + 2y(t - \frac{\pi}{500})}, \end{aligned}$$

and take

$$\begin{aligned} &\delta(t) = \frac{1}{4\pi}, \qquad \tau(t) = \frac{\pi}{500}, \\ &f(t,x,y) = \sin^2 t + x^{\frac{1}{4}} + y^{-\frac{1}{3}} + 3, \qquad g(t,x) = \cos^2 t + \frac{x}{1+2x}. \end{aligned}$$

Then $f(t,x,y) \in C(\mathbf{R} \times [0,\infty) \times [0,\infty), [0,\infty))$ is π-periodic with respect to the first variable, increasing with respect to the second variable and decreasing with respect to the third variable. $g(t,y) \in C(\mathbf{R} \times [0,\infty), [0,\infty))$ is π-periodic with respect to the first variable and increasing with respect to the second variable. Moreover, for $\lambda \in (0,1)$, $t \in \mathbf{R}$, $x, y \in [0,\infty)$, we have

$$f\bigl(t, \lambda x, \lambda^{-1} y\bigr) = \sin^2 t + \lambda^{\frac{1}{4}} x^{\frac{1}{4}} + \lambda^{\frac{1}{3}} y^{-\frac{1}{3}} + 3 \ge \lambda^{\frac{1}{3}}\bigl(\sin^2 t + x^{\frac{1}{4}} + y^{-\frac{1}{3}} + 3\bigr) = \lambda^{\beta} f(t,x,y),$$

where $\beta = \frac{1}{3}$, and

$$g(t, \lambda x) = \cos^2 t + \frac{\lambda x}{1 + 2\lambda x} \ge \cos^2 t + \frac{\lambda x}{1+2x} \ge \lambda\left(\cos^2 t + \frac{x}{1+2x}\right) = \lambda g(t,x).$$

Further, we take $\delta_0 \in (0,1]$, then

$$f(t,x,y) = \sin^2 t + x^{\frac{1}{4}} + y^{-\frac{1}{3}} + 3 \ge 3 \ge \delta_0\left(\cos^2 t + \frac{x}{1+2x}\right) = \delta_0 g(t,x).$$

Hence all of the conditions of Theorem 3.4 are satisfied. An application of Theorem 3.4 implies that problem (4.2) has a unique positive solution in $P_h = \mathring{P} = \{x \in C(\mathbf{R}, \mathbf{R}) : x(t+T) = x(t) \text{ and } x(t) > 0, t \in [0,\pi]\}$, where $h(t) = \pi$.

Finally, we present another example to illustrate Theorem 3.5.

Example 4.3 Consider the following equation:

$$y'(t) = F\big(t, y(t), y\big(t - \tau(t)\big)\big). \tag{4.6}$$

In this example, we choose $T = \pi$,

$$F\big(t, y(t), y\big(t-\tau(t)\big)\big) = -\frac{1}{4\pi}y(t) + \sin^2 t + \frac{y(t-\frac{\pi}{500})}{1+y(t-\frac{\pi}{500})} + \frac{1}{1+y(t-\frac{\pi}{500})}$$
$$+ \cos^2 t + \frac{y(t-\frac{\pi}{500})^{\frac{1}{2}}}{1+y(t-\frac{\pi}{500})^{\frac{1}{2}}} + 4,$$

and take

$$\delta(t) = \frac{1}{4\pi}, \qquad \tau(t) = \frac{\pi}{500},$$
$$f(t,x,y) = \sin^2 t + \frac{x}{1+x} + \frac{1}{1+y}, \qquad g(t,x) = \cos^2 t + \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}} + 4.$$

Then $f(t,x,y) \in C(\mathbf{R} \times [0,\infty) \times [0,\infty), [0,\infty))$ is π-periodic with respect to the first variable, increasing with respect to the second variable and decreasing with respect to the third variable. $g(t,y) \in C(\mathbf{R} \times [0,\infty), [0,\infty))$ is π-periodic with respect to the first variable and increasing with respect to the second variable. Moreover, for $\lambda \in (0,1)$, $t \in \mathbf{R}$, $x, y \in [0,\infty)$, we have

$$f\big(t, \lambda x, \lambda^{-1}y\big) = \sin^2 t + \frac{\lambda x}{1+\lambda x} + \frac{1}{1+\lambda^{-1}y} \geq \lambda\left(\sin^2 t + \frac{x}{1+x} + \frac{1}{1+y}\right) = \lambda f(t,x,y),$$
$$g(t,\lambda x) = \cos^2 t + \frac{\lambda^{\frac{1}{2}}x^{\frac{1}{2}}}{1+\lambda^{\frac{1}{2}}x^{\frac{1}{2}}} + 4 \geq \lambda^{\frac{1}{2}}\left(\cos^2 t + \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}} + 4\right) = \lambda^{\beta} g(t,x),$$

where $\beta = \frac{1}{2}$. Further, we take $\delta_0 \in (0,1]$, then

$$g(t,x) = \cos^2 t + \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}} + 4 \geq 4 \geq \delta_0\left(\sin^2 t + \frac{x}{1+x} + \frac{1}{1+y}\right) = \delta_0 f(t,x,y).$$

Hence all of the conditions of Theorem 3.5 are satisfied. An application of Theorem 3.5 implies that problem (4.3) has a unique positive solution in $P_h = \mathring{P} = \{x \in C(\mathbf{R}, \mathbf{R}) : x(t+T) = x(t) \text{ and } x(t) > 0, t \in [0,\pi]\}$, where $h(t) = \pi$.

Remark 4.4 To summarize, Examples 4.1-4.3 illustrate different situations in which the conditions of Theorems 3.3-3.5 are satisfied for some particular functions. In addition, the conditions of Theorems 3.3-3.5 are also easy to check for any given periodicity $T > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors? contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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Acknowledgements

The first author was supported by the Science Foundation of Shanxi Province (2013011003-3) and the Science Foundation of Business College of Shanxi University (2012050). The second author was partially supported by the Youth Science Foundations of China (11201272) and Shanxi Province (2010021002-1).

Received: 14 October 2014 Accepted: 23 December 2014 Published online: 14 January 2015

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